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On Some Problems of Difference Functions and Difference Equations

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Abstract. This paper is devoted to studying uniqueness of q-difference polynomials sharing values or fixed points. The results generalize those given by Zhang and Korhonen. Some examples are given to show the sharpness of our results. Moreover, value distribution of q-difference functions, and uniqueness of entire solutions of a certain type of q-difference equation are also considered.

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1. Introduction and main results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ be a non-constant meromorphic function in $\mathbb{C}$. It is assumed that the reader is familiar with the standard notion used in Nevanlinna value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$ (see, e.g. [8, 10, 22, 23]), and $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function of $f(z)$, provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $a(z)$ be a small function of $f(z)$ and $g(z)$ (Specially, $a(z)$ may be a constant or ∞). We say that f and g share the a CM provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. Let b and c be constants or ∞. If $f=b$ whenever $g=c$, we write $g=c \Rightarrow f=b$. If $g=c \Rightarrow f=b$ and $f=b \Rightarrow g=c$, we write $f=b \Leftrightarrow g=c$. We denote by $N_{k)}(r,1/(f-a))$ the counting function for zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}(r,1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,1/(f-a))$ be the counting function for zeros of $f-a$ with multiplicity $\geq k$, and $\overline{N}_{(k}(r,1/(f-a))$ be the corresponding one for which multiplicity is not counted. Moreover, we set $N_k(r,1/(f-a)) = \overline{N}(r,1/(f-a)) + \overline{N}_{(2}(r,1/(f-a)) + \overline{N}_{(3}(r,1/(f-a)) + \cdots + \overline{N}_{(k}(r,1/(f-a))$. We say that a finite value z_0 is a fixed point of $f(z)$ if $f(z_0) = z_0$. If $f-z$

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and $g - z$ assume the same zeros with the same multiplicities, then we say f and g share z CM.

The key result of difference counterparts of Nevanlinna Theory is the difference analogue of the lemma on the logarithmic derivative, which was obtained by Halburd-Korhonen [5] and Chiang-Feng [3], independently. Related to q-difference equations, Barnett *et al.* [1] got the corresponding logarithmic derivative lemma as follows.

Theorem 1.1. *Let $f(z)$ be a non-constant zero-order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m(r, f(qz)/f(z)) = S(r, f).$$

They also gave an example to show the sharpness of the condition of zero-order.

Recently, Zhang and Korhonen used Theorem 1.1 to investigate the relationship between $T(r, f(qz))$ and $T(r, f(z))$, as applications, corresponding to uniqueness theorems of meromorphic functions sharing values (see, e.g. [2, 4, 9, 11, 12, 13, 14, 16, 17]), they studied value distribution and uniqueness of some q-difference functions. Specially, they got

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (resp. entire) functions of zero-order. Suppose that q is a nonzero complex constant and n is an integer satisfying $n \geq 8$ (resp. $n \geq 4$). If $f^n(z)f(qz)$ and $g^n(z)g(qz)$ share 1, ∞ CM, then $f(z) \equiv tg(z)$ for $t^{n+1} = 1$.*

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of zero-order. Suppose that q is a nonzero complex constant and $n \geq 6$ is an integer. If $f^n(z)(f(z)-1)f(qz)$ and $g^n(z)(g(z)-1)g(qz)$ share 1 CM, then $f(z) \equiv g(z)$.*

We improve Theorem 1.2 as follows.

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions of zero-order. Suppose that q is a nonzero complex constant and $n \geq 14$ is an integer. If $f^n(z)f(qz)$ and $g^n(z)g(qz)$ share 1 CM, f and g have at least one common pole, then $f(z) \equiv tg(z)$, where t is a constant such that $t^{n+1} = 1$.*

Example 1.1. Let t be a nonzero constant. $f(z) = p(z)/r(z)$ and $g(z) = t \cdot r(z)/p(z)$ are two meromorphic functions of zero-order, where $p(z)$ and $r(z)$ are nonconstant entire functions (Specially, $f(z)$ and $g(z)$ can be rational functions). Then $f^n(z)f(qz)$ and $g^n(z)g(qz)$ share 1 CM, but $f \neq tg$.

This example shows that the condition "f and g have at least one common pole" is sharp, without such a condition, one can get $fg \equiv t$ besides $f \equiv tg$, in which t is a constant such that $t^{n+1} = 1$.

