

## $N(k)$ -Quasi Einstein Manifolds Satisfying Certain Curvature Conditions

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**Abstract.** The object of the present paper is to study  $N(k)$ -quasi Einstein manifolds satisfying certain curvature conditions. Two examples have been constructed to prove the existence of such a manifold. Finally, a physical example of an  $N(k)$ -quasi Einstein manifold is given.

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### 1. Introduction

A Riemannian or a semi-Riemannian manifold  $(M^n, g)$ ,  $n = \dim M \geq 2$ , is said to be an Einstein manifold if the following condition

$$(1.1) \quad S = \frac{r}{n}g,$$

holds on  $M$ , where  $S$  and  $r$  denote the Ricci tensor and the scalar curvature of  $(M^n, g)$ , respectively. According to ([1], p. 432), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([1], p. 432–433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds  $(M^n, g)$  realizing the following relation :

$$(1.2) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where  $a, b$  are smooth functions and  $\eta$  is a non-zero 1-form such that

$$(1.3) \quad g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1,$$

for all vector fields  $X, Y$ .

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A non-flat Riemannian manifold  $(M^n, g)$  ( $n > 2$ ) is defined to be a quasi Einstein manifold [2] if its Ricci tensor  $S$  of type  $(0, 2)$  is not identically zero and satisfies the condition (1.2). We shall call  $\eta$  the associated 1-form and the unit vector field  $\xi$  is called the generator of the manifold.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. So many studies about Einstein field equations are done. For example, in [11], Naschie turned the tables on the theory of elementary particles and showed that we could derive the expectation number of elementary particles of the standard model using Einstein's unified field equation or more precisely his somewhat forgotten strength criteria directly and without resorting to quantum field theory [12]. He also discussed possible connections between Gödel's classical solution of Einstein's field equations and E-infinity in [10]. Also quasi Einstein manifolds have some importance in the general theory of relativity. For instance, the Robertson-Walker spacetime are quasi Einstein manifolds [9]. Further, quasi Einstein manifolds can be taken as a model of the perfect fluid spacetime in general relativity [6].

The study of quasi Einstein manifolds was continued by Chaki [3], Guha [13], De and Ghosh [7, 8] and many others. The notion of quasi Einstein manifolds have been generalized in several ways by several authors. In recent papers, Özgür studied super quasi Einstein manifolds [19] and generalized quasi Einstein manifolds [20].

Let  $R$  denote the Riemannian curvature tensor of a Riemannian manifold  $M$ . The  $k$ -nullity distribution  $N(k)$  of a Riemannian manifold  $M$  is defined by [23]

$$(1.4) \quad N(k) : p \longrightarrow N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

$k$  being some smooth function. In a quasi Einstein manifold  $M$ , if the generator  $\xi$  belongs to some  $k$ -nullity distribution  $N(k)$ , then  $M$  is said to be a  $N(k)$ -quasi Einstein manifold [25]. In fact  $k$  is not arbitrary as the following:

In an  $n$ -dimensional  $N(k)$ -quasi Einstein manifold it follows that

$$(1.5) \quad k = \frac{a+b}{n-1}.$$

Now, it is immediate to note that in an  $n$ -dimensional  $N(k)$ -quasi Einstein manifold [17]

$$(1.6) \quad R(X, Y)\xi = \frac{a+b}{n-1}[\eta(Y)X - \eta(X)Y],$$

which is equivalent to

$$(1.7) \quad R(X, \xi)Y = \frac{a+b}{n-1}[\eta(Y)X - g(X, Y)\xi] = -R(\xi, X)Y.$$

From (1.4) we get

$$(1.8) \quad R(\xi, X)\xi = \frac{a+b}{n-1}[\eta(X)\xi - X].$$

In [25] it was shown that an  $n$ -dimensional conformally flat quasi Einstein manifold is an  $N((a+b)/(n-1))$ -quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an  $N((a+b)/2)$ -quasi Einstein manifold. Also in [18], Özgür, cited some physical examples of  $N(k)$ -quasi Einstein manifolds. In 2011, Taleshian and Hosseinzadeh [24] studied  $N(k)$ -quasi Einstein manifolds satisfying certain curvature conditions. Nagaraja [16] also studied  $N(k)$ -mixed quasi Einstein manifolds.

