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θ-Semi-Generalized Closed Sets in Fuzzy Topological Spaces

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Abstract. In 1997, Balasubramanian has introduced and investigated the notion of fuzzy generalized closed set. In this paper, we define and obtain a new notion of fuzzy generalized closed set called fuzzy θ-semi-generalized closed set and its characterizations are investigated. Moreover, as applications of fuzzy θ-semi-generalized closed set, we introduce fuzzy θ-semi-generalized continuity and fuzzy θ-semi-generalized irresolute mapping. Furthermore, we also introduce fuzzy θ-semi-generalized closed mapping and characterize them.

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1. Introduction

The notion of fuzzy sets due to Zadeh [22] plays important role in the study of fuzzy topological spaces which introduced by Chang [7]. In 1992, Azad [3] introduced and investigated fuzzy semi open sets and fuzzy semi-closed sets. Furthermore, Levine [12] initiated the study of generalized closed set in topological spaces whose closure of the set contained in every open superset of the set and Kılıçman and Salleh [11] obtained some further results on $(\delta\text{-}pre, s)$-continuous maps. Recently, Al-Omari *et al.* [1] introduced generalized b-closed sets in topological space. In 1997, Balasubramanian and Sundaram [4] defined the concepts of fuzzy generalized closed set in fuzzy topological spaces. Later, El-Shafei [10] introduced semi-generalized closed sets and semi-generalized continuous functions in fuzzy topological spaces and some of their properties.

In this paper, we introduced another new notion of fuzzy generalized closed set called fuzzy θ-semi-generalized closed sets, an alternative generalization of fuzzy semi-closed set by utilizing semi-θ-closure operator in fuzzy topological spaces and we also discuss the relations between fuzzy semi-θ-closed set, fuzzy semi-generalized closed set and fuzzy θ-semi-generalized closed set. Moreover, as applications of fuzzy θ-semi-generalized closed sets, we introduce fuzzy θ-semi-generalized continuity, fuzzy θ-semi-generalized irresolute

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mapping, and fuzzy θ-semi-generalized closed mapping. Some properties are given and the relationships between this new notion and other notions of fuzzy continuity are obtained.

2. Preliminaries

Throughout this paper, let X be a set and I the unit interval. A fuzzy set in X is an element of the set of all functions from X to I. The family of all fuzzy sets in X is denoted by I^X. A fuzzy singleton x_α is a fuzzy set in X define by $x_\alpha(x) = \alpha$, $x_\alpha(y) = 0$ for all $y \neq x$, $\alpha \in (0,1]$. The set of all fuzzy singletons in X is denoted by $S(X)$. For every $x_\alpha \in S(X)$ and $\mu \in I^X$, we define $x_\alpha \in \mu$ if and only if $\alpha \leq \mu(x)$. The members of τ are called fuzzy open sets and their complements are fuzzy closed sets. Spaces (X,τ) and (Y,δ) (or simply, X and Y) always mean fuzzy topological spaces in the sense of Chang [7], and by $f:(X,\tau)\to(Y,\delta)$ (or simply, $f: X \to Y$) we denote a mapping f of a space X into a space Y. By 1_X and 0_X, we mean the fuzzy sets with constant function 1 (unit function) and 0 (zero function), respectively.

For a fuzzy set μ of (X,τ), fuzzy closure and fuzzy interior of μ denoted by $\mathrm{cl}(\mu)$ and $\mathrm{int}(\mu)$, respectively. The operators fuzzy closure and fuzzy interior are defined by $\mathrm{cl}(\mu) = \wedge\{\lambda : \lambda \geq \mu, 1-\lambda \in \tau\}$ where λ is fuzzy closed set in (X,τ) and $\mathrm{int}(\mu) = \vee\{\eta : \eta \leq \mu, \eta \in \tau\}$ [15] where η is fuzzy open set in (X,τ). Fuzzy semi-closure [15] of μ denoted by $\mathrm{scl}(\mu) = \wedge\{\eta : \mu \leq \eta, \eta \in FSC(X,\tau)\}$ and fuzzy θ-closure of μ denoted by $\mathrm{cl}_\theta(\mu) = \wedge\{\mathrm{cl}(\eta) : \mu \leq \eta, \eta \in \tau\}$ [9]. Jafari and Caldas [5] introduced θ-semi-generalized closed set in topological spaces and now we proceed to introduce θ-semi-generalized closed set in fuzzy topological spaces.

Now we give some basic notions that used in the sequel.

Definition 2.1. *A fuzzy subset μ of (X,τ) is called*

(1) *fuzzy semi-open* [3] *if $\eta \leq \mu \leq \mathrm{cl}(\eta)$ where η is fuzzy open or equivalently* $\mathrm{cl}(\mathrm{int}(\mu)) \geq \mu$;
(2) *fuzzy semi-closed* [3] *if* $\mathrm{int}(\eta) \leq \mu \leq \eta$ *where η is fuzzy closed or equivalently* $\mathrm{int}(\mathrm{cl}(\mu)) \leq \mu$;
(3) *fuzzy regular closed* [3] *if* $\mathrm{cl}(\mathrm{int}(\mu)) = \mu$ *and fuzzy regular open if* $\mathrm{int}(\mathrm{cl}(\mu)) = \mu$;
(4) *fuzzy θ-closed* [9] *if* $\mu = \mathrm{cl}_\theta(\mu)$ *and fuzzy θ-open if* $\mu = \mathrm{int}_\theta(\mu)$;
(5) *fuzzy semi-θ-closed* [21] *if* $\mu = \mathrm{scl}_\theta(\mu)$ *and fuzzy semi-θ-open if* $\mu = \mathrm{sint}_\theta(\mu)$.

The family of all fuzzy semi open, fuzzy semi closed, fuzzy semi-θ-open and fuzzy semi-θ-closed sets in (X,τ) will be denoted by $FSO(X,\tau)$, $FSC(X,\tau)$, $FS\theta O(X,\tau)$ and $FS\theta C(X,\tau)$, respectively.

Definition 2.2. *A fuzzy set μ in (X,τ) is called:*

(1) *Fuzzy generalized closed set* [4] (*briefly, fg-closed set*) *if* $\mathrm{cl}(\mu) \leq \eta$ *whenever $\mu \leq \eta$ and η is fuzzy open while fuzzy generalized open set (briefly, fg-open set) if* $\mathrm{int}(\mu) \geq \eta$ *whenever $\mu \geq \eta$ and η is fuzzy closed.*
(2) *Fuzzy semi-generalized closed set* [10] (*briefly, fsg-closed set*) *if* $\mathrm{scl}(\mu) \leq \eta$ *whenever $\mu \leq \eta$ and η is fuzzy semi open.*
(3) *Fuzzy generalized semi-closed set* [15] (*briefly, fgs-closed set*) *if* $\mathrm{scl}(\mu) \leq \eta$ *whenever $\mu \leq \eta$ and η is fuzzy open.*
(4) *Fuzzy θ-generalized closed set* [9] (*briefly, f-θg-closed set*) *if* $\mathrm{cl}_\theta(\mu) \leq \eta$ *whenever $\mu \leq \eta$ and η is fuzzy open.*

The complement of fg-closed (resp. fsg-closed, fgs-closed, f-θg-closed) set is fg-open (resp. fsg-open, fgs-open, f-θg-open) set.