Considering fixed points, we obtain a similar theorem as above.

Theorem 1.5. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero-order. Suppose that q is a nonzero complex constant and $n \geq 14$ is an integer. If $f^n(z)f(qz)$ and $g^n(z)g(qz)$ share z CM, then $f(z) \equiv tg(z)$ for $t^{n+1} = 1$.*

Remark 1.1. In Theorem 1.5, it seems that if z is replaced by a polynomial $p(z)$ with $\deg(p(z)) = p \geq 1$, and let $n \geq \max\{2p+2, 14\}$, the conclusion still holds.

It's natural to ask whether Theorem 1.3 holds if f and g are meromorphic functions. The answer is negative. We give the following:

Example 1.2. Let q be a constant($|q| \neq 0, 1$), n be a positive integer. Suppose that

$$f(z) = \frac{K(z)^{n+1}K(qz) - K(z)}{K(z)^{n+1}K(qz) - 1}, \quad g(z) = \frac{K(z)^{n}K(qz) - 1}{K(z)^{n+1}K(qz) - 1},$$

where $K(z)$ is a nonconstant entire function of zero-order (Specially, $K(z)$ can be a nonconstant polynomial).

We deduce that $f^n(z)(f(z)-1)f(qz)$ and $g^n(z)(g(z)-1)g(qz)$ share 1 CM and $f(z)$ and $g(z)$ share ∞ CM, moreover, $f^n(z)(f(z)-1)f(qz) \equiv g^n(z)(g(z)-1)g(qz)$, but $f(z) \not\equiv g(z)$. This example shows that one cannot get $f(z) \equiv g(z)$ from $f^n(z)(f(z)-1)f(qz) \equiv g^n(z)(g(z) -1)g(qz)$. Thus Theorem 1.3 does not hold if $f(z)$ and $g(z)$ are non-entire meromorphic functions, even if $f(z)$ and $g(z)$ share ∞ CM.

In fact, we prove:

Theorem 1.6. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions of zero-order. Suppose that q is a constant($|q| \neq 0, 1$) and $n \geq 15$ is an integer. If $f^n(z)(f(z)-1)f(qz)$ and $g^n(z)(g(z)-1)g(qz)$ share 1 CM, $f(z)$ and $g(z)$ share ∞ IM, then $f^n(z)(f(z)-1)f(qz) \equiv g^n(z)(g(z)-1)g(qz)$.*

In 1959, Hayman [7] proved:

Theorem 1.7. *Let f be a transcendental meromorphic function and $a(\neq 0)$, b be finite complex constants. Then $f^n + af' - b$ has infinitely many zeros for $n \geq 5$. If f is transcendental entire, this holds for $n \geq 3$, resp. $n \geq 2$, if $b = 0$.*

We give an analogous result in q-difference as follows.

Theorem 1.8. *Let f be a transcendental meromorphic function of zero-order, and $a(z)(\not\equiv 0, \infty)$, $b(z)(\not\equiv \infty)$ be small functions of $f(z)$. Then $f^n(z) + a(z)f(qz) - b(z)$ has infinitely many zeros for $n \geq 6$. If $f(z)$ is transcendental entire, this holds for $n \geq 2$.*

Example 1.3. Let $f(z) = e^z$, $n = q$, $a(z) = a$ be a nonzero constant. Then $f^n(z) + af(qz) = (a+1)e^{nz}$ has no zero. This example shows that the condition "f is of zero-order" in Theorem 1.8 can not be removed.

Difference of Nevanlinna theory has been widely used to consider growth, oscillation, and existence of entire or meromorphic solutions of linear and nonlinear difference equations in complex domains (see e.g. [3, 6, 15, 18, 19, 21, 26]). Recently, Yang and Laine [21] studied the existence and uniqueness of finite order entire solutions of the nonlinear differential equations and differential difference equations. Specially, they got

Theorem 1.9. *Let $n \geq 4$ be an integer, $M(z, f)$ be a linear differential-difference polynomial of f, not vanishing identically, and h be a meromorphic function of finite order. Then the differential difference equation*

$$f^n + M(z, f) = h,$$

possesses at most one admissible transcendental entire solution of finite order such that all coefficients of $M(z, f)$ are small functions of f. If such a solution f exists, then f is of the same order as h.