In 1968, Yano and Sawaki [22] defined and studied a tensor  $\tilde{C}$  on a Riemannian manifold of dimensional  $n$  which includes both conformal curvature tensor and concircular curvature tensor as particular cases. This tensor is known as quasi-conformal curvature tensor and is defined by

$$(1.9) \quad \begin{aligned} \tilde{C}(X, Y)Z &= \lambda R(X, Y)Z + \mu \{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\} \\ &\quad - \frac{r}{n} \left\{ \frac{\lambda}{(n-1)} + 2\mu \right\} [g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where  $r$  is the scalar curvature and  $Q$  is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor  $S$ , that is,  $g(QX, Y) = S(X, Y)$ . Here  $\lambda$  and  $\mu$  are arbitrary constants. If  $\lambda = 1$  and  $\mu = -1/(n-2)$ , then the quasi-conformal curvature tensor is reduced to the conformal curvature tensor. For an  $n \geq 4$  dimensional Riemannian manifold, if  $\tilde{C} = 0$  then it is called quasi-conformally flat. Recently Mantica and Suh [15] studied quasi-conformally recurrent Riemannian manifolds.

The projective curvature tensor  $P$  and the concircular curvature tensor  $\tilde{Z}$  in a Riemannian manifold  $(M^n, g)$  are defined by [26]

$$(1.10) \quad P(X, Y)W = R(X, Y)W - \frac{1}{n-1} [S(Y, W)X - S(X, W)Y],$$

$$(1.11) \quad \tilde{Z}(X, Y)W = R(X, Y)W - \frac{r}{n(n-1)} [g(Y, W)X - g(X, W)Y],$$

respectively. In [25], the authors have proved that conformally flat quasi Einstein manifolds are certain  $N(k)$ -quasi Einstein manifolds. The derivation conditions  $R(\xi, X) \cdot R = 0$  and  $R(\xi, X) \cdot S = 0$  have been studied in [23], where  $R$  and  $S$  denote the curvature and Ricci tensor respectively. Özgür and Tripathi [17] continued the study of the  $N(k)$ -quasi Einstein manifolds. In [17], the derivation conditions  $\tilde{Z}(\xi, X) \cdot R = 0$  and  $\tilde{Z}(\xi, X) \cdot \tilde{Z} = 0$  on  $N(k)$ -quasi Einstein manifolds were studied, where  $\tilde{Z}$  is the concircular curvature tensor. Moreover in [17], for an  $N(k)$ -quasi Einstein manifold it was proved that  $k = (a + b)/(n - 1)$ . Özgür in [18] studied the condition  $R \cdot P = 0$ ,  $P \cdot S = 0$  and  $P \cdot P = 0$  for an  $N(k)$ -quasi Einstein manifold, where  $P$  denotes the projective curvature tensor and some physical examples of  $N(k)$ -quasi Einstein manifolds are given. Again, in 2008, Özgür and Sular [21] studied  $N(k)$ -quasi Einstein manifolds satisfying  $R \cdot C = 0$  and  $R \cdot \tilde{C} = 0$ , where  $C$  and  $\tilde{C}$  represent the conformal curvature tensor and the quasi-conformal curvature tensor, respectively. This paper is a continuation of previous studies.

The paper is organized as follows: After preliminaries in section 3, we study quasi-conformally recurrent  $N(k)$ -quasi Einstein manifolds. We prove that quasi-conformally recurrent manifold satisfies  $R(\xi, X) \cdot \tilde{C} = 0$ . In section 4, we prove that for an  $n \geq 4$  dimensional  $N(k)$ -quasi Einstein manifold, the conditions  $\tilde{C}(\xi, X) \cdot S = 0$ ,  $\tilde{C}(\xi, X) \cdot P = 0$ ,  $\tilde{C}(\xi, X) \cdot \tilde{Z} = 0$  hold on the manifold if and only if  $\lambda = \mu(2 - n)$ . Finally, we give two examples of an  $N(k)$ -quasi Einstein manifold and a physical example of an  $N(k)$ -quasi Einstein manifold.