Remark 2.1. Balasubramanian and Sundaram [4] (resp. El-Shafei and Zakari [9, 10]) call fuzzy generalized closed set in part (1) (resp. (2), (4)) of Definition 2.2 as generalized fuzzy closed (resp. semi-generalized fuzzy closed, θ-generalized fuzzy closed) set.

Lemma 2.1. [8] *Let μ be a fuzzy set in (X,τ). Then,*

$$\mu \leq \mathrm{scl}(\mu) \leq \mathrm{scl}_\theta(\mu)$$

and hence fuzzy semi-θ-closed set is a fuzzy semi-closed.

Definition 2.3. *A mapping $f:(X,\tau)\to(Y,\delta)$ is said to be*

(1) *fuzzy generalized continuous* [4] (*briefly, fg-continuous*) *if $f^{-1}(\lambda)$ is fuzzy generalized closed in X for each fuzzy closed set λ in Y;*
(2) *fuzzy semi-generalized continuous* [10] (*briefly, fsg-continuous*) *if $f^{-1}(\lambda)$ is fuzzy semi generalized closed in X for each fuzzy closed set λ in Y;*
(3) *fuzzy generalized semi-continuous* [19] (*briefly, fgs-continuous*) *if $f^{-1}(\lambda)$ is fuzzy generalized semi-closed in X for each fuzzy closed set λ in Y;*
(4) *fuzzy θ-generalized continuous* [9] (*briefly, f-θg-continuous*) *if $f^{-1}(\lambda)$ is fuzzy θ-generalized closed in X for each fuzzy closed set λ in Y.*

3. Fuzzy θ-semi-generalized closed sets

In this section, we introduce fuzzy θ-semi generalized closed sets in fuzzy topological space and we study some of their characterizations and relationships with other notions.

Definition 3.1. *Let μ be a fuzzy set in (X,τ). Then*

$$\wedge\{\mathrm{scl}(\eta):\mu\leq\eta,\eta\in FSO(X,\tau)\}$$

is called a fuzzy semi-θ-closure of μ denoted by $\mathrm{scl}_\theta(\mu)$. *Also, the fuzzy set*

$$\vee\{\mathrm{sint}(\eta):\eta\leq\mu,1-\eta\in FSO(X,\tau)\}$$

is fuzzy semi-θ-interior of μ denoted by $\mathrm{sint}_\theta(\mu)$.

Definition 3.2. *A fuzzy subset μ of (X,τ) is said to be fuzzy θ-semi generalized closed set* (*briefly, f-θsg-closed set*) *if* $\mathrm{scl}_\theta(\mu)\leq\eta$ *whenenever $\mu\leq\eta$ and $\eta\in FSO(X,\tau)$.*

The complement of fuzzy θ-semi generalized closed set is fuzzy θ-semi generalized open set (briefly, f-θsg-open set).

Lemma 3.1. *Every fuzzy semi-θ-closed set in a fuzzy topological space (X,τ) is fuzzy θ-semi generalized closed.*

Proof. Let μ be a fuzzy semi-θ-closed set, then $\mu=\mathrm{scl}_\theta(\mu)$. Suppose that $\mu\leq\eta$ and $\eta\in FSO(X,\tau)$. It follows that $\mathrm{scl}_\theta(\mu)\leq\eta$ and hence μ is fuzzy θ-semi generalized closed set in X. ∎

Examples 3.1 and 3.2 below show that the converse of Lemma 3.1 does not true.

Example 3.1. Let $X = \{x\}$ with fuzzy topology $\tau = \{0_X, x_{0.3}, 1_X\}$. So the family of all fuzzy semi-open sets in X is

$$FSO(X,\tau) = \left\{0_X, 1_X, x_p \text{ where } 0.3 \leq p \leq 0.7\right\}$$

and the family of all fuzzy semi-closed sets in X is

$$FSC(X,\tau) = \left\{0_X, 1_X, x_q \text{ where } 0.3 \leq q \leq 0.7\right\}.$$

If $\mu = x_{0.1}$ then μ is fuzzy θ-semi-generalized closed set, but not fuzzy semi-θ-closed since $\text{scl}_\theta(\mu) = x_{0.3} \neq \mu$.

Example 3.2. Let $X = \{a,b,c\}$ and $\tau = \{0_X, \mu_1, \mu_2, \mu_3, 1_X\}$ where $\mu_1 = a_0 \vee b_0 \vee c_{0.4}$, $\mu_2 = a_{0.9} \vee b_{0.6} \vee c_0$ and $\mu_3 = a_{0.9} \vee b_{0.6} \vee c_{0.4}$. The family of all fuzzy semi-open sets is

$$FSO(X,\tau) = \left\{ 0_X, 1_X, a_x \vee b_y \vee c_z \text{ either } \begin{array}{c} 0 \leq x \leq 0.1, \\ 0 \leq y \leq 0.4, \\ 0 \leq z \leq 0.6 \end{array} \text{ or } \begin{array}{c} 0.9 \leq x \leq 1, \\ 0.6 \leq y \leq 1, \\ 0 \leq z \leq 0.6 \end{array} \right\}.$$

Hence the family of all fuzzy semi-closed sets is

$$FSC(X,\tau) = \left\{ 0_X, 1_X, a_x \vee b_y \vee c_z \text{ either } \begin{array}{c} 0 \leq x \leq 0.1, \\ 0 \leq y \leq 0.4, \\ 0.4 \leq z \leq 1 \end{array} \text{ or } \begin{array}{c} 0.9 \leq x \leq 1, \\ 0.6 \leq y \leq 1, \\ 0.4 \leq z \leq 1 \end{array} \right\}.$$

If $\rho = a_{0.7} \vee b_{0.7} \vee c_{0.7}$, then ρ is fuzzy θ-semi-generalized closed sets since the only fuzzy semi-open superset of ρ is 1_X. But ρ is not fuzzy semi-θ-closed since $\text{scl}_\theta(\rho) = 1_X \neq \rho$.

Lemma 3.2. *Every fuzzy θ-semi-generalized closed set is fuzzy semi-generalized closed.*

Proof. Let μ be a fuzzy θ-semi-generalized closed set of (X,τ). Let $\mu \leq \eta$ and $\eta \in FSO(X,\tau)$. Since μ is a fuzzy θ-semi-generalized closed, $\text{scl}_\theta(\mu) \leq \eta$. By Lemma 2.1, it follows that $\text{scl}(\mu) \leq \eta$ since $\text{scl}(\mu) \leq \text{scl}_\theta(\mu)$. Hence μ is fuzzy semi-generalized closed. ∎

The following example shows that the converse of Lemma 3.2 is not true.