By Theorem 1.1, using the similar proof of Theorem 1.9, we obtain a similar theorem.

Theorem 1.10. *Let $n \geq 4$ be an integer, $M(z,f)(=a(z)f(qz)+b(z))$ be a linear q-difference polynomial of f, not vanishing identically, and h be a meromorphic function of zero order. Then the q-difference equation*

$$f^n + M(z,f) = h, \tag{1.1}$$

possesses at most one admissible transcendental entire solution of zero-order such that $a(z)$ and $b(z)$ are small functions of f.

As an application of Theorem 1.10, we give a uniqueness theorem related to Theorem 1.8.

Theorem 1.11. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of zero-order, $n \geq 7$ be an integer, q be a nonzero constant, and $a(z)(\not\equiv 0,\infty)$, $b(z)(\not\equiv 0,\infty)$ be small functions with respect to both $f(z)$ and $g(z)$. $a(z)$ and $b(z)$ have finitely many poles. If $f^n(z)+a(z)f(qz)$ and $g^n(z)+a(z)g(qz)$ share $b(z)$ CM, then $f(z) \equiv g(z)$.*

2. Preliminary lemmas

Lemma 2.1. [25] *Let $f(z)$ be a non-constant zero-order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$T(r,f(qz)) = (1+o(1))T(r,f(z))$$

on a set of lower logarithmic density 1.

Remark 2.1. Zhang and Korhonen gave an example to show that the zero-order growth restriction in Lemma 2.1 cannot be, in general, generalize to include any strictly positive order.

Lemma 2.2. [20] *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and n,k be two positive integers, a be a finite nonzero constant. If f and g share 1 CM, then one of the following cases holds:*

(i) *$T(r,f) \leq N_2(r,1/f)+N_2(r,1/g)+N_2(r,f)+N_2(r,g)+S(r,f)+S(r,g)$, the same inequality holding for $T(r,g)$;*
(ii) *$fg \equiv 1$;*
(iii) *$f \equiv g$.*

Lemma 2.3. [24] *Let f_j $(j=1,2,3)$ be meromorphic functions that satisfy*

$$\sum_{j=1}^{3} f_j = 1. \tag{2.1}$$

Assume that f_1 is not a constant, and

$$\sum_{j=1}^{3} N_2(r,1/f_j) + \sum_{j=1}^{3} \overline{N}(r,f_j) \leq (\lambda+o(1))T(r), \quad r \in I, \tag{2.2}$$

where $\lambda < 1$, $T(r) = \max\{T(r,f_j), j=1,2,3\}$, I is a set of $r \in (0,\infty)$ with infinite linear measure. Then

$$f_2 = 1 \quad or \quad f_3 = 1. \tag{2.3}$$

3. Proof of Theorem 1.4

Let $F = f^n(z)f(qz)$, $G = g^n(z)g(qz)$. Then F and G share 1 CM. By Lemma 2.2 we have

(i) $T(r,F) \le N_2(r,1/F) + N_2(r,1/G) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G)$;
(ii) $FG \equiv 1$;
(iii) $F \equiv G$.

By Lemma 2.1 we obtain

$$nT(r, f(z)) = T(r, f^n(z)) = T(r, F/f(qz)) \le T(r,F) + T(r, f(z)) + S(r, f),$$

namely

$$(n-1)T(r, f(z)) \le T(r,F) + S(r, f). \tag{3.1}$$

For Case (i), we get

$$\begin{aligned} T(r,F(z)) \le\ & 2\overline{N}(r,1/f(z)) + N(r,1/f(qz)) + 2\overline{N}(r,1/g(z)) + N(r,1/g(qz)) + 2\overline{N}(r, f(z)) \\ & + N(r, f(qz)) + 2\overline{N}(r, g(z)) + N(r, g(qz)) + S(r, f) + S(r, g) \\ \le\ & 6(T(r, f(z)) + T(r, g(z))) + S(r, f) + S(r, g). \end{aligned}$$

This together with (3.1) implies

$$(n-1)T(r, f(z)) \le 6(T(r, f(z)) + T(r, g(z))) + S(r, f) + S(r, g). \tag{3.2}$$

Similarly we have

$$(n-1)T(r, g(z)) \le 6(T(r, f(z)) + T(r, g(z))) + S(r, f) + S(r, g). \tag{3.3}$$

Combining (3.2) and (3.3) gives

$$n(T(r, f(z)) + T(r, g(z))) \le 13(T(r, f(z)) + T(r, g(z))) + S(r, f) + S(r, g),$$

which contradicts the assumption. Case (i) has been ruled out.