## 2. Preliminaries

From (1.2) and (1.3) it follows that

$$(2.1) \quad S(X, \xi) = (a + b)\eta(X),$$

and

$$(2.2) \quad r = an + b,$$

where  $r$  is the scalar curvature of  $M^n$ .

From the definition of the quasi-conformal curvature tensor, we can write

$$\begin{aligned} \tilde{C}(\xi, X)Y &= \lambda R(\xi, X)Y + \mu\{S(X, Y)\xi - S(\xi, Y)X + g(X, Y)Q\xi - g(\xi, Y)QX\} \\ &\quad - \frac{r}{n}\left\{\frac{\lambda}{(n-1)} + 2\mu\right\}[g(X, Y)\xi - g(\xi, Y)X]. \end{aligned}$$

Here using (1.7) and (2.1), we find

$$\tilde{C}(\xi, X)Y = \left\{\lambda k - \frac{r}{n}\left\{\frac{\lambda}{(n-1)} + 2\mu\right\} + \mu(2a + b)\right\}\{g(X, Y)\xi - \eta(Y)X\}.$$

Now using  $r = an + b$ , we find

$$\lambda k - \frac{r}{n}\left\{\frac{\lambda}{(n-1)} + 2\mu\right\} + \mu(2a + b) = \lambda - \mu(2 - n).$$

Then we obtain

$$(2.3) \quad \tilde{C}(\xi, X)Y = \{\lambda - \mu(2 - n)\}\{g(X, Y)\xi - \eta(Y)X\}.$$

The curvature conditions  $\tilde{C} \cdot S$ ,  $\tilde{C} \cdot P$  and  $\tilde{C} \cdot \tilde{Z}$  are defined by

$$(2.4) \quad (\tilde{C}(U, X) \cdot S)(Y, Z) = -S(\tilde{C}(U, X)Y, Z) - S(Y, \tilde{C}(U, X)Z),$$

$$(2.5) \quad \begin{aligned} (\tilde{C}(U, X) \cdot P)(Y, Z, W) &= \tilde{C}(U, X)P(Y, Z)W - P(\tilde{C}(U, X)Y, Z)W \\ &\quad - P(Y, \tilde{C}(U, X)Z)W - P(Y, Z)\tilde{C}(U, X)W, \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} (\tilde{C}(U, X) \cdot \tilde{Z})(Y, Z, W) &= \tilde{C}(U, X)\tilde{Z}(Y, Z)W - \tilde{Z}(\tilde{C}(U, X)Y, Z)W \\ &\quad - \tilde{Z}(Y, \tilde{C}(U, X)Z)W - \tilde{Z}(Y, Z)\tilde{C}(U, X)W, \end{aligned}$$

respectively.

### 3. Quasi-conformally recurrent $N(k)$ -quasi Einstein manifolds

In [21], Özgür and Sular proved that in an  $N(k)$ -quasi Einstein manifold the condition  $R(\xi, X) \cdot \tilde{C} = 0$  holds on  $M^n$  if and only if either  $a = -b$  or,  $M^n$  is conformally flat with  $\lambda = \mu(2 - n)$ . In this section we study quasi-conformally recurrent  $N(k)$ -quasi Einstein manifolds.