Example 3.3. Let $X = \{y\}$ with fuzzy topology $\tau = \{0_X, y_{\frac{2}{3}}, y_{\frac{3}{4}}, 1_X\}$. So

$$FSO(X,\tau) = \left\{0_X, 1_X, y_p \text{ where } \frac{2}{3} \leq p < 1\right\}$$

and

$$FSC(X,\tau) = \left\{0_X, 1_X, y_q \text{ where } 0 < q \leq \frac{1}{3}\right\}.$$

Let $\mu = y_{\frac{1}{3}}$ then μ is fuzzy semi-generalized closed set. But $\text{scl}_\theta(\mu) = 1_X \nleq y_{\frac{2}{3}}$ where $y_{\frac{2}{3}} \in FSO(X,\tau)$. Hence μ is not fuzzy θ-semi-generalized closed.

It is obvious that every fuzzy θ-generalized closed set is fuzzy θ-semi-generalized closed but the converse need not be true in general as the following examples show.

Example 3.4. Consider the fuzzy topological spaces (X,τ) in Example 3.1 and let $\mu = x_{0.2}$. Then μ is fuzzy θ-semi-generalized closed set but not fuzzy θ-generalized closed since $\text{cl}_\theta(\mu) = x_{0.7} \nleq x_{0.3}$.

Example 3.5. Let $X = \{a,b\}$ and $\tau = \{0_X, \mu_1, \mu_2, 1_X\}$ where $\mu_1 = a_{0.4} \vee b_{0.5}$ and $\mu_2 = a_{0.7} \vee b_{0.5}$. The family of all fuzzy semi-open sets is

$$FSO(X,\tau) = \left\{ 0_X, 1_X, a_x \vee b_y \text{ either } \begin{array}{c} 0.4 \leq x \leq 0.6, \\ y = 0.5 \end{array} \text{ or } \begin{array}{c} 0.7 \leq x \leq 1, \\ 0.5 \leq y \leq 1 \end{array} \right\}.$$

Hence the family of all fuzzy semi-closed sets is

$$FSC(X,\tau) = \left\{ 0_X, 1_X, a_x \vee b_y \text{ either } \begin{array}{c} 0.4 \leq x \leq 0.6, \\ y = 0.5 \end{array} \text{ or } \begin{array}{c} 0 \leq x \leq 0.3, \\ 0 \leq y \leq 0.5 \end{array} \right\}.$$

If $\rho = a_{0.2} \vee b_{0.4}$, then ρ is fuzzy θ-semi-generalized closed set but not fuzzy θ-generalized closed since $\mathrm{cl}_\theta(\rho) = a_{0.6} \vee b_{0.5} \nleq \mu_1$.

We summarize that every fuzzy closed set is fuzzy semi-closed and fuzzy generalized closed set but the converse are not true as in [10] and [4]. Every fuzzy semi-closed set is fuzzy semi-generalized closed but the converse is not true (see [10]). Moreover, fuzzy semi-θ-closed implies fuzzy θ-semi-generalized closed but the converse may not be true as in Examples 3.1 and 3.2 above. Lemma 3.2 shows that fuzzy θ-semi-generalized closed set implies fuzzy semi-generalized closed set but the reverse is not true in general as in Example 3.3. Examples 3.4 and 3.5 above show that fuzzy θ-semi-generalized closed does not implies fuzzy θ-generalized closed. Furthermore, fuzzy θ-closed implies fuzzy θ-generalized closed but the converse is not true (see [9]).

The Figure 1 below summarize the relationships among some fuzzy generalized closed sets discussed above where none of these implications is reversible. The abbreviation "F" in the diagram means "fuzzy".

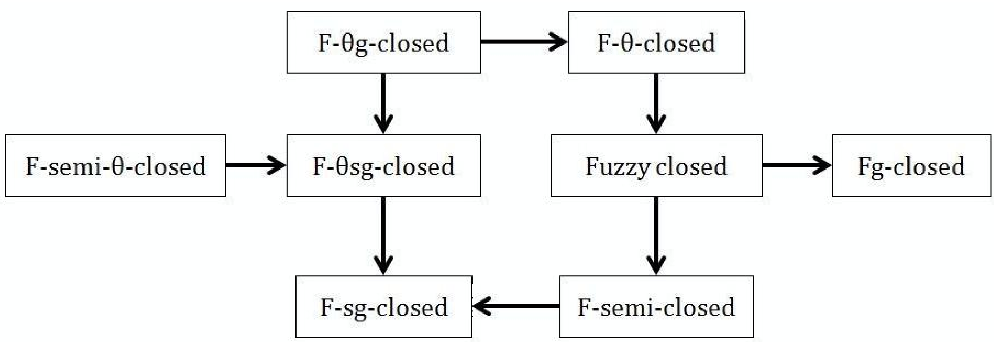

Figure 1. Relationship among some fuzzy generalized closed sets.

Lemma 3.3. [14] *If μ is a fuzzy semi-open set in a fuzzy topological space X, then* $\mathrm{scl}_\theta(\mu) = \mathrm{scl}(\mu)$.

Theorem 3.1. *Let μ be a fuzzy semi-open set in a fuzzy topological space (X,τ). The fuzzy set μ is a fuzzy θ-semi-generalized closed if and only if μ is fuzzy semi-generalized closed.*

Proof. Necessity. Let μ be a fuzzy θ-semi-generalized closed set in X and let $\mu \leq \eta$ where $\eta \in FSO(X,\tau)$. Hence $\mathrm{scl}_\theta(\mu) \leq \eta$ and since μ is fuzzy semi-open, $\mathrm{scl}(\mu) \leq \eta$ by Lemma 3.3. Hence, μ is fuzzy semi-generalized closed set.

Sufficiency. Let μ be a fuzzy semi-generalized closed set and let $\mu \leq \eta$ where $\eta \in FSO(X,\tau)$. Hence $\operatorname{scl}(\mu) \leq \eta$ and since μ is fuzzy semi-open, $\operatorname{scl}_\theta(\mu) \leq \eta$. Thus μ is fuzzy θ-semi-generalized closed set. ∎

Theorem 3.2. *A fuzzy set μ is fuzzy θ-semi generalized open if and only if $\eta \leq \operatorname{sint}_\theta(\mu)$ whenever η is fuzzy semi-closed in X and $\eta \leq \mu$.*

Proof. Necessity. Let μ be f-θsg-open set in X and $\eta \leq \mu$ where η is fuzzy semi-closed. It is obvious that μ^c is contained in η^c. Since μ^c is f-θsg-closed set then $\operatorname{scl}_\theta(\mu^c) \leq \eta^c$ and hence $\operatorname{scl}_\theta(\mu^c) = (\operatorname{sint}_\theta(\mu))^c \leq \eta^c$ such that $\eta \leq \operatorname{sint}_\theta(\mu)$.

