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On *AAM*'s Conjecture for $D_n(3)$

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Abstract. The noncommuting graph of a finite nonabelian group G, denoted $\nabla(G)$, is defined as follows: its vertices are the non-central elements of G, and two vertices are adjacent when they do not commute. Problem 16.1 in *the Kourovka Notebook* contains the following conjecture: If M is a finite nonabelian simple group and G is a group such that $\nabla(G) \cong \nabla(M)$, then $G \cong M$. The validity of this conjecture is still unknown for most of finite simple groups with connected prime graphs even though it is known to hold for all finite simple groups with disconnected prime graphs and only a few of finite simple groups with connected prime graphs, for example, A_{10} and $L_4(9)$. In the present paper, it is proved that the finite simple group of Lie type $D_n(3)$, where $n \geq 5$ is an odd integer or $n = p+1$ for a prime $p > 3$, is quasirecognizable by its prime graph. In particular, *AAM*'s conjecture is true for it. Thus it is an example of an infinite series of finite simple groups recognizable by their noncommuting graphs, whose prime graphs are connected for some n.

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1. Introduction

One of the well-known graphs which has deserved a lot of attention is the *prime graph* or *Gruenberg-Kegel graph* $\Gamma(G)$ of a finite group G. In this graph, the vertices are the prime numbers dividing the order of the group G and two different vertices p and q are connected when G possesses an element of order pq (see [18]). Another graph associated with a finite nonabelian group is called the *noncommuting graph*. Actually, the noncommuting graph of a finite nonabelian group G, denoted $\nabla(G)$, is defined as follows: its vertices are the non-central elements of G, and two vertices are adjacent when they do not commute (see [2]).

For a graph X, we denote the sets of its vertices and edges by $V(X)$ and $E(X)$, respectively. Two graphs X and Y are said to be isomorphic if there exists a bijective map ϕ: $V(X) \to V(Y)$ such that x and y are adjacent in X if and only if $\phi(x)$ and $\phi(y)$ are adjacent in Y. If two graphs X and Y are isomorphic, we denote it by $X \cong Y$. It is easy to see that if $X \cong Y$, then $|V(X)| = |V(Y)|$ and $|E(X)| = |E(Y)|$.

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In 2006, Abdollahi, Akbari and Maimani posed the following conjecture in [2], which is also compiled as Problem 16.1 in *the Kourovka Notebook* (see [10]).

***AAM*'s Conjecture:** If M is a finite nonabelian simple group and G is a group such that $\nabla(G) \cong \nabla(M)$, then $G \cong M$.

In fact, *AAM*'s conjecture is true for all finite simple groups with disconnected prime graphs, A_{10}, $L_4(q)(q \leq 17)$, $SL(2,q)$ and $PGL(2,q)$, where q is a prime power (see [1–3, 6, 9, 17, 20–26]). However, it is still unknown whether *AAM*'s conjecture is true for most of the finite (almost) simple groups with connected prime graphs.

The *spectrum* of a finite group G, which is denoted by $\pi_e(G)$, is the set of its element orders. A finite nonabelian simple group S is called *quasirecognizable* by its prime graph (resp. by spectrum) if every finite group G with $\Gamma(G) = \Gamma(S)$ (resp. $\pi_e(G) = \pi_e(S)$) has a unique nonabelian composition factor isomorphic to S. We denote by $k(\Gamma(G))$ (resp. $h(\pi_e(G))$) the number of isomorphism classes of finite groups H satisfying $\Gamma(G) = \Gamma(H)$ (resp. $\pi_e(G) = \pi_e(H)$). Given a natural number r, a finite group G is called *r-recognizable* by its prime graph (resp. by spectrum) if $k(\Gamma(G)) = r$ (resp. $h(\pi_e(G)) = r$) and *irrecognizable* if $k(\Gamma(G))$ (resp. $h(\pi_e(G))$) is infinite. Usually a 1-*recognizable* group by its prime graph (resp. by spectrum) is called a *recognizable* group by its prime graph (resp. by spectrum) (see [4, 12]).

Let M be a finite nonabelian simple group. If G is a group such that $\nabla(G) \cong \nabla(M)$, then $\Gamma(G) \cong \Gamma(M)$ by Corollary 2.2. Thus there is a close relation between *AAM*'s conjecture and recognition by its prime graph for a finite simple group.

In the present paper, we focus our attention on the finite simple group $D_n(3)$, where $n \geq 5$ is an odd integer, and prove that it is quasirecognizable by its prime graph. As a consequence of this result and another known one, we have that *AAM*'s conjecture is also true for $D_n(3)$, where $n \geq 5$ is an odd integer or $n = p+1$ for a prime $p > 3$. Thus it is an example of an infinite series of finite simple groups recognizable by their noncommuting graphs, whose prime graphs are connected for some n. To prove these results, we use the classification of finite simple groups and some special properties of their prime graphs.

2. Preliminaries and lemmas

In the sequel, we denote by $\pi(n)$ the set of prime divisors of a natural number n. Let G be a finite group. For short, we define $\pi(G) := \pi(|G|)$. Moreover, we denote by $\pi_i(G)$ ($i = 1, 2, \ldots, s(G)$) the ith connected component of $\Gamma(G)$. When $|G|$ is even, we always assume that $2 \in \pi_1(G)$. Let q be a prime power and p be an odd prime. By [11, 18], we have the following statements about the finite simple group $D_n(q)$.

(1) When $n = p \geq 5$ and $q = 2, 3, 5$, $\Gamma(D_p(q))$ has two connected components: $\pi_1(D_p(q)) = \pi(q\prod_{i=1}^{p-1}(q^{2i}-1))$; $\pi_2(D_p(q)) = \pi((q^p-1)/(q-1))$.

(2) When $n = p+1$ and $q = 2, 3$, $\Gamma(D_{p+1}(q))$ has two connected components: $\pi_1(D_{p+1}(q)) = \pi(q(q^p+1)(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1))$; $\pi_2(D_{p+1}(q)) = \pi(q^p-1)$.

(3) Except for the above two cases, $\Gamma(D_n(q))$ has only one connected component: $\pi_1(D_n(q)) = \pi(q(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1))$.

Next we state some lemmas which are particularly useful in our analysis.

Lemma 2.1. [5] *Let G be a finite simple group of Lie type over a finite field of order q, where q is a prime power. Then $|G| = N/d$, where N and d are itemized in Table* 1.