A non-flat Riemannian manifold  $M$  is said to be quasi-conformally recurrent [15] if the quasi conformal curvature tensor  $\tilde{C}$  satisfies the condition  $\nabla \tilde{C} = A \otimes \tilde{C}$ , where  $A$  is an everywhere non-zero 1-form. We now define a function  $f$  on  $M$  by  $f^2 = g(\tilde{C}, \tilde{C})$ , where the metric  $g$  is extended to the inner product between the tensor fields in the standard fashion. Then we know that  $f(Yf) = f^2A(Y)$ . So from this we have  $Yf = fA(Y)$ , because  $f \neq 0$ . This implies that

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + (XA(Y))f.$$

Hence

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X)\}f.$$

Therefore we get

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})f = \{XA(Y) - YA(X) - A([X,Y])\}f.$$

Since the left hand side of the above equation is identically zero and  $f \neq 0$  on  $M$  by our assumption, we obtain

$$(3.1) \quad dA(X, Y) = 0, \text{ that is, the 1-form } A \text{ closed.}$$

Now from

$$(\nabla_X \tilde{C})(U, V)Z = A(X)\tilde{C}(U, V)Z,$$

we get

$$(\nabla_U \nabla_V \tilde{C})(X, Y)Z = \{UA(V) + A(U)A(V)\}\tilde{C}(X, Y)Z.$$

Hence using (3.1) we get

$$(R(X, Y)\tilde{C})(U, V)Z = [2dA(X, Y)]\tilde{C}(U, V)Z = 0.$$

Therefore, for a quasi-conformally recurrent manifold, we have

$$(3.2) \quad R(X, Y)\tilde{C} = 0 \text{ for all } X, Y.$$

An equivalent proof can be given as follows: From the condition  $\nabla_i \tilde{C}_{jkl}^m = A_i \tilde{C}_{jkl}^m$  one gets easily  $\nabla_i (\tilde{C}_{jkl}^m \tilde{C}_m^{jkl}) = 2A_i (\tilde{C}_{jkl}^m \tilde{C}_m^{jkl})$  and thus putting  $f = \tilde{C}_{jkl}^m \tilde{C}_m^{jkl}$ , we recover locally the closedness of the 1-form  $A$ .

Hence by Theorem 4.3 of Özgür and Sular [21], we can state the following:

**Theorem 3.1.** *An  $N(k)$ -quasi Einstein manifold is quasi-conformally recurrent if and only if either  $a = -b$  or,  $M^n$  is conformally flat with  $\lambda = \mu(2 - n)$ .*

#### 4. Main results

In this section we give the main results of the paper. At first we give the following:

**Theorem 4.1.** *Let  $M^n$  be an  $n$ -dimensional,  $n \geq 4, N(k)$ -quasi Einstein manifold. Then  $M^n$  satisfies the condition  $\tilde{C}(\xi, X) \cdot S = 0$  if and only if  $\lambda = \mu(2 - n)$ .*

*Proof.* Assume that an  $N(k)$ -quasi Einstein manifold satisfies

$$\tilde{C}(\xi, X) \cdot S = 0.$$

Then we get from (2.4)

$$(4.1) \quad S(\tilde{C}(\xi, X)Y, Z) + S(Y, \tilde{C}(\xi, X)Z) = 0.$$

Using (2.3) in (4.1) we get

$$\{\lambda - \mu(2 - n)\} [g(X, Y)S(\xi, Z) - \eta(Y)S(X, Z) + g(X, Z)S(Y, \xi) - \eta(Z)S(Y, X)] = 0.$$

Then either

$$\lambda - \mu(2 - n) = 0,$$

or

$$(4.2) \quad g(X, Y)S(\xi, Z) - \eta(Y)S(X, Z) + g(X, Z)S(Y, \xi) - \eta(Z)S(Y, X) = 0.$$

Putting  $Y = \xi$  in (4.2) we find

$$(4.3) \quad S(X, Z) = (a + b)g(X, Z),$$

which implies that the manifold is an Einstein manifold which contradicts the definition of  $N(k)$ -quasi Einstein manifold. Then only  $\lambda - \mu(2 - n) = 0$  holds.