For Case (ii), we have $FG \equiv 1$, namely

$$f(z)^n f(qz)g(z)^n g(qz) \equiv 1. \tag{3.4}$$

Let $h = f(z)g(z)$. Then (3.4) implies

$$h^n(z)h(qz) \equiv 1. \tag{3.5}$$

If $h(z)$ is not a constant, then from (3.5) we deduce

$$nT(r, h(z)) = T(r, h^n(z)) = T(r, 1/h(qz)) = T(r, h(z)) + S(r, h), \tag{3.6}$$

which is a contradiction. Thus $h(z) \equiv t$, where t is a nonzero constant. We have $f(z)g(z) \equiv t$. Since $f(z)$ and $g(z)$ have at least one common pole, we immediately get a contradiction. Thus $f(z) \ne \infty$ and $g(z) \ne \infty$. Moreover, we obtain $f(z) \ne 0$ and $g(z) \ne 0$. Thus $f = e^{\alpha(z)}$, where $\alpha(z)$ is a non-constant entire function. Since $f(z)$ is of zero-order, we get that $\alpha(z)$ is a constant, a contradiction. Case (ii) has been ruled out.

Therefore, $F \equiv G$, namely, $f(z)^n f(qz) \equiv g(z)^n g(qz)$. Let $H(z) = f(z)/g(z)$, similarly as the proof in Case 2, we get $H(z) \equiv t$, thus $f \equiv tg$, where t is a constant such that $t^{n+1} = 1$. This completes the proof of Theorem 1.4. ∎

4. Proof of Theorem 1.5

Let $F = f^n(z)f(qz)/z$, $G = g^n(z)g(qz)/z$. Then F and G share 1 CM. Note that f and g are transcendental and z is a small function of f and g. Similar to the proof in Theorem 1.4, we obtain $f(z)^n f(qz)g(z)^n g(qz) \equiv z^2$ or $f(z)^n f(qz) \equiv g(z)^n g(qz)$. The last one implies $f \equiv tg$, where t is a constant such that $t^{n+1} = 1$. Now we only need to consider the case $f(z)^n f(qz)g(z)^n g(qz) \equiv z^2$. Let $h(z) = f(z)g(z)$ and we get

$$h^n(z)h(qz) \equiv z^2. \tag{4.1}$$

Obviously $h(z)$ can not be a constant, but note that

$$T(r,h(z)) \geq \log r + O(1) \tag{4.2}$$

and we still get a contradiction from the following

$$\begin{aligned} nT(r,h(z)) &= T(r,h^n(z)) = T(r,z^2/h(qz)) \\ &\leq 2\log r + T(r,h(z)) + O(1) \leq 3T(r,h(z)) + S(r,h). \end{aligned}$$

This proves Theorem 1.5. ∎

5. Proof of Theorem 1.6

Let $F = f^n(z)(f(z)-1)f(qz)$, $G = g^n(z)(g(z)-1)g(qz)$. Then F and G share 1 CM. By Lemma 2.1 we obtain

$$\begin{aligned} (n+1)T(r,f(z)) &= T(r,f^n(z)(f(z)-1)) + S(r,f) = T(r,F/f(qz)) + S(r,f) \\ &\leq T(r,F) + T(r,f(qz)) + S(r,f) \leq T(r,F) + T(r,f(z)) + S(r,f), \end{aligned}$$

namely

$$nT(r,f(z)) \leq T(r,F) + S(r,f). \tag{5.1}$$

By Lemma 2.2, similar to the proof of Theorem 1.4, we obtain three cases.
Case 1. We get

$$\begin{aligned} T(r,F(z)) \leq\ & 2\overline{N}(r,1/f(z)) + N(r,1/(f(z)-1)) + N(r,1/f(qz)) + 2\overline{N}(r,1/g(z)) \\ & + N(r,1/(g(z)-1)) + N(r,1/g(qz)) + 2\overline{N}(r,f(z)) + N(r,f(qz)) \\ & + 2\overline{N}(r,g(z)) + N(r,g(qz)) + S(r,f) + S(r,g) \\ \leq\ & 7(T(r,f(z)) + T(r,g(z))) + S(r,f) + S(r,g). \end{aligned}$$