Table 1. Orders of finite simple groups of Lie type

G	N	d
$A_n(q)(n \geq 1)$	$q^{\frac{n(n+1)}{2}} \prod_{i=1}^{n}(q^{i+1}-1)$	$(n+1, q-1)$
$B_n(q)(n \geq 2)$	$q^{n^2} \prod_{i=1}^{n}(q^{2i}-1)$	$(2, q-1)$
$C_n(q)(n \geq 3)$	$q^{n^2} \prod_{i=1}^{n}(q^{2i}-1)$	$(2, q-1)$
$D_n(q)(n \geq 4)$	$q^{n(n-1)}(q^n-1) \prod_{i=1}^{n-1}(q^{2i}-1)$	$(4, q^n-1)$
$G_2(q)$	$q^6(q^6-1)(q^2-1)$	1
$F_4(q)$	$q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	1
$E_6(q)$	$q^{36}(q^{12}-1)(q^9-1)(q^8-1)\times$ $(q^6-1)(q^5-1)(q^2-1)$	$(3, q-1)$
$E_7(q)$	$q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)\times$ $(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)$	$(2, q-1)$
$E_8(q)$	$q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)\times$ $(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	1
${}^2A_n(q)(n \geq 2)$	$q^{\frac{n(n+1)}{2}} \prod_{i=1}^{n}(q^{i+1}-(-1)^{i+1})$	$(n+1, q+1)$
${}^2B_2(q)(q = 2^{2m+1})$	$q^2(q^2+1)(q-1)$	1
${}^2D_n(q)(n \geq 4)$	$q^{n(n-1)}(q^n+1) \prod_{i=1}^{n-1}(q^{2i}-1)$	$(4, q^n+1)$
${}^3D_4(q)$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	1
${}^2G_2(q)(q = 3^{2m+1})$	$q^3(q^3+1)(q^2-1)$	1
${}^2F_4(q)(q = 2^{2m+1})$	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	1
${}^2E_6(q)$	$q^{36}(q^{12}-1)(q^9+1)(q^8-1)\times$ $(q^6-1)(q^5+1)(q^2-1)$	$(3, q+1)$

If q is a natural number, r is an odd prime and $(r,q) = 1$, then by $e(r,q)$ we denote the minimal natural number n with $q^n \equiv 1 \pmod{r}$. If q is odd, let $e(2,q) = 1$ if $q \equiv 1 \pmod{4}$ and $e(2,q) = 2$ if $q \equiv -1 \pmod{4}$.

Lemma 2.2. [27, Corollary to Zsigmondy's theorem] *Let q be a natural number greater than* 1. *For every natural number m there exists a prime r with $e(r,q) = m$ but for the cases $q = 2$ and $m = 1$, $q = 3$ and $m = 1$, and $q = 2$ and $m = 6$.*

The prime r with $e(r,q) = n$ is said to be a Zsigmondy prime of $q^n - 1$. By Lemma 2.2 it always exists except for the cases indicated above. If q is fixed, we denote by $r_n(q)$ or r_n some Zsigmondy prime of $q^n - 1$. Obviously, $q^n - 1$ can have more than one such divisor. Note that according to our definition every prime divisor of $q - 1$ is a Zsigmondy prime of $q - 1$ with sole exception: 2 is not a Zsigmondy prime of $q - 1$ if $e(2,q) = 2$. In the last case 2 is a Zsigmondy prime of $q^2 - 1$.

Let G be a finite group and $r \in \pi(G)$. We denote by $\rho(G)$ (by $\rho(r,G)$) some independence set in $\Gamma(G)$ (containing r) with maximal number of vertices. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise non-adjacent in $\Gamma(G)$. In other words, $t(G)$ is a maximal number of vertices in independent sets of $\Gamma(G)$ and is called an independence number of the graph. By analogy we denote by $t(r,G)$ the maximal number of vertices in independent sets of $\Gamma(G)$ containing the prime r. We call this number an r-independence number. Obviously,

$|\rho(G)| = t(G)$ and $|\rho(r,G)| = t(r,G)$ (see [15]). It is not hard to see that $\rho(G)$ and $\rho(r,G)$ are generally not uniquely determined.

For natural numbers m and r, $[m]$ denotes the integral part of m, and m_r stands for the r-part of m, which is the greatest divisor of m with $\pi(m_r) \subseteq \pi(r)$. Moreover, we define

$$\eta(m) = \begin{cases} m, & \text{if m is odd;} \\ \frac{m}{2}, & \text{otherwise;} \end{cases} \qquad \text{and} \qquad \nu(m) = \begin{cases} m, & m \equiv 0 \pmod 4; \\ \frac{m}{2}, & m \equiv 2 \pmod 4; \\ 2m, & m \equiv 1 \pmod 2). \end{cases}$$

The following lemmas describe a connection between the structure of a finite simple group and the properties of its prime graph.

Lemma 2.3. [15, Proposition 2.2] *Let $G \cong {}^2A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Suppose that r and s are odd primes with $r,s \in \pi(G)\backslash\{p\}, k = e(r,q), l = e(s,q)$ and $2 \le \nu(k) \le \nu(l)$. Then r and s are non-adjacent if and only if $\nu(k)+\nu(l) > n$ and $\nu(k) \nmid \nu(l)$.*

Lemma 2.4. [16, Proposition 2.5] *Suppose $\varepsilon \in \{+,-\}$. Let $G = D_n^{\varepsilon}(q)$ be a finite simple group of Lie type over a field of characteristic p. Suppose that r and s are odd primes with $r,s \in \pi(G)\backslash\{p\}, k = e(r,q), l = e(s,q)$ and $1 \le \eta(k) \le \eta(l)$. Then r and s are non-adjacent if and only if $2\eta(k)+2\eta(l) > 2n-(1-\varepsilon(-1)^{k+l})$, l/k is not an odd natural number, and for $\varepsilon = +$, a chain of equalities like $n = l = 2\eta(l) = 2\eta(k) = 2k$ is not true.*

Lemma 2.5. [13, Propositions 1 and 2], [14, Theorem 1] *Let G be a finite group such that $t(G) \ge 3$ and $t(2,G) \ge 2$. Then the following assertions hold.*

(a) *There is a finite nonabelian simple group S such that $S \lesssim \overline{G} := G/K \lesssim \mathrm{Aut}\,(S)$ for the maximal normal solvable subgroup K of G.*
(b) *For every independent subset ρ of primes in $\pi(G)$ with $|\rho| \ge 3$ at most one prime in ρ divides the product $|K|\cdot|\overline{G}/S|$. In particular, $t(S) \ge t(G)-1$.*
(c) *One of the following statements holds:*
  (1) *$S \cong A_7$ or $L_2(q)$ for some odd prime power q and $t(S) = t(2,S) = 3$;*
  (2) *For every prime $r \in \pi(G)$ non-adjacent to 2 in $\Gamma(G)$ a Sylow r-subgroup of G is isomorphic to a Sylow r-subgroup of S. In particular, $t(2,S) \ge t(2,G)$.*

An immediate corollary arises from the above lemma as follows.

Corollary 2.1. *Let (G,S,K) be as in Lemma 2.5. Then the following statements hold.*

(a) *$S \lesssim G/K \lesssim \mathrm{Aut}\,(S)$. In particular, $\pi(S) \subseteq \pi(G)$ and $|S|\,\big|\,|G|$.*
(b) *If $S \ncong A_7$ and $S \ncong L_2(q)$, there always exists an independent subset $\rho(2,S)$ of $\pi(S)$ containing a fixed independent subset $\rho(2,G)$ of $\pi(G)$. For convenience we write $\rho(2,G) \subseteq \rho(2,S)$ to denote the above relation.*
(c) *$t(S) \ge t(G)-1$. Moreover, for every odd prime $r \in \pi(S)$, we have that $t(r,S) \ge t(r,G)-1$.*

In this paper, we will repeatedly use [15, Tables 2–9] and their corrections in [16]. For convenience we only display some of them here.