Sufficiency. If μ is a fuzzy semi-closed set with $\eta \leq \operatorname{sint}_\theta(\mu)$ whenever $\eta \leq \mu$, then it follows that $\mu^c \leq \eta^c$ and $(\operatorname{sint}_\theta(\mu))^c \leq \eta^c$ such that $\operatorname{scl}_\theta(\mu^c) \leq \eta^c$. Hence μ^c is f-θsg-closed and therefore μ is f-θsg-open. ∎

Lemma 3.4. *If μ and ν are two fuzzy subsets of a fuzzy topological space (X,τ), then*:

(a) $\operatorname{scl}_\theta(\operatorname{scl}_\theta(\mu)) = \operatorname{scl}_\theta(\mu)$;
(b) $\operatorname{scl}_\theta(\mu \vee \nu) = \operatorname{scl}_\theta(\mu) \vee \operatorname{scl}_\theta(\nu)$;
(c) $\operatorname{scl}_\theta(\mu \wedge \nu) \leq \operatorname{scl}_\theta(\mu) \wedge \operatorname{scl}_\theta(\nu)$;
(d) $(\operatorname{scl}_\theta(\mu))^c = \operatorname{sint}_\theta(\mu^c)$;
(e) $\operatorname{sint}_\theta(\operatorname{sint}_\theta(\mu)) = \operatorname{sint}_\theta(\mu)$;
(f) $\operatorname{sint}_\theta(\mu \vee \nu) \geq \operatorname{sint}_\theta(\mu) \vee \operatorname{sint}_\theta(\nu)$;
(g) $\operatorname{sint}_\theta(\mu \wedge \nu) = \operatorname{sint}_\theta(\mu) \wedge \operatorname{sint}_\theta(\nu)$;
(h) $(\operatorname{sint}_\theta(\mu))^c = \operatorname{scl}_\theta(\mu^c)$.

Proof. (a) Let μ be a fuzzy set in (X,τ). It is obvious that $\operatorname{scl}_\theta(\mu) \leq \operatorname{scl}_\theta(\operatorname{scl}_\theta(\mu))$. Since $\mu \leq \operatorname{scl}_\theta(\mu)$ we will have

$$\begin{aligned}\operatorname{scl}_\theta(\operatorname{scl}_\theta(\mu)) &= \wedge\{\operatorname{scl}(\eta) : \operatorname{scl}_\theta(\mu) \leq \eta, \eta \in FSO(X,\tau)\}\\ &\leq \wedge\{\operatorname{scl}(\eta) : \mu \leq \eta, \eta \in FSO(X,\tau)\}\\ &= \operatorname{scl}_\theta(\mu).\end{aligned}$$

Hence part (a) proved.
(b) Since

$$\mu \leq \mu \vee \nu \text{ and } \nu \leq \mu \vee \nu,$$

then

$$\operatorname{scl}_\theta(\mu) \leq \operatorname{scl}_\theta(\mu \vee \nu) \text{ and } \operatorname{scl}_\theta(\nu) \leq \operatorname{scl}_\theta(\mu \vee \nu).$$

Thus

$$\operatorname{scl}_\theta(\mu) \vee \operatorname{scl}_\theta(\nu) \leq \operatorname{scl}_\theta(\mu \vee \nu).$$

On the other hand,

$$\mu \leq \operatorname{scl}_\theta(\mu) \text{ and } \nu \leq \operatorname{scl}_\theta(\nu),$$

then

$$\mu \vee \nu \leq \operatorname{scl}_\theta(\mu) \vee \operatorname{scl}_\theta(\nu).$$

Since $\operatorname{scl}_\theta(\mu) \vee \operatorname{scl}_\theta(\nu)$ is a fuzzy semi-θ-closed set and $\operatorname{scl}_\theta(\mu \vee \nu)$ is the smallest fuzzy semi-θ-closed set containing $\mu \vee \nu$, hence

$$\operatorname{scl}_\theta(\mu \vee \nu) \leq \operatorname{scl}_\theta(\mu) \vee \operatorname{scl}_\theta(\nu).$$

This gives the equality.

(c) Since $\mu \wedge \nu \leq \mu$ and $\mu \wedge \nu \leq \nu$, then

$$\mathrm{scl}_\theta(\mu \wedge \nu) \leq \mathrm{scl}_\theta(\mu) \text{ and } \mathrm{scl}_\theta(\mu \wedge \nu) \leq \mathrm{scl}_\theta(\nu).$$

Combining, we obtain

$$\mathrm{scl}_\theta(\mu \wedge \nu) \leq \mathrm{scl}_\theta(\mu) \wedge \mathrm{scl}_\theta(\nu).$$

(d) Observe that,

$$\begin{aligned}(\mathrm{scl}_\theta(\mu))^c &= 1 - \mathrm{scl}_\theta(\mu) \\ &= 1 - \wedge\{\mathrm{scl}(\lambda) : \mu \leq \lambda, \lambda \in FSO(X,\tau)\} \\ &= \vee\{\mathrm{sint}(1-\lambda) : 1-\mu \geq 1-\lambda, \lambda \in FSO(X,\tau)\}.\end{aligned}$$

By letting $\eta = 1-\lambda$, we have

$$\begin{aligned}(\mathrm{scl}_\theta(\mu))^c &= \vee\{\mathrm{sint}(\eta) : 1-\mu \geq \eta, 1-\eta \in FSO(X,\tau)\} \\ &= \mathrm{sint}_\theta(1-\mu).\end{aligned}$$

(e) The proof is similar with part (a), by using the Definition 3.1.
(f) is the complement of (c).
(g) is the complement of (b).
(h) Observe that

$$\begin{aligned}(\mathrm{sint}_\theta(\mu))^c &= 1 - \mathrm{sint}_\theta(\mu) \\ &= 1 - \vee\{\mathrm{sint}(\lambda) : \mu \geq \lambda, 1-\lambda \in FSO(X,\tau)\} \\ &= \wedge\{\mathrm{scl}(1-\lambda) : 1-\mu \leq 1-\lambda, 1-\lambda \in FSO(X,\tau)\}.\end{aligned}$$

Let $\eta = 1-\lambda$, then we have

$$(\mathrm{sint}_\theta(\mu))^c = \wedge\{\mathrm{scl}(\eta) : 1-\mu \leq \eta, \eta \in FSO(X,\tau)\} = \mathrm{scl}_\theta(1-\mu).$$ ∎

Part (b) and (g) of Lemma 3.4 can be extended to a finite case as follows.

Corollary 3.1. *If $\mu_1, \mu_2, \ldots, \mu_n$ are fuzzy subsets of a fuzzy topological space (X,τ), then*
(a) $\mathrm{scl}_\theta(\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n) = \mathrm{scl}_\theta(\mu_1) \vee \mathrm{scl}_\theta(\mu_2) \vee \cdots \vee \mathrm{scl}_\theta(\mu_n)$;
(b) $\mathrm{sint}_\theta(\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n) = \mathrm{sint}_\theta(\mu_1) \wedge \mathrm{sint}_\theta(\mu_2) \wedge \cdots \wedge \mathrm{sint}_\theta(\mu_n)$.

Theorem 3.3. *Let (X,τ) be a fuzzy topological space. The collection of all fuzzy semi-θ-open sets in (X,τ) is a fuzzy topological space.*

Proof. (i) Note that $0_X,\ 1_X \in FS\theta O(X,\tau)$ since $\mathrm{sint}_\theta(0_X) = 0_X$ and $\mathrm{sint}_\theta(1_X) = 1_X$ according to the Definition 2.1.