Conversely, let  $\lambda = \mu(2 - n)$ , then from (2.3) we have  $\tilde{C}(\xi, X)Y = 0$ . Hence we get  $\tilde{C}(\xi, X) \cdot S = 0$ . This completes the proof.  $\blacksquare$

If  $\lambda = \mu(2 - n)$ , then from the definition of quasi-conformal curvature tensor it follows that  $\tilde{C} = \lambda C$ . Thus we can state the following:

**Corollary 4.1.** *In an  $N(k)$ -quasi Einstein manifold satisfying the condition  $\tilde{C}(\xi, X) \cdot S = 0$ , conformally flatness and quasi-conformally flatness are equivalent.*

Now we give the following:

**Theorem 4.2.** *Let  $M^n$  be an  $n$ -dimensional,  $n \geq 4$ ,  $N(k)$ -quasi Einstein manifold. Then  $M^n$  satisfies the condition  $\tilde{C}(\xi, X) \cdot P = 0$  if and only if  $\lambda = \mu(2 - n)$ .*

*Proof.* Suppose that the  $N(k)$ -quasi Einstein manifold satisfies

$$\tilde{C}(\xi, X) \cdot P = 0.$$

Then from (2.5), we get

$$\tilde{C}(\xi, X)P(Y, Z)W - P(\tilde{C}(\xi, X)Y, Z)W - P(Y, \tilde{C}(\xi, X)Z)W - P(Y, Z)\tilde{C}(\xi, X)W = 0.$$

Using (1.10) and (2.3) we obtain

$$\begin{aligned} & \{\lambda - \mu(2 - n)\} \{g(X, P(Y, Z)W)\xi - \eta(P(Y, Z)W)X - g(X, Y)P(\xi, Z)W \\ & \quad + \eta(Y)P(X, Z)W - g(X, Z)P(Y, \xi)W + \eta(Z)P(Y, X)W - g(X, W)P(Y, Z)\xi \\ & \quad + \eta(W)P(Y, Z)X\} = 0, \end{aligned}$$

which implies either  $\lambda - \mu(2 - n) = 0$  or,

$$(4.4) \quad \begin{aligned} & g(X, P(Y, Z)W)\xi - \eta(P(Y, Z)W)X - g(X, Y)P(\xi, Z)W \\ & \quad + \eta(Y)P(X, Z)W - g(X, Z)P(Y, \xi)W + \eta(Z)P(Y, X)W - g(X, W)P(Y, Z)\xi \\ & \quad + \eta(W)P(Y, Z)X = 0. \end{aligned}$$

Taking the inner product of both sides of (4.4) with  $\xi$ , we have

$$(4.5) \quad \begin{aligned} & g(X, P(Y, Z)W) - \eta(P(Y, Z)W)\eta(X) - g(X, Y)\eta(P(\xi, Z)W) \\ & \quad + \eta(Y)\eta(P(X, Z)W) - g(X, Z)\eta(P(Y, \xi)W) + \eta(Z)\eta(P(Y, X)W) \\ & \quad + \eta(W)\eta(P(Y, Z)X) = 0. \end{aligned}$$

Hence with the help of (1.10), the equation (4.5) is reduced to

$$(4.6) \quad 0 = P(Y, Z, W, X) + \frac{b}{n-1} \{g(X, Z)g(Y, W) - g(X, Y)g(Z, W)\},$$

where  $P(Y, Z, W, X) = g(X, P(Y, Z)W)$ .

Then by using (1.10) and putting  $X = Y = e_i$  in (4.6), where  $\{e_i\}$  is orthonormal basis at each point of the manifold and taking summation over  $i$ ,  $1 \leq i \leq n$ , we obtain

$$bg(Z, W) = 0.$$

This means that  $b = 0$  which implies that the manifold is an Einstein manifold which contradicts the definition of an  $N(k)$ -quasi Einstein manifold. Then only the relation  $\lambda - \mu(2 - n) = 0$  holds. Conversely, let  $\lambda = \mu(2 - n)$ , then from (2.3), we have  $\tilde{C}(\xi, X)Y = 0$ . Hence  $\tilde{C}(\xi, X) \cdot P = 0$ . This completes the proof. ■