This together with (5.1) implies

$$nT(r,f(z)) \leq 7(T(r,f(z)) + T(r,g(z))) + S(r,f) + S(r,g). \tag{5.2}$$

Similarly we have

$$nT(r,g(z)) \leq 7(T(r,f(z)) + T(r,g(z))) + S(r,f) + S(r,g). \tag{5.3}$$

Combining (5.2) and (5.3) gives

$$n(T(r,f(z)) + T(r,g(z))) \leq 14(T(r,f(z)) + T(r,g(z))) + S(r,f) + S(r,g),$$

which contradicts the assumption $n \geq 15$. Case 1 has been ruled out.
Case 2. We have $FG \equiv 1$, namely

$$f^n(z)(f(z)-1)f(qz)g^n(z)(g(z)-1)g(qz) \equiv 1. \tag{5.4}$$

Rewrite (5.4) as

$$[h^{n+1}(z)-h^n(z)(f(z)+g(z))+h^n(z)]h(qz)\equiv 1, \tag{5.5}$$

where $h=f(z)g(z)$. Next we distinguish into two subcases below.

Subcase 2.1. $h(z)$ is a constant. Note that $f(z)$ and $g(z)$ share ∞ IM, then $f(z)\neq\infty$ and $g(z)\neq\infty$. Moreover, we obtain $f(z)\neq 0$ and $g(z)\neq 0$. Thus $f=e^{\beta(z)}$, where $\beta(z)$ is a non-constant entire function. Since $f(z)$ is of zero-order, we get that $\beta(z)$ is a constant, a contradiction. Subcase 2.1 has been ruled out.

Subcase 2.2. $h(z)$ is not a constant. Suppose that $h(z_0)=0$, then $f(z_0)g(z_0)=0$, since $f(z)$ and $g(z)$ share ∞ IM, we obtain $f(z_0)\neq\infty$ and $g(z_0)\neq\infty$. Thus from (5.5) we deduce that $h(qz_0)=\infty$. This implies that

$$h(z)=0\Rightarrow h(qz)=\infty. \tag{5.6}$$

Suppose that $h(qz_1)=0$, we obtain $h(z_1)=\infty$, or else, if $h(z_1)\neq\infty$, we deduce that $f(z_1)\neq\infty$ and $g(z_1)\neq\infty$. Then we get a contradiction with (5.5). Thus

$$h(qz)=0\Rightarrow h(z)=\infty. \tag{5.7}$$

Suppose that $h(z_2)=\infty$, then we get $f(z_2)=\infty$ and $g(z_2)=\infty$. Suppose that z_2 is a pole of $f(z)$ of multiplicity s, and is a pole of $g(z)$ of multiplicity t, then z_2 is a pole of $h^{n+1}(z)$ of multiplicity $(n+1)(s+t)$, a pole of $h^n(z)(f(z)+g(z))$ of multiplicity $<(n+1)(s+t)$, and a pole of $h^n(z)$ of multiplicity $n(s+t)(<(n+1)(s+t))$. Thus from (5.5) we get $h(qz_2)=0$. So we deduce that

$$h(z)=\infty\Rightarrow h(qz)=0. \tag{5.8}$$

Similarly we obtain

$$h(qz)=\infty\Rightarrow h(z)=0\, or\, (f-1)(g-1)=0. \tag{5.9}$$

Combining (5.6), (5.7) and (5.8) yields

$$h(z)=0\Rightarrow h(qz)=\infty. \tag{5.10}$$

and

$$h(qz)=0\Leftrightarrow h(z)=\infty \tag{5.11}$$

If $|q|<1$, from (5.10) and (5.11) we deduce that

$$h(z)=0\Rightarrow h(qz)=\infty\Rightarrow h(q^2z)=0\Rightarrow\cdots\Rightarrow h(q^{2k}z)=0\cdots, \tag{5.12}$$

where k is a positive integer. For sufficiently large k, we derive

$$h(0)=0, \tag{5.13}$$

again from (5.10) we get $h(0)=\infty$, a contradiction.