(ii) Suppose that $\{\mu_\alpha : \alpha \in \Delta\}$ be a collection of fuzzy semi-θ-open sets in X. Then $\mu_\alpha = \mathrm{sint}_\theta(\mu_\alpha)$ for each $\alpha \in \Delta$. Let $\mu = \vee\{\mu_\alpha : \alpha \in \Delta\}$. It is obvious that $\mathrm{sint}_\theta(\mu) \leq \mu$. On the other hand, since $\mu_\alpha \leq \mu$ we have that $\mathrm{sint}_\theta(\mu_\alpha) \leq \mathrm{sint}_\theta(\mu)$ for each $\alpha \in \Delta$. So $\vee\{\mathrm{sint}_\theta(\mu_\alpha) : \alpha \in \Delta\} \leq \mathrm{sint}_\theta(\mu)$. Thus we have $\mu = \vee\{\mu_\alpha : \alpha \in \Delta\} \leq \mathrm{sint}_\theta(\mu)$. Hence we have $\mu = \mathrm{sint}_\theta(\mu)$ and this shows that the arbitrary union of fuzzy semi-θ-open sets is a fuzzy semi-θ-open set.

(iii) Suppose that μ_1 and μ_2 be two fuzzy semi-θ-open sets in X. Then $\mu_1 = \mathrm{sint}_\theta(\mu_1)$ and $\mu_2 = \mathrm{sint}_\theta(\mu_2)$. Let $\mu = \mu_1 \wedge \mu_2$. It is obvious that $\mathrm{sint}_\theta(\mu) \leq \mu$. On the other hand, since $\mathrm{sint}_\theta(\mu_1) \wedge \mathrm{sint}_\theta(\mu_2) \leq \mu_1 \wedge \mu_2$, then by Lemma 3.4,

$$\mathrm{sint}_\theta(\mathrm{sint}_\theta(\mu_1) \wedge \mathrm{sint}_\theta(\mu_2)) \leq \mathrm{sint}_\theta(\mu_1 \wedge \mu_2)$$

$$\begin{aligned} &\Longrightarrow \operatorname{sint}_\theta(\operatorname{sint}_\theta(\mu_1)) \wedge \operatorname{sint}_\theta(\operatorname{sint}_\theta(\mu_2)) \leq \operatorname{sint}_\theta(\mu) \\ &\Longrightarrow \operatorname{sint}_\theta(\mu_1) \wedge \operatorname{sint}_\theta(\mu_2) \leq \operatorname{sint}_\theta(\mu) \\ &\Longrightarrow \mu_1 \wedge \mu_2 \leq \operatorname{sint}_\theta(\mu) \\ &\Longrightarrow \mu \leq \operatorname{sint}_\theta(\mu). \end{aligned}$$

Hence we have $\mu = \operatorname{sint}_\theta(\mu)$ and this shows that the intersection of two fuzzy semi-θ-open sets is also fuzzy semi-θ-open set. This completes the proof. ∎

By Theorem 3.3, the collection of all fuzzy semi-θ-open sets in (X,τ) is a fuzzy topological space. We shall denote this new fuzzy topology by $\tau_{s\theta}$. By the similar argument that has been discussed in Theorem 3.3, we have the following remark.

Remark 3.1. Let (X,τ) be a fuzzy topological space. The collection of all fuzzy semi-θ-closed sets in (X,τ) is also a fuzzy topological space.

Proposition 3.1. *The union of two fuzzy θ-semi-generalized closed sets is always fuzzy θ-semi-generalized closed set.*

Proof. Suppose that μ and ν are fuzzy θ-semi-generalized closed sets in X and let $\eta \in FSO(X,\tau)$ such that $\mu \vee \nu \leq \eta$. Since μ and ν are fuzzy θ-semi-generalized closed, then we have $\operatorname{scl}_\theta(\mu) \vee \operatorname{scl}_\theta(\nu) \leq \eta$ and by Lemma 3.4(b), $\operatorname{scl}_\theta(\mu \vee \nu) \leq \eta$. Hence, $\mu \vee \nu$ is fuzzy θ-semi-generalized closed. ∎

By utilizing Corollary 3.1, we obtain the following corollary.

Corollary 3.2. *The union of finite fuzzy θ-semi-generalized closed sets is always fuzzy θ-semi-generalized closed set.*

The intersection of two fuzzy θ-semi-generalized closed sets is not necessarily a fuzzy θ-semi-generalized closed set as the following example shows.

Example 3.6. Let $X = \{a,b\}$ and $\tau = \{0_X, a_{0.4}, b_{0.5}, a_{0.4} \vee b_{0.5}, 1_X\}$. The family of all fuzzy semi-open sets is

$$FSO(X,\tau) = \left\{ 0_X, 1_X, a_x \vee b_y \text{ either } \begin{matrix} 0.4 \leq x \leq 0.6, \\ 0 \leq y < 0.5 \end{matrix} \text{ or } \begin{matrix} 0 \leq x \leq 0.6, \\ y = 0.5 \end{matrix} \right\}.$$

Hence the family of all fuzzy semi-closed sets is

$$FSC(X,\tau) = \left\{ 0_X, 1_X, a_x \vee b_y \text{ either } \begin{matrix} 0.4 \leq x \leq 0.6, \\ 0.5 < y \leq 1 \end{matrix} \text{ or } \begin{matrix} 0.4 \leq x \leq 1, \\ y = 0.5 \end{matrix} \right\}.$$

If $\mu = a_{0.8} \vee b_{0.3}$ and $\rho = a_{0.5} \vee b_{0.7}$, then μ and ρ are fuzzy θ-semi-generalized closed sets since the only fuzzy semi-open superset of μ and ρ is 1_X. But $\mu \wedge \rho = a_{0.5} \vee b_{0.3}$ is not fuzzy θ-semi-generalized closed set since $\operatorname{scl}_\theta(\mu \wedge \rho) = a_{0.5} \vee b_{0.5} \nleq a_{0.5} \vee b_{0.3}$ where $a_{0.5} \vee b_{0.3} \in FSO(X,\tau)$.

Theorem 3.4. *If μ be a fuzzy θ-semi-generalized closed set and $\mu \leq \beta \leq \operatorname{scl}_\theta(\mu)$ then β is a fuzzy θ-semi-generalized closed set.*

Proof. Let η be a fuzzy semi-open subset of X such that $\beta \leq \eta$. Then $\mu \leq \eta$. Since μ is fuzzy θ-semi-generalized closed, it follows that $\operatorname{scl}_\theta(\mu) \leq \eta$. Now, $\beta \leq \operatorname{scl}_\theta(\mu)$ implies $\operatorname{scl}_\theta(\beta) \leq \operatorname{scl}_\theta(\operatorname{scl}_\theta(\mu)) = \operatorname{scl}_\theta(\mu)$. Thus, $\operatorname{scl}_\theta(\beta) \leq \eta$. This prove that β is also fuzzy θ-semi-generalized closed subset of X. ∎

Corollary 3.3. *Let μ be fuzzy θ-semi-generalized open set in X and* $\text{sint}_\theta(\mu) \le \beta \le \mu$, *then β is also fuzzy θ-semi-generalized open set.*