Lemma 2.6. [15, Tables 6 and 8], [16, Tables 6 and 8] *Let q be a power of a prime p and r_i be a Zsigmondy prime of q^i-1. Let G be a finite simple classical group over a field of order q. If $p \ne 2$, then the 2-independence number of G is as displayed in Table 2. Moreover, the independence number of G is as displayed in Table 3.*

Table 2. 2-independence numbers for finite simple classical groups with $p \neq 2$

G	Conditions	$t(2,G)$	$\rho(2,G)$
$A_{n-1}(q)$	$n=2, q \equiv 1 \pmod 4$	3	$\{2,p,r_2\}$
	$n=2, q \neq 3, q \equiv 3 \pmod 4$	3	$\{2,p,r_1\}$
	$n \geq 3$ and $n_2 < (q-1)_2$	2	$\{2,r_n\}$
	$n \geq 3$ and either $n_2 > (q-1)_2$ or $n_2 = (q-1)_2 = 2$	2	$\{2,r_{n-1}\}$
	$2 < n_2 = (q-1)_2$	3	$\{2,r_{n-1},r_n\}$
${}^2A_{n-1}(q)$	$n_2 > (q+1)_2$	2	$\{2,r_{2n-2}\}$
	$n_2 = 1$	2	$\{2,r_{2n}\}$
	$2 < n_2 < (q+1)_2$	2	$\{2,r_n\}$
	$n \geq 3, 2 = n_2 \leq (q+1)_2$	2	$\{2,r_{\frac{n}{2}}\}$
	$2 < n_2 = (q+1)_2$	3	$\{2,r_{2n-2},r_n\}$
$B_n(q)$ or $C_n(q)$	$n > 1$ is odd and $(q-1)_2 = 2$	2	$\{2,r_n\}$
	n is even or $(q-1)_2 > 2$	2	$\{2,r_{2n}\}$
$D_n(q)$	$n \equiv 0 \pmod 2, n \geq 4, q \equiv 3 \pmod 4$	2	$\{2,r_{n-1}\}$
	$n \equiv 0 \pmod 2, n \geq 4, q \equiv 1 \pmod 4$	2	$\{2,r_{2n-2}\}$
	$n \equiv 1 \pmod 2, n > 4, q \equiv 3 \pmod 4$	2	$\{2,r_n\}$
	$n \equiv 1 \pmod 2, n > 4, q \equiv 1 \pmod 8$	2	$\{2,r_{2n-2}\}$
	$n \equiv 1 \pmod 2, n > 4, q \equiv 5 \pmod 8$	3	$\{2,r_n,r_{2n-2}\}$
${}^2D_n(q)$	$n \equiv 0 \pmod 2, n \geq 4$	2	$\{2,r_{2n}\}$
	$n \equiv 1 \pmod 2, n \geq 4, q \equiv 1 \pmod 4$	2	$\{2,r_{2n}\}$
	$n \equiv 1 \pmod 2, n > 4, q \equiv 7 \pmod 8$	2	$\{2,r_{2n-2}\}$
	$n \equiv 1 \pmod 2, n > 4, q \equiv 3 \pmod 8$	3	$\{2,r_{2n-2},r_{2n}\}$

At the end of this section we quote some lemmas on noncommuting graph of a finite group.

Lemma 2.7. [6, Theorem 1] *Let M be a finite nonabelian simple group. If G is a finite group such that $\nabla(M) \cong \nabla(G)$, then $|M| = |G|$.*

Lemma 2.8. [7, Corollary 5] *Let S be a finite nonabelian simple group. If G is a finite group such that $\nabla(G) \cong \nabla(S)$, then $\Gamma(S) = \Gamma(G)$. In particular, the recognizability by prime graph of S implies the recognizability by noncommuting graph of S.*

By the above two lemmas, the following corollary follows immediately.

Corollary 2.2. *Let M be a finite nonabelian simple group. If G is a finite group such that $\nabla(M) \cong \nabla(G)$, then $|M| = |G|$ and $\Gamma(G) = \Gamma(M)$.*

3. On *AAM*'s conjecture for $D_n(3)$

In this section we will prove that AAM's conjecture is valid for $D_n(3)$ for some n. First let us state a lemma and its corollary which will be used often later.

Lemma 3.1. *Let $S \in \{A_{m-1}(q), {}^2A_{m-1}(q), B_m(q), C_m(q), D_m(q), {}^2D_m(q)\}$ be a simple classical group of Lie type over a field $GF(q)$, where q is a power of a prime p and m is a natural*

Table 3. Independence numbers for finite simple classical groups

G	Conditions	$t(G)$	$\rho(G)$
$A_{n-1}(q)$	$n=2, q>3$	3	$\{p, r_1, r_2\}$
	$n=3, (q-1)_3=3,\ q+1\neq 2^k$	4	$\{p, 3, r_2\neq 2, r_3\}$
	$n=3, (q-1)_3\neq 3,\ q+1\neq 2^k$	3	$\{p, r_2\neq 2, r_3\}$
	$n=3, (q-1)_3=3,\ q+1=2^k$	3	$\{p, 3, r_3\}$
	$n=3, (q-1)_3\neq 3,\ q+1=2^k$	2	$\{p, r_3\}$
	$n=4$	3	$\{p, r_{n-1}, r_n\}$
	$n=5,6, q=2$	3	$\{5, 7, 31\}$
	$7\leq q\leq 11, q=2$	$[\frac{n-1}{2}]$	$\{r_i \mid i\neq 6, [\frac{n}{2}]<i\leq n\}$
	$n>5$ and $q>2$, or $n>12$ and $q=2$	$[\frac{n+1}{2}]$	$\{r_i \mid [\frac{n}{2}]<i\leq n\}$
${}^2A_{n-1}(q)$	$n=3, q\neq 2, (q+1)_3=3$ and $q-1\neq 2^k$	4	$\{p, 3, r_1\neq 2, r_6\}$
	$n=3, (q+1)_3\neq 3,\ q-1\neq 2^k$	3	$\{p, r_1\neq 2, r_6\}$
	$n=3, (q+1)_3=3,\ q-1=2^k$	3	$\{p, 3, r_6\}$
	$n=3, (q+1)_3\neq 3,\ q-1=2^k$	2	$\{p, r_6\}$
	$n=4, q=2$	2	$\{2, 5\}$
	$n=4, q>2$	3	$\{p, r_4, r_6\}$
	$n=5, q=2$	3	$\{2, 5, 11\}$
	$n\geq 5$ and $(n,q)\neq(5,2)$	$[\frac{n+1}{2}]$	$\{r_{\frac{i}{2}} \mid [\frac{n}{2}]<i\leq n, i\equiv 2 \pmod 4\}$ $\cup\{r_{2i} \mid [\frac{n}{2}]<i\leq n, i\equiv 1 \pmod 2\}$ $\cup\{r_i \mid [\frac{n}{2}]<i\leq n, i\equiv 0 \pmod 4\}$
$B_n(q)$ or $C_n(q)$	$n=2, q>2$	2	$\{p, r_4\}$
	$n=3$ and $q=2$	2	$\{5, 7\}$
	$n=4$ and $q=2$	3	$\{5, 7, 17\}$
	$n=5$ and $q=2$	4	$\{7, 11, 17, 31\}$
	$n=6$ and $q=2$	5	$\{7, 11, 13, 17, 31\}$
	$n>2$ and $(n,q)\neq(3,2),(4,2),(5,2),(6,2)$	$[\frac{3n+5}{4}]$	$\{r_{2i} \mid [\frac{n+1}{2}]\leq i\leq n\}$ $\cup\{r_i \mid [\frac{n}{2}]<i\leq n, i\equiv 1 \pmod 2\}$
$D_n(q)$	$n=4$ and $q=2$	2	$\{5, 7\}$
	$n=5$ and $q=2$	4	$\{5, 7, 17, 31\}$
	$n=6$ and $q=2$	4	$\{7, 11, 17, 31\}$
	$n\geq 4, n\neq 3 \pmod 4$ $(n,q)\neq(4,2),(5,2),(6,2)$	$[\frac{3n+1}{4}]$	$\{r_{2i} \mid [\frac{n+1}{2}]\leq i<n\}$ $\cup\{r_i \mid [\frac{n}{2}]<i\leq n, i\equiv 1 \pmod 2\}$
	$n\equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\{r_{2i} \mid [\frac{n+1}{2}]\leq i<n\}$ $\cup\{r_i \mid [\frac{n}{2}]\leq i\leq n, i\equiv 1 \pmod 2\}$
${}^2D_n(q)$	$n=4$ and $q=2$	3	$\{5, 7, 17\}$
	$n=5$ and $q=2$	3	$\{7, 11, 17\}$
	$n=6$ and $q=2$	5	$\{7, 11, 13, 17, 31\}$
	$n=7$ and $q=2$	5	$\{11, 13, 17, 31, 43\}$
	$n\geq 4, n\neq 1 \pmod 4$, $(n,q)\neq(4,2),(6,2),(7,2)$	$[\frac{3n+4}{4}]$	$\{r_{2i} \mid [\frac{n}{2}]\leq i\leq n\}$ $\cup\{r_i \mid [\frac{n}{2}]<i<n, i\equiv 1 \pmod 2\}$
	$n>4, n\equiv 1 \pmod 4$, $(n,q)\neq(5,2)$	$[\frac{3n+4}{4}]$	$\{r_{2i} \mid [\frac{n}{2}]<i\leq n\}$ $\cup\{r_i \mid [\frac{n}{2}]<i<n, i\equiv 1 \pmod 2\}$