If $|q|>1$ and if

$$h(qz)=\infty\Rightarrow h(z)=0, \tag{5.14}$$

From (5.11) and (5.14) we deduce that

$$h(qz)=0\Rightarrow h(z)=\infty\Rightarrow h(z/q)=0\Rightarrow\cdots\Rightarrow h(z/q^{2k-1})=0\cdots, \tag{5.15}$$

where k is a positive integer. For sufficiently large k, with similar discussion as above, we derive a contradiction again.
If

$$h(qz) = \infty \Rightarrow h(z) \neq 0, \tag{5.16}$$

from (5.12) we deduce that

$$h(qz) = \infty \Rightarrow h(q^3 z) = \infty \Rightarrow h(q^2 z) \neq 0, \tag{5.17}$$

which contradicts (5.12). Case 2 has been ruled out. Thus we obtain $F \equiv G$, namely

$$f^n(z)(f(z)-1)f(qz) \equiv g^n(z)(g(z)-1)g(qz). \tag{5.18}$$

This completes the proof of Theorem 1.6. ∎

6. Proof of Theorem 1.8

Case 1. $f(z)$ is meromorphic. Let

$$\phi = (b(z) - a(z)f(qz))/f^n(z). \tag{6.1}$$

We only need to prove $\phi - 1$ has infinitely many zeros. By Lemma 2.1, we obtain

$$\begin{aligned} T(r, f^n(z)) &= T(r, \phi/(b(z) - a(z)f(qz))) + O(1) \\ &\leq T(r,\phi) + T(r, f(qz)) + S(r,f) \leq T(r,\phi) + T(r, f(z)) + S(r,f), \end{aligned}$$

which implies that

$$T(r,\phi) \geq (n-1)T(r,f) + S(r,f). \tag{6.2}$$

Furthermore

$$\overline{N}(r,\phi) \leq \overline{N}(r, f(qz)) + \overline{N}(r, 1/f) + S(r,f). \tag{6.3}$$

$$\overline{N}(r, 1/\phi) \leq \overline{N}(r, 1/(b(z) - a(z)f(qz))) + \overline{N}(r,f). \tag{6.4}$$

By the second fundamental theorem, we deduce

$$\begin{aligned} T(r,\phi) &\leq \overline{N}(r,1/\phi) + \overline{N}(r,\phi) + \overline{N}(r, 1/(\phi-1)) + S(r,\phi) \\ &\leq \overline{N}(r, 1/(b(z) - a(z)f(qz))) + \overline{N}(r,f) + \overline{N}(r, f(qz)) \\ &\quad + \overline{N}(r,1/f) + \overline{N}(r,1/(\phi-1)) + S(r,f) \\ &\leq 4T(r,f) + \overline{N}(r, 1/(\phi-1)) + S(r,f). \end{aligned} \tag{6.5}$$

In view of $n \geq 6$, by (6.2) and (6.5) we obtain the conclusion.
Case 2. $f(z)$ is entire. Suppose, to the contrary, that $f^n(z) - a(z)f(qz) - b(z)$ has only finitely many zeros, then

$$f^n(z) + a(z)f(qz) - b(z) = p(z)/K(z), \tag{6.6}$$

where $p(z)$ is a polynomial and $K(z)$ is an entire function such that $K(z)$ and $p(z)$ have no common factors and $T(r,(K(z))) \leq T(r,a(z)) + T(r,b(z)) = S(r,f)$. Thus we have

$$nT(r, f(z)) = T(r, f^n(z)) = T(r, p(z)/K(z) - (a(z)f(qz) - b(z))) \leq T(r, f(z)) + S(r,f).$$

Note that $n \geq 2$, we derive a contradiction. This completes the proof of Theorem 1.8. ∎

7. Proof of Theorem 1.11

Under the assumptions we have

$$\frac{f^n(z)+a(z)f(qz)-b(z)}{g^n(z)+a(z)g(qz)-b(z)}=c(z). \tag{7.1}$$

If $c(z)$ is not a constant, then we have $T(r,c)=O(T(r,f)+T(r,g))$. Thus $\sigma(c)\le\sigma(f+g)\le\max\{\sigma(f),\sigma(g)\}=0$. Therefore, $c(z)$ must have zeros or poles.