Proof. Let μ be a fuzzy θ-semi-generalized open set in X and $\text{sint}_\theta(\mu) \le \beta \le \mu$. Then $1-\mu$ is fuzzy θ-semi-generalized closed set and $1-\mu \le 1-\beta \le \text{scl}_\theta(1-\mu)$. By Theorem 3.4, $1-\beta$ is fuzzy θ-semi-generalized closed set. Hence, β is fuzzy θ-semi-generalized open set. ∎

Theorem 3.5. *Let μ be a f-θsg-closed subset of (X,τ). Then*

(i) $\text{scl}_\theta(\mu) - \mu$ *does not contain a nonzero fuzzy semi-closed set*;
(ii) $\text{scl}_\theta(\mu) - \mu$ *is f-θsg-open set.*

Proof. (i) Let μ be a fuzzy set of (X,τ) and suppose that there exists a nonzero fuzzy semi-closed subset υ of X such that $\upsilon \le \text{scl}_\theta(\mu) - \mu$ and $\upsilon \ne 0_X$. Now, $\upsilon \le \text{scl}_\theta(\mu) - \mu$, i.e., $\upsilon \le \mu^c$ which implies $\mu \le \upsilon^c$. Since υ^c is fuzzy semi-open and μ is f-θsg-closed set, $\text{scl}_\theta(\mu) \le \upsilon^c$, i.e. $\upsilon \le (\text{scl}_\theta(\mu))^c$. Then $\upsilon \le (\text{scl}_\theta(\mu)) \wedge (\text{scl}_\theta(\mu))^c = 0_X$ and hence $\upsilon = 0_X$ which is contradiction.

(ii) Suppose that μ be f-θsg-closed and υ be a fuzzy semi-closed set such that $\upsilon \le \text{scl}_\theta(\mu) - \mu$. Then by (i), υ is zero and therefore $\upsilon \le \text{sint}_\theta(\text{scl}_\theta(\mu) - \mu)$. Hence, $\text{scl}_\theta(\mu) - \mu$ is f-θsg-open by Theorem 3.2. ∎

Lemma 3.5. *Let μ be a fuzzy subset of the fuzzy topological space (X,τ) and $x_\alpha \in S(X)$. Then $x_\alpha \in \text{scl}(\mu)$ if and only if $\nu \wedge \mu \ne 0$ for each $\nu \in FSO(X,\tau)$ and $x_\alpha \in \nu$.*

Proof. We prove using contrapositive. If $x_\alpha \notin \text{scl}(\mu)$, the fuzzy set $1-\text{scl}(\mu)$ is a fuzzy semi-open set such that $x_\alpha \in 1-\text{scl}(\mu)$. Choose $\nu = 1-\text{scl}(\mu)$, we see that $\nu \wedge \mu = 0$. Conversely, if there exists a fuzzy semi-open set ν such that $x_\alpha \in \nu$ and $\nu \wedge \mu = 0$, then $1-\nu$ is a fuzzy semi-closed set containing μ. By definition of the fuzzy semi-closure $\text{scl}(\mu)$, the fuzzy set $1-\nu \ge \text{scl}(\mu)$. Therefore $x_\alpha \notin \text{scl}(\mu)$. ∎

Recall that a fuzzy topological space (X,τ) is said to be fuzzy semi-$T_{\frac{1}{2}}$ [13] if and only if every fuzzy semi-generalized closed set in X is fuzzy semi-closed.

Theorem 3.6. *A fuzzy topological space (X,τ) is said to be fuzzy semi-$T_{\frac{1}{2}}$ if and only if*

(i) *every fuzzy singleton is fuzzy semi-open or fuzzy semi-closed.*
(ii) *every fuzzy θ-semi-generalized closed set is fuzzy semi-closed.*

Proof. (i) Let (X,τ) be a fuzzy semi-$T_{\frac{1}{2}}$-space and for some $x_\alpha \in S(X)$, x_α is not fuzzy semi-closed. Then $1-x_\alpha$ is not fuzzy semi-open and hence 1_X is the only fuzzy semi-open set containing $1-x_\alpha$. Therefore, $1-x_\alpha$ is fsg-closed in (X,τ). Since (X,τ) is a fuzzy semi-$T_{\frac{1}{2}}$-space, then $1-x_\alpha$ is fuzzy semi-closed set or equivalently x_α is fuzzy semi-open set.

Conversely, assume that every fuzzy singleton of (X,τ) is either fuzzy semi-closed or fuzzy semi-open set. Let μ be a fsg-closed set of (X,τ). Let $x_\alpha \in S(X)$ and by hypothesis we have two cases:

Case I: Suppose that x_α is a fuzzy semi-closed and let $x_\alpha \in \text{scl}(\mu)$. If $x_\alpha \notin \mu$, then $x_\alpha \in \text{scl}_\theta(\mu) - \mu$. Now $\text{scl}_\theta(\mu) - \mu$ contains a nonzero fuzzy semi-closed set. Since μ is fsg-closed set, it is a contradiction by part (i) of Theorem 3.5. Hence $x_\alpha \in \mu$.

Case II: Assume that x_α is a fuzzy semi-open and let $x_\alpha \in \text{scl}(\mu)$, then $x_\alpha \wedge \mu \ne 0_X$ by Lemma 3.5. So, $x_\alpha \in \mu$.

Thus in both cases $x_\alpha \in \mu$. So $\mathrm{scl}(\mu) \leq \mu$. Therefore $\mu = \mathrm{scl}(\mu)$, i.e., μ is a fuzzy semi-closed set. Hence, (X,τ) is fuzzy semi-$T_{\frac{1}{2}}$-space.

(ii) *Necessity.* Let μ be a f-θsg-closed set in (X,τ). By Lemma 3.2, μ is fsg-closed set. Since (X,τ) is a fuzzy semi-$T_{\frac{1}{2}}$-space, μ is fuzzy semi-closed set.

Sufficiency. Let $x_\alpha \in S(X)$. If x_α is not fuzzy semi-closed, then $1-x_\alpha$ is not fuzzy semi-open set and thus the only superset of $1-x_\alpha$ is 1_X. So, $1-x_\alpha$ is f-θsg-closed. By hypothesis, $1-x_\alpha$ is fuzzy semi-closed or equivalently x_α is fuzzy semi-open. Hence (X,τ) is a fuzzy semi-$T_{\frac{1}{2}}$-space. ∎

4. Fuzzy θ-semi-generalized continuous maps

As application of the concept of fuzzy θ-semi-generalized closed set, we identify some types of fuzzy mappings and introducing some of their properties as follows.