number. Let $r \in \pi(S) \setminus \{2,p\}$ and $k = e(r,q)$. We give some upper boundaries U and lower boundaries L for $t(r,S)$ in Tables 4 *and* 5.

Table 4. Boundaries of $t(r,S)$ of the simple classical group of Lie type (I)

S	Conditions	k	U	L
$A_{m-1}(q)$	$m \geq 5$ and $q > 2$, or $m \geq 12$ and $q = 2$	$2 \leq k \leq [\frac{m}{2}]$	k	k
$^2A_{m-1}(q)$	$m \geq 5$, $(m,q) \neq (5,2)$	$2 \leq \nu(k) \leq [\frac{m}{2}]$	$\nu(k)$	$\nu(k)$
$B_m(q)$ or $C_m(q)$	$m > 2, (m,q) \neq (3,2), (4,2), (5,2), (6,2)$	$\eta(k) < \frac{m+1}{2}$, k odd	$\frac{3k+3}{2}$	$\frac{3k-3}{2}$
$B_m(q)$ or $C_m(q)$	$m > 2, (m,q) \neq (3,2), (4,2), (5,2), (6,2)$	$\eta(k) < \frac{m+1}{2}$, k even	$\frac{k}{2} + [\frac{k}{4}] + 2$	$\frac{k}{2} + [\frac{k}{4}] - 1$
$D_m(q)$	$m \geq 4, (m,q) \neq (4,2), (5,2), (6,2)$	$\eta(k) < \frac{m+1}{2}$, k odd	$\frac{3k+3}{2}$	$\frac{3k-3}{2}$
$D_m(q)$	$m \geq 4, (m,q) \neq (4,2), (5,2), (6,2)$	$\eta(k) < \frac{m+1}{2}$, k even	$\frac{k}{2} + [\frac{k+2}{4}] + 1$	$\frac{k}{2} + [\frac{k+2}{4}] - 2$
$^2D_m(q)$	$m \geq 4, (m,q) \neq (4,2), (5,2), (6,2), (7,2)$	$\eta(k) \leq \frac{m}{2}$, k odd	$\frac{3k+3}{2}$	$\frac{3k-3}{2}$
$^2D_m(q)$	$m \geq 4, (m,q) \neq (4,2), (5,2), (6,2), (7,2)$	$\eta(k) \leq \frac{m}{2}$, k even	$\frac{k}{2} + [\frac{k-2}{4}] + 3$	$\frac{k}{2} + [\frac{k-2}{4}]$

Table 5. Boundaries of $t(r,S)$ of the simple classical group of Lie type (II)

S	Conditions	$t(r,S)$
$A_{m-1}(q)$; $m \geq 5$ and $q > 2$, or $m \geq 12$ and $q = 2$	$k > [\frac{m}{2}], k \geq 2$	$t(S) = [\frac{m+1}{2}]$
$^2A_{m-1}(q)$; $m \geq 5$ and $(m,q) \neq (5,2)$	$\nu(k) > [\frac{m}{2}], \nu(k) \geq 2$	$t(S) = [\frac{m+1}{2}]$
$B_m(q)$ or $C_m(q)$; $m > 2, (m,q) \neq (3,2), (4,2), (5,2), (6,2)$	$\eta(k) \geq \frac{m+1}{2}$	$t(S) = [\frac{3m+5}{4}]$
$D_m(q); m \geq 4, (m,q) \neq (4,2), (5,2), (6,2)$	$\eta(k) \geq \frac{m+1}{2}$	$t(S) = [\frac{3m+1}{4}]$ or $\frac{3m+3}{4}$
$^2D_m(q)$; $m \geq 4, (m,q) \neq (4,2), (5,2), (6,2), (7,2)$	$\eta(k) > \frac{m}{2}$	$t(S) = [\frac{3m+4}{4}]$

Proof. Let $s \in \pi(S) \setminus \{2,p\}$ and $l = e(s,q)$.

A. Let $S \cong {}^2A_{m-1}(q)$ such that $m \geq 5$ and $(m,q) \neq (5,2)$.

We note that if $\nu(k) > [m/2]$ and $k \geq 2$, then we can assume that $r \in \rho(S)$ by Table 3 and so $t(r,S) = t(S) = [(m+1)/2]$.

If $2 \leq \nu(k) \leq [m/2]$, then $\nu(k) < (m+2)/2$. Let $r \nsim s$ in $\Gamma(S)$. Therefore $\nu(l) > m - \nu(k)$ and $\nu(k) \nmid \nu(l)$ by Lemma 2.3. Considering the order of S, we have that $\nu(l) \leq m$ and thus

$\nu(l) \in I := [m-\nu(k)+1, m]$. Because there are only $\nu(k)$ integers in the interval I, it is obvious that $\nu(k)$ divides exactly one element in I. So there exist at most $\nu(k)-1$ elements of $\pi(S)$ non-adjacent to r. Therefore $t(r,S) \leq (\nu(k)-1)+1 = \nu(k)$.