**Remark 4.1.** The Corollary 4.1 also holds in this case.

**Theorem 4.3.** *Let  $M^n$  be an  $n$ -dimensional,  $n \geq 4$ ,  $N(k)$ -quasi Einstein manifold. Then  $M^n$  satisfies the condition  $\tilde{C}(\xi, X) \cdot \tilde{Z} = 0$  if and only if  $\lambda = \mu(2 - n)$ .*

*Proof.* We suppose that

$$\tilde{C}(\xi, X) \cdot \tilde{Z} = 0.$$

Then we get from (2.6)

$$\tilde{C}(\xi, X)\tilde{Z}(Y, V)W - \tilde{Z}(\tilde{C}(\xi, X)Y, V)W - \tilde{Z}(Y, \tilde{C}(\xi, X)V)W - \tilde{Z}(Y, V)\tilde{C}(\xi, X)W = 0.$$

So from (1.11) and (2.3), we obtain

$$\begin{aligned} & \{\lambda - \mu(2 - n)\} \{g(X, \tilde{Z}(Y, V)W)\xi - \eta(\tilde{Z}(Y, V)W)X - g(X, Y)\tilde{Z}(\xi, V)W \\ & + \eta(Y)\tilde{Z}(X, V)W - g(X, V)\tilde{Z}(Y, \xi)W + \eta(V)\tilde{Z}(Y, X)W - g(X, W)\tilde{Z}(Y, V)\xi \\ & + \eta(W)\tilde{Z}(Y, V)X\} = 0. \end{aligned}$$

Then either  $\lambda - \mu(2 - n) = 0$  or,

$$\begin{aligned} & g(X, \tilde{Z}(Y, V)W)\xi - \eta(\tilde{Z}(Y, V)W)X - g(X, Y)\tilde{Z}(\xi, V)W \\ (4.7) \quad & + \eta(Y)\tilde{Z}(X, V)W - g(X, V)\tilde{Z}(Y, \xi)W + \eta(V)\tilde{Z}(Y, X)W - g(X, W)\tilde{Z}(Y, V)\xi \\ & + \eta(W)\tilde{Z}(Y, V)X = 0. \end{aligned}$$

Taking inner product with  $\xi$  of equation (4.7), we get

$$\begin{aligned} & g(X, \tilde{Z}(Y, V)W) - \eta(\tilde{Z}(Y, V)W)\eta(X) - g(X, Y)\eta(\tilde{Z}(\xi, V)W) \\ & + \eta(Y)\eta(\tilde{Z}(X, V)W) - g(X, V)\eta(\tilde{Z}(Y, \xi)W) \\ & + \eta(V)\eta(\tilde{Z}(Y, X)W) - g(X, W)\eta(\tilde{Z}(Y, V)\xi) + \eta(W)\eta(\tilde{Z}(Y, V)X) = 0. \end{aligned}$$

Using (2.6) we obtain

$$(4.8) \quad g(X, R(Y, V)W) - k\{g(X, Y)g(V, W) - g(X, V)g(Y, W)\} = 0.$$

Taking  $X = Y = e_i$  in (4.8), we obtain

$$S(Y, W) = (a + b)g(Y, W),$$

which implies that the manifold is an Einstein manifold which contradicts the definition of an  $N(k)$ -quasi Einstein manifold. Then we have  $\lambda = \mu(2 - n)$ .