Definition 4.1. *A mapping $f:(X,\tau)\to(Y,\delta)$ is called*

(a) *fuzzy θ-semi-generalized continuous (briefly f-θsg-continuous) if $f^{-1}(\mu)$ is f-θsg-closed in (X,τ) for every fuzzy semi-closed set μ in (Y,δ);*

(b) *fuzzy θ-semi-generalized irresolute (briefly f-θsg-irresolute) if $f^{-1}(\mu)$ is fuzzy θ-semi-generalized closed in (X,τ) for every fuzzy θ-semi-generalized closed set μ in (Y,δ).*

Theorem 4.1. *A mapping $f:(X,\tau)\to(Y,\delta)$ is f-θsg-continuous if and only if the inverse image of each fuzzy semi-open subset of (Y,δ) is f-θsg-open in (X,τ).*

Proof. Straightforward. ∎

Theorem 4.2. *If a mapping $f:(X,\tau)\to(Y,\delta)$ is f-θsg-continuous, then for each fuzzy point x_α of (X,τ) and each fuzzy semi-open set λ in (Y,δ) such that $f(x_\alpha)\in\lambda$, there exists a f-θsg-open set μ of (X,τ) such that $x_\alpha\in\mu$ and $f(\mu)\leq\lambda$.*

Proof. Suppose that f is f-θsg-continuous. Let x_α be a fuzzy point of (X,τ) and λ be fuzzy semi-open set in (Y,δ) such that $f(x_\alpha)\in\lambda$. Then $f^{-1}(\lambda)$ is f-θsg-open set in (X,τ) and $x_\alpha\in f^{-1}(\lambda)$. Take $\mu=f^{-1}(\lambda)$ then $x_\alpha\in\mu$ and $f(\mu)=f(f^{-1}(\lambda))\leq\lambda$. Hence, $f(\mu)\leq\lambda$. ∎

Theorem 4.3. *If $f:(X,\tau)\to(Y,\delta)$ is fuzzy θ-semi-generalized continuous, then f is fuzzy semi-generalized continuous.*

Proof. Let λ be a fuzzy closed set in (Y,δ) and thus λ is also fuzzy semi-closed set. Since f is f-θsg-continuous, then $f^{-1}(\lambda)$ is f-θsg-closed set in (X,τ). Since every f-θsg-closed set is fuzzy semi-generalized closed set by Lemma 3.2, then $f^{-1}(\lambda)$ is fuzzy semi-generalized closed in (X,τ). Thus, f is fuzzy semi-generalized continuous. ∎

Example 4.1. Suppose that $X=\{x,y\}$ with fuzzy topology $\tau=\{0_X, x_{0.6}\vee y_{0.1}, 1_X\}$ and $Y=\{a,b\}$ with fuzzy topology $\delta=\{0_X, a_{0.5}\vee b_{0.6}, 1_X\}$. Let $f:(X,\tau)\to(Y,\delta)$ be defined by $f(x)=b$ and $f(y)=a$. Now, the families of all fuzzy semi-open and fuzzy semi-closed sets in X and Y, respectively, are as follow:

$$FSO(X,\tau)=\left\{0_X, 1_X, x_a\vee y_b \quad \text{where} \quad \begin{array}{c} 0.6\leq a\leq 1 \\ 0.1\leq b\leq 1 \end{array}\right\},$$

$$FSC(X,\tau)=\left\{0_X,1_X,x_a\vee y_b \quad \text{where} \quad \begin{array}{c}0\le a\le 0.4\\ 0\le b\le 0.9\end{array}\right\};$$
$$FSO(Y,\delta)=\left\{0_Y,1_Y,a_x\vee b_y \quad \text{where} \quad \begin{array}{c}0.5\le x\le 1\\ 0.6\le y\le 1\end{array}\right\},$$
$$FSC(Y,\delta)=\left\{0_Y,1_Y,a_x\vee b_y \quad \text{where} \quad \begin{array}{c}0\le x\le 0.5\\ 0\le y\le 0.4\end{array}\right\}.$$

Then f is fuzzy semi-generalized continuous. However f is not fuzzy θ-semi-generalized continuous since $f^{-1}(a_{0.1}\vee b_{0.4})=x_{0.4}\vee y_{0.1}$ is not fuzzy θ-semi-generalized closed set in X for $a_{0.1}\vee b_{0.4}$ is fuzzy semi-closed set in Y because, $x_{0.4}\vee y_{0.1}\le x_{0.6}\vee y_{0.1}\in FSO(X,\tau)$ but $\mathrm{scl}_\theta(x_{0.4}\vee y_{0.1})=1_X\nleq x_{0.6}\vee y_{0.1}$.

We have observed that every fuzzy continuous function is a fuzzy semi-continuous but the converse is not true in general (see [3]). Every fuzzy continuous function is fuzzy generalized continuous but the converse is not true as in [4]. Moreover, Theorem 4.3 shows that every fuzzy θ-semi-generalized continuous is fuzzy semi-generalized continuous but Example 4.1 shows that the converse of the implication is not true.

The following Figure 2 summarizes the discussion above which none of these implications is reversible. The abbreviation "F" stands for "fuzzy".

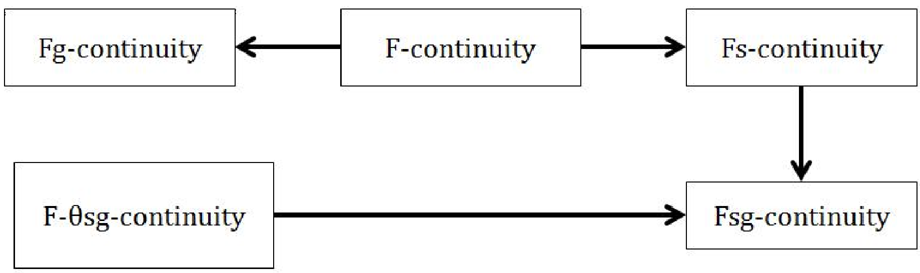

Figure 2. Relationships among some fuzzy generalized continuities.

Definition 4.2. *A mapping $f:(X,\tau)\to(Y,\delta)$ is said to be fuzzy θ-semi-generalized closed (resp. fuzzy θ-semi-generalized open) if $f(\lambda)$ is fuzzy θ-semi-generalized closed (resp. fuzzy θ-semi-generalized open) in (Y,δ) for every fuzzy semi-closed (resp. fuzzy semi-open) set λ in (X,τ).*

Theorem 4.4. *A mapping $f:(X,\tau)\to(Y,\delta)$ is fuzzy θ-semi-generalized closed if and only if for each fuzzy subset λ of (Y,δ) and for each fuzzy semi-open set μ in (X,τ) containing $f^{-1}(\lambda)$ there is a fuzzy θ-semi-generalized open subset ν of (Y,δ) such that $\lambda\le\nu$ and $f^{-1}(\nu)\le\mu$.*

Proof. Assume that f is fuzzy θ-semi-generalized closed map. Let λ be a fuzzy subset of (Y,δ) and μ be a fuzzy semi-open set of (X,τ) such that $f^{-1}(\lambda)\le\mu$. Now, $1_X-\mu$ is fuzzy semi-closed set in X. Then $f(1_X-\mu)$ is fuzzy θ-semi-generalized closed set in (Y,δ), since f is fuzzy θ-semi-generalized closed. So, $1_Y-f(1_X-\mu)$ is fuzzy θ-semi-generalized open in (Y,δ). Thus, choose $\nu=1_Y-f(1_X-\mu)$ is a fuzzy θ-semi-generalized open set such that $\lambda\le\nu$ and $f^{-1}(\nu)\le\mu$.