On the other hand, we suppose that

$$X := \{s \in \pi(S) | 2 \leq \nu(e(s,q)) \in I, \nu(k) \nmid \nu(e(s,q))\}$$

such that $s, s' \in X$ and $s \neq s'$ imply that $e(s,q) \neq e(s',q)$. Let $s, s' \in X$ and $s \neq s'$. So $e(s,q) \neq e(s',q)$. Since $m \geq 5$, it follows that $\nu(e(s,q)), \nu(e(s',q)) \geq m-\nu(k)+1 > m-(m+2)/2+1 = m/2 \geq 2$ and so $\nu(e(s,q))+\nu(e(s',q)) > m$. Assume that $e(s,q) = m-\nu(k)+j$ and $e(s',q) = m-\nu(k)+j'$, where $1 \leq j < j' \leq \nu(k) \leq m$. If $(m-\nu(k)+j) \big| (m-\nu(k)+j')$, there exists an integer $t \geq 2$ such that $m-\nu(k)+j' = (m-\nu(k)+j)t \geq (m-\nu(k)+1)t > m/2 \cdot 2 = m$, a contradiction by the choice of $e(s',q)$. Thus $\nu(e(s,q)) \nmid \nu(e(s',q))$. Hence $s \nsim s'$ in $\Gamma(S)$ by Lemma 2.3. Furthermore, it is clear that $r \notin X$ and so $r \nsim s$ in $\Gamma(S)$ for each $s \in X$ by the same lemma. Therefore $X \cup \{r\}$ is an independent subset and thus $t(r,S) \geq (\nu(k)-1)+1 = \nu(k)$. Consequently, $U = L = \nu(k)$.

B. Let $S \cong D_m(q)$ such that $m \geq 4$ and $(m,q) \neq (4,2), (5,2), (6,2)$.

We note that if $\eta(k) \geq m+1/2$, then we can assume that $r \in \rho(S)$ by Table 3 and so $t(r,S) = t(S) = [(3m+1)/4]$ or $(3m+3)/4$.

Assume that $\eta(k) < (m+1)/2$. Let $r \nsim s$ in $\Gamma(S)$. Therefore $2\eta(l)+2\eta(k) > 2m-(1-(-1)^{k+l})$ such that one of the conditions (a) and (b) in Lemma 2.4 holds. Thus $\eta(l) > m-(1-(-1)^{k+l})/2-\eta(k)$. On the other hand, since $|S| = (q^{m(m-1)}(q^m-1)\prod_{i=1}^{m-1}(q^{2i}-1))/((4, q^m-1))$ by Table 1, we have that $\eta(l) \leq m$. Thus $\eta(l) \in I := [m-(1-(-1)^{k+l})/2-\eta(k)+1, m]$.

Case 1. Let k be an even number and l an odd number. Then $l \in I = [m-k/2, m]$. Suppose

$$X := \{s \in \pi(S) | 1 \leq e(s,q) \in I, e(s,q) \textit{ is odd}\}$$

such that $s, s' \in X$ and $s \neq s'$ imply that $e(s,q) \neq e(s',q)$.

Let $s, s' \in X$ and $s \neq s'$. So $e(s,q) \neq e(s',q)$. Since $m \geq 4$, it follows that $\eta(e(s,q)), \eta(e(s',q)) \geq m-\eta(k) > m-(m+1)/2 = (m-1)/2 \geq 1$. Thus $2\eta(e(s,q))+2\eta(e(s',q)) \geq 2(m-k/2)+2(m-k/2+2) = 4m-2k+4 > 4m-2(m+1)+4 = 2m+2 > 2m$. Let $e(s,q) = m-k/2+j$ and $e(s',q) = m-k/2+j'$, where $0 \leq j < j' \leq k/2 \leq m$. If $(e(s',q))/(e(s,q))$ is an odd number, then there exists an integer $t \geq 3$ such that $m-k/2+j' = (m-k/2+j)t \geq (m-k/2)t \geq (m-1)/2 \cdot 3 > m$, a contradiction by the choice of $e(s',q)$. Thus $(e(s',q))/(e(s,q))$ is not an odd number. Hence $s \nsim s'$ in $\Gamma(S)$ by Lemma 2.4. Furthermore, it is clear that $r \notin X$ since k is even and so $r \nsim s$ in $\Gamma(S)$ for each $s \in X$ by the same lemma. Therefore $X \cup \{r\}$ is an independent subset such that

$$|X| = \begin{cases} [\frac{k+2}{4}]+1, & \text{if } m \text{ is odd and } \frac{k}{2} \text{ is even;} \\ [\frac{k+2}{4}], & \text{otherwise.} \end{cases}$$

Case 2. Let k and l be two even numbers. Then $l/2 \in I := [m-k/2+1, m]$. We have that $l \in I' := [2m-k+2, 2(m-1)]$ considering the order of S. Suppose

$$X' := \{s \in \pi(S) | 1 \leq e(s,q) \in I', e(s,q) \textit{ is even}\}$$

such that the following statements hold:

(1) $s, s' \in X$ and $s \neq s'$ imply that $e(s,q) \neq e(s',q)$.

(2) $(e(s,q))/k$ is not an odd integer.

Let $s, s' \in X'$ and $s \neq s'$. So $e(s,q) \neq e(s',q)$. Since $m \geq 4$, it follows that $\eta(e(s,q)), \eta(e(s',q)) > m - \eta(k) > m - (m+1)/2 = (m-1)/2 > 1$. Thus $2\eta(e(s,q)) + 2\eta(e(s',q)) \geq 2(m-k/2+1) + 2(m-k/2+2) = 4m - 2k + 6 > 4m - 2(m+1) + 6 > 2m$. Let $e(s,q) = 2m - k + j$ and $e(s',q) = 2m - k + j'$, where $2 \leq j < j' \leq k-2$. If $(e(s',q))/(e(s,q))$ is an odd number, then there exists an odd integer $t \geq 3$ such that $2m - k + j' = (2m - k + j)t \geq (2m - (m+1) + 2)t \geq 3(m+1) > 2(m-1)$, a contradiction by the choice of $e(s',q)$. Hence $(e(s,q))/(e(s',q))$ is not an odd number. By Lemma 2.4, $s \nsim s'$ in $\Gamma(S)$. Therefore X' is an independent subset such that

$$|X'| = \begin{cases} \frac{k-2}{2}, & \text{if all odd multiples of } k \text{ are not in } I'; \\ \frac{k-2}{2} - 1, & \text{if an odd multiple of } k \text{ is in } I'. \end{cases}$$

Note that $s \nsim s'$ in $\Gamma(S)$ for each $s \in X$ and each $s' \in X'$ by Lemma 2.4. From Cases 1 and 2, we can conclude that if $\eta(k) < (m+1)/2$ and k is even, then

$$U = \max\{|X \cup \{r\} \cup X'|\} = \left(\left[\frac{k+2}{4}\right] + 1\right) + \frac{k-2}{2} + 1 = \left[\frac{k+2}{4}\right] + \frac{k}{2} + 1$$

and

$$L = \min\{|X \cup X'|\} = \left[\frac{k+2}{4}\right] + \left(\frac{k-2}{2} - 1\right) = \left[\frac{k+2}{4}\right] + \frac{k}{2} - 2.$$

Case 3. Let k be an odd number and l an even number. Then $l/2 \in I := [m-k, m]$. We have that $l \in I' := [2m-2k, 2(m-1)]$ considering the order of S. Suppose

$$X' := \{s \in \pi(S) | 1 \leq e(s,q) \in I', e(s,q) \textit{ is even }\}$$

such that the following statements hold:

(1) $s, s' \in X$ and $s \neq s'$ imply that $e(s,q) \neq e(s',q)$.