Conversely, let  $\lambda = \mu(2 - n)$ , then from (2.3), we have  $\tilde{C}(\xi, X)Y = 0$ . Hence  $\tilde{C}(\xi, X) \cdot \tilde{Z} = 0$ . ■

**Remark 4.2.** The Corollary 4.1 also holds in this case.

**Corollary 4.2.** *From Theorems 3.1–4.3 the following statements are equivalent:*

- (i)  $\tilde{C}(\xi, X) \cdot S = 0$ ,
- (ii)  $\tilde{C}(\xi, X) \cdot P = 0$ ,

- (iii)  $\tilde{C}(\xi X) \cdot \tilde{Z} = 0,$
- (iv)  $\lambda = \mu(2 - n).$

**5. Examples of an  $N(k)$ -quasi Einstein manifold**

**Example 5.1.** Let us consider a semi-Riemannian metric  $g$  on  $\mathbb{R}^4$  by

$$(5.1) \quad ds^2 = g_{ij}dx^i dx^j = x^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2.$$

Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\Gamma_{11}^2 = \Gamma_{33}^2 = -\frac{1}{2x^2}, \quad \Gamma_{22}^2 = \Gamma_{12}^1 = \Gamma_{23}^3 = \frac{1}{2x^2},$$

$$R_{1221} = R_{2332} = -\frac{1}{2x^2}, \quad R_{1331} = \frac{1}{4x^2}, \quad R_{1232} = 0,$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensor  $R_{ij}$  are

$$R_{11} = R_{33} = -\frac{1}{4(x^2)^2}, \quad R_{22} = -\frac{1}{(x^2)^2}, \quad R_{44} = 0.$$

It can be easily shown that the scalar curvature of the resulting manifold  $(\mathbb{R}^4, g)$  is

$$-\frac{3}{2(x^2)^3} \neq 0.$$

We choose the 1-form  $A$  as follows

$$A_i(x) = \begin{cases} \sqrt{\frac{\{4(x^2)^2+1\}x^2}{6\{(x^2)^2+1\}}}, & \text{for } i = 1, 3 \\ \sqrt{\frac{2x^2}{3}}, & \text{for } i = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

at any point  $x \in \mathbb{R}^4$ . We take the associated scalars as follows:

$$a = \frac{1}{x^2} \text{ and } b = -\frac{3}{2} \frac{1 + (x^2)^2}{(x^2)^3}.$$

Here we have

$$(5.2) \quad R_{11} = ag_{11} + bA_1A_1,$$

$$(5.3) \quad R_{22} = ag_{22} + bA_2A_2,$$

$$(5.4) \quad R_{33} = ag_{33} + bA_3A_3.$$

R.H.S. of (5.2) is  $ag_{11} + bA_1A_1 = -1/(4(x^2)^2) = R_{11} = L.H.S$  of (5.2). Similarly, we can verify (5.3) and (5.4). Now,

$$\frac{a+b}{n-1} = \frac{\frac{1}{x^2} - \frac{3}{2} \frac{1+(x^2)^2}{(x^2)^3}}{3} = -\frac{3+(x^2)^2}{6(x^2)^3}.$$

In an  $n$ -dimensional  $N(k)$ -quasi Einstein manifold, the relation

$$r = na + b,$$

holds. Here we find that  $r = 4a + b$  holds for this example. Therefore,  $(M^4, g)$  is an  $N(-(3 + (x^2)^2)/(6(x^2)^3))$ -quasi Einstein manifold.



**Example 5.2.** We consider the Riemannian metric  $g$  on  $\mathbb{R}^4$

$$(5.5) \quad ds^2 = g_{ij}dx^i dx^j = x^1(x^3)^4(dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 + (dx^4)^2,$$

where  $i, j = 1, 2, 3, 4$ . Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are following :

$$\begin{aligned} \Gamma_{11}^3 &= -2x^1(x^3)^3, \quad \Gamma_{11}^2 = \frac{1}{2}(x^3)^4, \quad \Gamma_{13}^2 = 2x^1(x^3)^3, \\ \Gamma_{12}^1 &= \Gamma_{23}^3 = \frac{1}{2x^2}, \quad R_{1331} = 6x^1(x^3)^2, \quad R_{11} = 6x^1(x^3)^2. \end{aligned}$$