Conversely, suppose that m is fuzzy semi-closed set in (X,τ). Then $1_X - m$ is fuzzy semi-open and $f^{-1}(1_Y - f(m)) \leq 1_X - m$. Then there exists a fuzzy θ-semi-generalized open set ν of (Y,δ) such that $1_Y - f(m) \leq \nu$ and $f^{-1}(\nu) \leq 1_X - m$ and so $m \leq 1_X - f^{-1}(\nu)$. Hence $1_Y - \nu \leq f(m) \leq f\left(1_X - f^{-1}(\nu)\right) \leq 1_Y - \nu$ which implies $f(m) = 1_Y - \nu$. Since $1_Y - \nu$ is fuzzy θ-semi-generalized closed, $f(m)$ is fuzzy θ-semi-generalized closed and thus f is fuzzy θ-semi-generalized closed map. ∎

Definition 4.3. *A mapping $f:(X,\tau)\to(Y,\delta)$ is said to be fuzzy pre-semi-open (resp. fuzzy pre-semi-closed) if $f(\mu)\in FSO(Y,\delta)$ (resp. $f(\mu)\in FSC(Y,\delta)$) for every $\mu\in FSO(X,\tau)$ (resp. $\mu\in FSC(X,\tau)$).*

Theorem 4.5. *If the surjective mapping $f:(X,\tau)\to(Y,\delta)$ is fuzzy θ-semi generalized irresolute, fuzzy pre-semi-closed, and (X,τ) is fuzzy semi-$T_{\frac{1}{2}}$-space, then (Y,δ) is also fuzzy semi-$T_{\frac{1}{2}}$-space.*

Proof. Let λ be a f-θsg-closed set in Y. Since f is f-θsg-irresolute map, $f^{-1}(\lambda)$ is f-θsg-closed in X. As X is fuzzy semi-$T_{\frac{1}{2}}$-space, $f^{-1}(\lambda)$ is fuzzy semi-closed in X by Theorem 3.6(ii). Also since f is fuzzy pre-semi-closed map, $f\left(f^{-1}(\lambda)\right)$ is fuzzy semi-closed in Y. Since f is surjective, $f\left(f^{-1}(\lambda)\right)=\lambda$. Thus λ is fuzzy semi-closed in Y. Hence, Y is fuzzy semi-$T_{\frac{1}{2}}$-space. ∎

Theorem 4.6. *Let $f:(X,\tau)\to(Y,\delta)$ and $g:(Y,\delta)\to(Z,\gamma)$ be two maps. Then*

(1) *$g\circ f:(X,\tau)\to(Z,\gamma)$ is f-θsg-irresolute if f and g are f-θsg-irresolute.*
(2) *$g\circ f:(X,\tau)\to(Z,\gamma)$ is f-θsg-continuous if f is f-θsg-irresolute and g is f-θsg-continuous.*

Proof. (1) Let k be f-θsg-closed set in (Z,γ). Since $g:(Y,\delta)\to(Z,\gamma)$ is f-θsg-irresolute, $g^{-1}(k)$ is f-θsg-closed subset of (Y,δ). Now, $f:(X,\tau)\to(Y,\delta)$ is f-θsg-irresolute, therefore $f^{-1}\left(g^{-1}(k)\right)$ is f-θsg-closed in (X,τ). Since $(g\circ f)^{-1}(k)=f^{-1}\left(g^{-1}(k)\right)$. Then $g\circ f$ is f-θsg-irresolute.

(2) Let h be a fuzzy semi-closed set in (Z,γ). Since g is f-θsg-continuous, $g^{-1}(h)$ is f-θsg-closed in (Y,δ). Now, $f:(X,\tau)\to(Y,\delta)$ is f-θsg-irresolute, therefore $f^{-1}\left(g^{-1}(h)\right)$ is f-θsg-closed in (X,τ). Since $(g\circ f)^{-1}(h)=f^{-1}\left(g^{-1}(h)\right)$, then $g\circ f$ is f-θsg-continuous. ∎

The results in Theorem 4.6 can be extended to finite compositions of maps as follows.

Corollary 4.1. *If for each $i=1,2,\ldots,n$, $f_i:(X_i,\tau_i)\to(X_{i+1},\tau_{i+1})$ are f-θsg-irresolute maps, then $f_n\circ f_{n-1}\circ\cdots\circ f_2\circ f_1:(X_1,\tau_1)\to(X_{n+1},\tau_{n+1})$ is f-θsg-irresolute.*

Corollary 4.2. *If for each $i=1,2,\ldots,n$, $f_i:(X_i,\tau_i)\to(X_{i+1},\tau_{i+1})$ are f-θsg-irresolute maps and $g:(X_{n+1},\tau_{n+1})\to(Z,\gamma)$ is f-θsg-continuous map, then $g\circ f_n\circ f_{n-1}\circ\cdots\circ f_2\circ f_1:(X_1,\tau_1)\to(Z,\gamma)$ is f-θsg-continuous.*

Theorem 4.7. *For any bijection mapping $f:(X,\tau)\to(Y,\delta)$, the following statements are equivalent:*

(a) *f^{-1} is f-θsg-continuous.*
(b) *f is a f-θsg-open.*
(c) *f is a f-θsg-closed.*

Proof. (a) $\Longrightarrow$ (b) : Let μ be a fuzzy semi-open set in (X,τ). Assume that the inverse of f is f-θsg-continuous, thus we have $\left(f^{-1}\right)^{-1}(\mu) = f(\mu)$ is f-θsg-open in (Y,δ) and hence f is f-θsg-open map.

(b) $\Longrightarrow$ (c) : Suppose that μ is fuzzy semi-closed subset of (X,τ), then $1-\mu$ is fuzzy semi-open subset of (X,τ). By (b), $f(1-\mu)$ is f-θsg-open in (Y,δ). So, $f(1-\mu) = 1-f(\mu)$ is f-θsg-open in (Y,δ). Therefore $f(\mu)$ is f-θsg-closed in (Y,δ). Hence, f is f-θsg-closed map.

(c) $\Longrightarrow$ (a) : Let λ be a fuzzy semi-closed set in (X,τ). By (c), $f(\lambda)$ is f-θsg-closed in (Y,δ). Then, $f(\lambda) = \left(f^{-1}\right)^{-1}(\lambda)$ is f-θsg-closed and therefore f^{-1} is f-θsg-continuous by Definition 4.2. ∎

5. Conclusion

In this paper, we introduce fuzzy θ-semi-generalized closed set to create some applications which is fuzzy θ-semi-generalized continuity, fuzzy θ-semi-generalized irresolute and fuzzy θ-semi-generalized closed maps. We also investigate the relationship of some generalized closed sets which is related to fuzzy θ-semi-generalized closed sets. Those will give some new relationships which have be found to be useful in the study of generalized closed sets and generalized continuities in fuzzy topological spaces. Recently, Kılıçman and Salleh [11] obtained some further results on $(\delta\text{-}pre,s)$-continuous maps in topological spaces. Moreover, Xu *et al.* [20] investigated about generalized fuzzy compactness in *L*-topological spaces and Saadati *et al.* [18] gained some common fixed point theorems in complete *L*-fuzzy metric spaces which are generalizations of fuzzy metric spaces and intuitionistic fuzzy metric spaces. It is an open problem to extend these new concepts to the fuzzy topological spaces.

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