(2) the chain of equalities $m = e(s,q) = 2k$ is not true.

Let $s, s' \in X'$ and $s \neq s'$. So $e(s,q) \neq e(s',q)$. Since $m \geq 4$, it follows that $\eta(e(s,q)), \eta(e(s',q)) \geq m - \eta(k) > m - (m+1)/2 = (m-1)/2 \geq 1$. Thus $2\eta(e(s,q)) + 2\eta(e(s',q)) \geq 2(m-k) + 2(m-k+1) = 4m - 4k + 2 > 4m - 4 \cdot (m+1)/2 + 2 = 2m$. Let $e(s,q) = 2m - 2k + j$ and $e(s',q) = 2m - 2k + j'$, where $2 \leq j < j' \leq 2k-2$. If $(e(s',q))/(e(s,q))$ is an odd number, then there exists an odd integer $t \geq 3$ such that $2m - 2k + j' = (2m - 2k + j)t \geq (2m - (m+1)/2 \cdot 2)t \geq 3(m-1) > 2(m-1)$, a contradiction by the choice of $e(s',q)$. Hence $(e(s',q))/(e(s,q))$ is not an odd number. By Lemma 2.4, $s \nsim s'$ in $\Gamma(S)$. Furthermore, it is clear that $r \notin X'$ since k is odd and so $r \nsim s$ in $\Gamma(S)$ for each $s \in X'$ by the same lemma. Therefore $X' \cup \{r\}$ is an independent subset such that

$$|X'| = k.$$

Case 4. Let k and l be two odd numbers. Then $l \in I := [m-k+1, m]$. Suppose

$$X := \{s \in \pi(S) | 1 \leq e(s,q) \in I, e(s,q) \textit{ is odd}\}$$

such that the following statements hold:

(1) $s, s' \in X$ and $s \neq s'$ imply that $e(s,q) \neq e(s',q)$.

(2) $e(s,q)/k$ is not an odd integer.

Let $s, s' \in X$ and $s \neq s'$. So $e(s,q) \neq e(s',q)$. Since $m \geq 4$, it follows that $\eta(e(s,q)), \eta(e(s',q)) \geq m-k+1 > m-(m+1)/2+1 = (m+1)/2 \geq 1$. Thus $2\eta(e(s,q)) + 2\eta(e(s',q)) \geq 2(m-k+1)+2(m-k+2) = 4m-4k+6 > 4m-4\cdot(m+1)/2+6 = 2m+4 > 2m$. Let $e(s,q) = m-k+j$ and $e(s',q) = m-k+j'$, where $1 \leq j < j' \leq k$. If $(e(s,q))/(e(s',q))$ is an odd number, then there exists an odd integer $t \geq 3$ such that $m-k+j' = (m-k+j)t > (m-(m+1)/2+1)t \geq (m+1)/2\cdot 3 > m$, a contradiction by the choice of $e(s',q)$. Hence $(e(s',q))/(e(s,q))$ is not an odd number. By Lemma 2.4, $s \nsim s'$ in $\Gamma(S)$. Therefore X is an independent subset such that

$$|X| = \begin{cases} \frac{k-1}{2} \ or \ \frac{k+1}{2}, & \text{if all odd multiples of } k \text{ are not in } I; \\ \frac{k-1}{2} - 1 \ or \ \frac{k+1}{2} - 1, & \text{if an odd multiple of } k \text{ is in } I. \end{cases}$$

Note that $s \nsim s'$ in $\Gamma(S)$ for each $s \in X$ and each $s' \in X'$ by Lemma 2.4. From Cases 3 and 4, we can conclude that if $\eta(k) < (m+1)/2$ and k is odd, then

$$U = \max\{|X \cup \{r\} \cup X'|\} = \frac{k+1}{2} + k + 1 = \frac{3k+3}{2}$$

and

$$L = \min\{|X \cup X'|\} = (\frac{k-1}{2} - 1) + k = \frac{3k-3}{2}.$$

For convenience we omit the proof of other cases since they are similar. See Propositions 2.1 − 2.2 in [15] and Proposition 2.4 in [16] if necessary. The lemma is proved. ∎

Corollary 3.1. *Let $S \in \{A_{m-1}(q), {}^2A_{m-1}(q), B_m(q), C_m(q), D_m(q), {}^2D_m(q)\}$ be a simple group of Lie type over a field $GF(q)$, where q is a power of a prime p and m is a positive integer. If $r \in \pi(S) \setminus \{2, p\}$ and $k = e(r,q)$, the following statements hold:*

(a) *If $k > 1$ is odd, then $t(r,S) \leq 2k$.*

(b) *If k is even, then $t(r,S) \leq \max\{[(k-2)/4] + k/2 + 3, k\}$.*

Proof. If (m,q) satisfies the conditions of Lemma 3.1, it follows from Lemma 3.1. Otherwise, it follows from Table 3 by a one-by-one check. ∎

Let $n \geq 5$ be an odd integer. Without loss of generality, by Tables 2 and 3, we may suppose that

$$\rho(2, D_n(3)) = \{2, r_n\}, t(2, D_n(3)) = 2 \quad \text{and} \quad t(D_n(3)) = \left[\frac{3n+1}{4}\right] \text{ or } \frac{3n+3}{4} \geq 4.$$

Before considering *AAM*'s conjecture for $D_n(3)$ we prove the following stronger result.

Theorem 3.1. *Let $n \geq 9$ is an odd natural number. If G is a finite group such that $\Gamma(G) = \Gamma(D_n(3))$, then $D_n(3) \lesssim G/K \lesssim Aut(D_n(3))$, where K is the maximal normal solvable subgroup of G. That is, $D_n(3)$ is quasirecognizable by its prime graph.*

Proof. Since $\Gamma(G) = \Gamma(D_n(3))$, it follows that $\pi(G) = \pi(D_n(3))$, $t(G) = t(D_n(3)) \geq 3$ and $t(2,G) = t(2,D_n(3)) = 2$. By Lemma 2.5(a), there exists a finite nonabelian simple group S such that $S \lesssim \overline{G} = G/K \lesssim Aut(S)$ for the maximal normal solvable subgroup K of G. In addition, by Corollary 2.1, we have that

(A) $S \lesssim G/K \lesssim Aut(S)$. In particular, $\pi(S) \subseteq \pi(G)$ and $|S| \mid |G|$;

(B) If $S \ncong A_7$ and $S \ncong L_2(q)$, then $\rho(2,G) \subseteq \rho(2,S)$;

(C) $t(S) \geq t(G) - 1$. Moreover, for every odd prime $r \in \pi(S)$, we have that $t(r,S) \geq t(r,G) - 1$.

In the sequel, we denote by r_i a Zsigmondy prime of 3^i-1. If S is a simple group of Lie type over a field $GF(q)$, where $q=p^{\alpha}$ is power of a prime p, then u_i denotes a Zsigmondy prime of q^i-1. According to the classification of finite simple groups, we consider each possibility for S.

Step 1. We prove that the simple group S is not isomorphic to an alternating group.