The zeros of $c(z)$ come from zeros of $f^n(z)+a(z)f(qz)-b(z)$ and poles of $g^n(z)+a(z)g(qz)-b(z)$. Note that $f^n(z)+a(z)f(qz)$ and $g^n(z)+a(z)g(qz)$ share $b(z)$ CM, the zeros of $c(z)$ can only come from the poles of $g^n(z)+a(z)g(qz)-b(z)$, namely they come from the poles of $a(z)$ and $b(z)$. Since $a(z)$ and $b(z)$ have finitely many poles, we get that $c(z)$ has finitely many zeros. Similarly, $c(z)$ has finitely many poles, and thus is a rational function. Note that $f(z)$ and $g(z)$ are transcendental, then $c(z)$ is a small function with respect to both $f(z)$ and $g(z)$. Rewrite (7.1) as

$$f^n(z)-c(z)g^n(z)+a(z)f(qz)-c(z)a(z)g(qz)=(1-c(z))b(z). \tag{7.2}$$

If $c(z)\not\equiv 1$, we have

$$\frac{f^n(z)}{(1-c(z))b(z)}+\frac{c(z)g^n(z)}{(c(z)-1)b(z)}+\frac{a(z)(f(qz)-c(z)g(qz))}{(1-c(z))b(z)}=1. \tag{7.3}$$

Let $f_1(z)=\frac{f^n(z)}{(1-c(z))b(z)}$, $f_2(z)=\frac{c(z)g^n(z)}{(c(z)-1)b(z)}$, $f_3(z)=\frac{a(z)(f(qz)-c(z)g(qz))}{(1-c(z))b(z)}$. Then

$$f_1(z)+f_2(z)+f_3(z)=1. \tag{7.4}$$

Obviously $f_1(z)$ is not a constant, or else we get $f^n(z)=d(1-c(z))b(z)$, where d is a constant. Thus

$$nT(r,f(z))=T(r,f^n(z))\le T(r,c(z))+T(r,b(z))+O(1)=S(r,f(z)), \tag{7.5}$$

which is a contradiction. Without loss of generality, suppose that $T(r,f(z))\le T(r,g(z))$, $r\in I$, in view of $n\ge 7$, by Lemma 2.1 we get

$$T(r)=(1+o(1))T(r,g^n(z)),\quad r\in I, \tag{7.6}$$

where $T(r)$ and I are defined as in Lemma 2.3. Then we deduce that

$$\begin{aligned}
&\sum_{j=1}^{3}N_2(r,1/f_j(z))+\sum_{j=1}^{3}\overline{N}(r,f_j(z))\\
&\le 2\overline{N}(r,1/f(z))+2\overline{N}(r,1/g(z))+N(r,1/(f(qz)+g(qz)))+S(r,f)+S(r,g)\\
&\le 3(T(r,f(z))+T(r,g(z)))+S(r,f)+S(r,g)\le 6(1+o(1))T(r,g(z))<T(r).
\end{aligned}$$

By Lemma 2.3 we obtain $f_2\equiv 1$ or $f_3\equiv 1$.
If $f_2\equiv 1$, it is easy to get a contradiction. If $f_3\equiv 1$, namely

$$\frac{a(z)(f(qz)-c(z)g(qz))}{(1-c(z))b(z)}\equiv 1. \tag{7.7}$$

Substituting (7.7) into (7.3), we derive $f^n(z)\equiv c(z)g^n(z)$. Thus we have

$$T\left(r,\frac{g}{f}\right)=S(r,f), \tag{7.8}$$

From (7.7) we have

$$f(qz) = \frac{(1-c(z))b(z)/a(z)}{1-c(z)g(qz)/f(qz)}. \tag{7.9}$$

Note that $c(z) \not\equiv 1$ and $a(z)$, $b(z)$ are non-vanishing small functions of $f(z)$, by Lemma 2.12.1, we get from (7.8) and (7.9) that

$$T(r, f(qz)) = (1+o(1))T(r, f(z)) = S(r, f(z)), \tag{7.10}$$

which is a contradiction. Thus $c(z) \equiv 1$. From (7.2) we get

$$f^n(z) + a(z)f(qz) = g^n(z) + a(z)g(qz). \tag{7.11}$$

By Theorem 1.10 we obtain $f(z) \equiv g(z)$. This completes the proof of Theorem 1.11. ∎

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