Also the scalar curvature  $r = 0$ . We take the scalars  $a$  and  $b$  as follows :

$$a = x^1x^3 \text{ and } b = -4x^1x^3.$$

We choose the 1-form  $A$  as follows :

$$\begin{aligned} A_i(x) &= \frac{1}{2}\sqrt{x^1(x^3)^4 - 6x^3} \text{ for } i = 1 \\ &= 0, \text{ otherwise.} \end{aligned}$$

From the definition we get

$$(5.6) \quad R_{11} = ag_{11} + bA_1A_1.$$

R.H.S. of (5.6) is  $ag_{11} + bA_1A_1 = 6x^1(x^3)^2 = R_{11} = L.H.S$  of (5.6). Now,

$$a + b = x^1x^3 - 4x^1x^3 = -3x^1x^3.$$

So, this is an example of  $N(-x^1x^3)$ -quasi Einstein manifold. In this example, we take the scalars  $a$  and  $b$ , such that the condition  $r = an + b$  is satisfied i.e., the condition  $4a + b = 0$  is satisfied.

### 6. Physical example of an $N(k)$ -quasi-Einstein manifold

This example is concerned with an  $N(k)$ -quasi-Einstein manifold in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold  $(M^4, g)$  with Lorentzian metric  $g$  with signature  $(-, +, +, +)$ . The geometry of the Lorentzian manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity. Here we consider a perfect fluid  $(PRS)_4$  spacetime of non-zero scalar curvature and having the basic vector field  $U$  as the timelike vector field of the fluid, that is,  $g(U, U) = -1$ . An  $n$ -dimensional semi-Riemannian manifold is said to be pseudo Ricci-symmetric [4] if the Ricci tensor  $S$  satisfies the condition

$$(6.1) \quad (\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X).$$

Such a manifold is denoted by  $(PRS)_n$ .

For the perfect fluid spacetime, we have the Einstein equation without cosmological constant as

$$(6.2) \quad S(X, Y) - \frac{1}{2}rg(X, Y) = \kappa T(X, Y),$$

where  $\kappa$  is the gravitational constant,  $T$  is the energy-momentum tensor of type  $(0, 2)$  given by

$$(6.3) \quad T(X, Y) = (\sigma + p)B(X)B(Y) + pg(X, Y),$$

with  $\sigma$  and  $p$  as the energy density and isotropic pressure of the fluid respectively.

Using (6.3) in (6.2) we get

$$(6.4) \quad S(X, Y) - \frac{1}{2}rg(X, Y) = \kappa[(\sigma + p)B(X)B(Y) + pg(X, Y)].$$

Taking a frame field and contracting (6.4) over  $X$  and  $Y$  we have

$$(6.5) \quad r = \kappa(\sigma - 3p).$$

Using (6.4) in (6.5), we see that

$$(6.6) \quad S(X, Y) = \kappa[(\sigma + p)B(X)B(Y) + \frac{(\sigma - p)}{2}g(X, Y)].$$

Putting  $Y = U$  in (6.6) and since  $g(U, U) = -1$ , we get

$$(6.7) \quad S(X, U) = -\frac{\kappa}{2}[\sigma + 3p]B(x).$$

Again for  $(PRS)_4$  spacetime [4],  $S(X, U) = 0$ . This condition will be satisfied by the equation (6.7) if

$$(6.8) \quad \sigma + 3p = 0 \quad \text{as } \kappa \neq 0 \text{ and } A(X) \neq 0.$$

Using (6.5) and (6.8) in (6.6), we see that

$$(6.9) \quad S(X, Y) = \frac{r}{3}[B(X)B(Y) + g(X, Y)].$$

Thus we can state the followings:

**Theorem 6.1.** *A perfect fluid pseudo Ricci-symmetric spacetime is an  $N(2r/9)$ -quasi-Einstein manifold.*

**Remark 6.1.** Equation (6.9) recovers a result of Guha [14] which says that a perfect fluid pseudo Ricci-symmetric spacetime is a quasi Einstein manifold with each of its associates scalars equal to  $r/3$ ,  $r$  being the scalar curvature. Also, this result has been mentioned by De and Gazi [5].

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