Let $S\cong A_m$. Since $n\geq 9$, it follows from (C) that $t(S)\geq t(G)-1\geq[(3\cdot 9+1)/4]-1=6$. Therefore $t(S)=t(A_m)=|\tau(m)|+1\geq 6$ or $t(S)=t(A_m)=|\tau(m)|\geq 6$ from Table 3 in [15], where $\tau(m):=\{s|s \textit{ is a prime such that } m/2<s\leq m\}$. By the definition of the function $\tau(m)$, we have that $m\geq 31$ and so $31\in\pi(S)$.

Let $s\in\pi(S)\backslash\{2\}$. Notice the fact that $s\nsim 31$ in $\Gamma(S)$ if and only if $s+31>m$. Therefore $s\in[m-30,m]$. It is obvious that there are at least 20 elements in $[m-30,m]$ which are divisible by 2 or 3 since $[31/2]+[31/3]-[31/6]=20$. Therefore there are at most $31-20=11$ odd prime numbers in $[m-30,m]$. If $2\sim 31$ in $\Gamma(S)$, we have that $t(31,S)\leq 12$. If $2\nsim 31$ in $\Gamma(S)$, then $31+4>m$ and so $m<35$. Thus there are at most 9 odd prime numbers in $[m-30,m]$ and we have that $t(31,S)\leq 11$. Hence, in both cases, $t(31,S)\leq 12$. On the other hand, it is obvious that $e(31,3)=30$. By Tables 4 and 5, we conclude that $t(31,G)\geq[(30+2)/4]+[30/2]-2=21$ if $n>29$; and $t(31,G)\geq[(3n+1)/4]\geq[(3\cdot 19+1)/4]=14$ if $19\leq n\leq 29$. In both cases, by (C), we have that $13\leq t(31,G)-1\leq t(31,S)\leq 12$, a contradiction. Hence $9\leq n\leq 18$. Since n is odd, it follows that $9\leq n\leq 17$. By Table 1, G is a finite simple $\{2,3,5,7,11,13,17,19,23,29,31,37,41,61,67,73,193,271,547,661,1093,1181,757,1871,16493,21523361,398581,797161,4561,6481,34511,3851\}$-group. Clearly $43\notin\pi(G)$, and so $31\leq m\leq 42$. Let $\rho=\{r_9,r_{14},r_{16}\}$ if $n=9,10$ and $\rho=\{r_{11},r_{18},r_{20}\}$ if $11\leq n\leq 17$, respectively. Thus ρ is an independent subset of $\pi(G)$ by Table 3. By Lemma 2.5(b), at most one number of ρ divides the product $|K||\overline{G}/S|=|G|/|S|$ and so at least two numbers in it divide $|S|$, which is impossible since $31\leq m\leq 42$.

Step 2. We prove that the simple group S is not isomorphic to a simple classical group over a field of characteristic $p\neq 3$.

Because $n\geq 9$ is odd, we can suppose that $B:=\{r_{2(n-1)},r_{2(n-2)},r_{2(n-3)},r_{2(n-4)},r_n\}$ is an independent subset in $\Gamma(G)=\Gamma(D_n(3))$ by Table 3. Therefore $|B\cap\pi(S)|\geq 4$ by Lemma 2.5(b). Notice that, by Table 4 in [15], $t(p,S)\leq 3$ for each simple classical group except exactly in the following cases:

(a) $S\cong A_2(q)$ such that $(q-1)_3=3$ and $q+1\neq 2^t$;
(b) $S\cong {}^2A_2(q)$ such that $q\neq 2,(q+1)_3=3$ and $q-1\neq 2^t$;
(c) $S\cong {}^2D_m(q)$ such that $m\equiv 0 \pmod 2$, $m\geq 4$ and $(m,q)\neq(4,2)$.

It follows that $p\notin B$ if S is not isomorphic to these exceptions. Otherwise, $t(p,S)\geq|B\cap\pi(S)|\geq 4$, a contradiction. Therefore p is adjacent to at least two elements of B in $\Gamma(S)$. (Otherwise, we can also deduce that $t(p,S)\geq 4$, a contradiction.) Hence p is also adjacent to them in $\Gamma(G)$ since $\pi(S)\subseteq\pi(G)$. Without loss of generality we may suppose $p\sim r_{2(n-1)}$ and $p\sim r_{2(n-4)}$ in $\Gamma(G)$. Let $l=e(p,3)$. Since $n\geq 9$ is odd, it follows that the equalities $n=l=2\eta(l)=2\eta(k)=2k$ can not be true, where $k\in\{2(n-1),2(n-2),2(n-3),2(n-4),n\}$.

Moreover, since $p \notin B$, we have that $l \notin \{2(n-1), 2(n-2), 2(n-3), 2(n-4), n\}$. Thus, by Lemma 2.4, we have that

(1) if l is even, then
  (a) $2\eta(l) + 2(n-1) \leq 2n$, or $(2(n-1))/l$ is an odd integer; and
  (b) $2\eta(l) + 2(n-4) \leq 2n$, or $2(n-4)/l$ is an odd integer;

or

(2) if l is odd, then
  (c) $2\eta(l) + 2(n-1) \leq 2n-2$; and
  (d) $2\eta(l) + 2(n-4) \leq 2n-2$.

Thus in each case we conclude that $\eta(l) \leq 4$ and so $l \in \{1,2,3,4,6,8\}$. Therefore $p \in \{2,5,7,13,41\}$.

Case 1. Let $S \cong A^{\varepsilon}_{m-1}(q)$, where $q = p^{\alpha}$.

Let $n \geq 9$. Since $t(S) \geq t(G) - 1 \geq [(3 \cdot 9 + 1)/4] - 1 = 6$ by (C), it follows that $6 \leq t(S) \in \{[(m+1)/2], [(m-1)/2]\}$ by Table 3. By an easy computation we get that $m \geq 11$ and so the exceptional cases (a) and (b) are ruled out.

Let $p = 2$. Clearly $e(31,2) = 5$ and $31 \in \pi(S)$ if $m \geq 11$ by Table 1. Therefore $31 \in \pi(G)$ by (A). Since $e(31,3) = 30$ and $\pi(G) = \pi(D_n(3))$, it follows that $n \geq 16$ by Table 1. By (C), we know that $t(31,G) - 1 \leq t(31,S)$. If $16 \leq n \leq 29$, then $\eta(30) = 15 \geq (n+1)/2$ and so $t(31,G) \geq [(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $11 = [(3 \cdot 16 + 1)/4] - 1 \leq [(3n+1)/4] - 1 \leq t(31,G) - 1 \leq t(31,S) \leq 2 \cdot 5 = 10$, which is a contradiction. Therefore $n > 29$. Similarly, we have that $20 = ([(30+2)/4] + 30/2 - 2) - 1 \leq t(31,G) - 1 \leq t(31,S) \leq 10$ by Table 4, a contradiction.

Let $p = 5$. Clearly $e(521,5) = 10$ and $521 \in \pi(S)$ if $m \geq 11$ by Table 1. Therefore $521 \in \pi(G)$. Since $e(521,3) = 520$, it follows that $n \geq 261$ by Table 1. By (C), we know that $t(521,G) - 1 \leq t(521,S)$. If $261 \leq n \leq 519$, then $\eta(520) = 260 \geq (n+1)/2$ and so $t(521,G) \geq [(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $195 = [(3 \cdot 261 + 1)/4] - 1 \leq [(3n+1)/4] - 1 \leq t(521,G) - 1 \leq t(521,S) \leq max\{[(10-2)/4] + 10/2 + 3, 10\} = 10$, which is a contradiction. Therefore $n > 519$. Similarly, we have that $387 = (520/2 + [(520+2)/4] - 2) - 1 \leq t(521,G) - 1 \leq t(521,S) \leq 10$ by Table 4, a contradiction.

Let $p = 7$. Clearly $e(191,7) = 10$ and $191 \in \pi(S)$ if $m \geq 11$ by Table 1. Therefore $191 \in \pi(G)$. Since $e(191,3) = 95$, it follows that $n \geq 95$ by Table 1. By (C), we know that $t(191,G) - 1 \leq t(191,S)$. If $95 \leq n \leq 189$, then $\eta(95) = 95 \geq (n+1)/2$ and so $t(191,G) \geq [(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $70 = [(3 \cdot 95 + 1)/4] - 1 \leq [(3n+1)/4] - 1 \leq t(191,G) - 1 \leq t(191,S) \leq max\{[(10-2)/4] + 10/2 + 3, 10\} = 10$, which is a contradiction. Therefore $n > 189$. Similarly, we have that $140 = (3 \cdot 95 - 3)/2 - 1 \leq t(191,G) - 1 \leq t(191,S) \leq 10$ by Table 4, a contradiction.

Let $p = 13$. Clearly $e(2411,13) = 10$ and $2411 \in \pi(S)$ if $m \geq 11$ by Table 1. Therefore $2411 \in \pi(G)$. Since $e(2411,3) = 1205$, it follows that $n \geq 1205$ by Table 1. By (C), we know that $t(2411,G) - 1 \leq t(2411,S)$. If $1205 \leq n \leq 2409$, then $\eta(1205) = 1205 \geq (n+1)/2$ and so $t(2411,G) \geq [(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $903 = [(3 \cdot 1205 + 1)/4] - 1 \leq [(3n+1)/4] - 1 \leq t(2411,G) - 1 \leq t(2411,S) \leq max\{[(10-2)/4] +$

$10/2+3,10\} = 10$, which is a contradiction. Therefore $n > 2409$. Similarly, we have $1805 = (3\cdot 1205-3)/2-1 \le t(2411,G)-1 \le t(2411,S) \le 10$ by Table 4, a contradiction.

Let $p = 41$. Clearly $e(4111,41) = 10$ and $4111 \in \pi(S)$ if $m \ge 11$ by Table 1. Therefore $4111 \in \pi(G)$. Since $e(4111,3) = 822$, it follows that $n \ge 412$ by Table 1. By (C), we know that $t(4111,G)-1 \le t(4111,S)$. If $412 \le n \le 821$, then $\eta(822) = 411 \ge (n+1)/2$ and so $t(4111,G) \ge [(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $308 = [(3\cdot 412+1)/4]-1 \le [(3n+1)/4]-1 \le t(4111,G)-1 \le t(4111,S) \le max\{[(10-2)/4]+10/2+3,10\} = 10$, which is a contradiction. Therefore $n > 821$. Similarly, we have $614 = (822/2+[(822+2)/4]-2)-1 \le t(4111,G)-1 \le t(4111,S) \le 10$ by Table 4, a contradiction.

Case 2. Similarly we can rule out the following cases: $S \cong B_m(q), C_m(q)$ or $D_m(q)$, where $q = p^\alpha$.

Thus we must focus our attention on the exceptional case (c).

Case 3. Let $S \cong {}^2D_m(q)$, where $q = p^\alpha$.

Let $n \ge 11$. Since $t(S) \ge t(G)-1 \ge [(3\cdot 11+1)/4]-1 = 7$, it follows that $t(S) = [(3m+4)/4] \ge 7$. By an easy computation we get that $m \ge 8$.

Since $n \ge 11$ is odd, we can suppose that $B' = \{r_{2(n-1)}, r_{2(n-2)}, r_{2(n-3)}, r_{2(n-4)}, r_{2(n-5)}, r_n\}$ is an independent set in $\Gamma(G)$ by Table 3. Therefore $|B\cap\pi(S)| \ge 5$ by Lemma 2.5(b). Notice that, by [15, Table 4], $t(p,S) \le 4$, which implies that $p \notin B$. Otherwise, $t(p,S) \ge |B\cap\pi(S)| \ge 5$, a contradiction. Therefore p is adjacent to at least two elements of B in $\Gamma(S)$. (Otherwise, we can also deduce that $t(p,S) \ge 5$, a contradiction.) Hence p is also adjacent to them in $\Gamma(G)$. Without loss of generality, we may suppose $p \sim r_{2(n-1)}$ and $p \sim r_{2(n-5)}$ in $\Gamma(G)$. Let $l = e(p,3)$. Since $n \ge 9$ is odd, it follows that the equalities $n = l = 2\eta(l) = 2\eta(k) = 2k$ can not be true, where $k \in \{2(n-1), 2(n-2), 2(n-3), 2(n-4), 2(n-5), n\}$. Moreover, since $p \notin B$, it follows that $l \notin \{2(n-1), 2(n-2), 2(n-3), 2(n-4), 2(n-5), n\}$. Thus, by Lemma 2.4, we have that

(1) if l is even, then
  (a) $2\eta(l)+2(n-1) \le 2n-2$, or $(2(n-1))/l$ is an odd integer; and
  (b) $2\eta(l)+2(n-5) \le 2n-2$, or $(2(n-5))/l$ is an odd integer;

or

(2) if l is odd, then
  (c) $2\eta(l)+2(n-1) \le 2n$; and
  (d) $2\eta(l)+2(n-5) \le 2n$.

Thus in each case we conclude that $\eta(l) \le 5$ and so $l \in \{1,2,3,4,5,6,8,10\}$. Therefore $p \in \{2,5,7,11,13,41,61\}$.

Let $p = 2$. Clearly $e(31,2) = 5$ and $31 \in \pi(S)$ if $m \ge 8$ by Table 1. Therefore $31 \in \pi(G)$. Since $e(31,3) = 30$, it follows that $n \ge 16$ by Table 1. By (C), we know that $t(31,G)-1 \le t(31,S)$. If $16 \le n \le 29$, then since $\eta(30) = 15 \ge (n+1)/2$ and so $t(31,G) \ge [(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $11 = [(3\cdot 16+1)/4]-1 \le [(3n+1)/4]-1 \le t(31,G)-1 \le t(31,S) \le 2\cdot 5$, which is a contradiction. Therefore $n > 29$. Similarly, we have that $20 = (30/2+[(30+2)/4]-2)-1 \le t(31,G)-1 \le t(31,S) \le 2\cdot 5$ by Table 4, a contradiction.

Let $p=5$. Clearly $e(521,5)=10$ and $521\in\pi(S)$ if $m\ge 8$ by Table 1. Therefore $521\in\pi(G)$. Since $e(521,3)=520$, it follows that $n\ge 261$ by Table 1. By (C), we know that $t(521,G)-1\le t(521,S)$. If $261\le n\le 519$, then $\eta(520)=260\ge(n+1)/2$ and so $t(521,G)\ge[(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $195 = [(3\cdot 261+1)/4]-1\le[(3n+1)/4]-1\le t(521,G)-1\le t(521,S)\le \max\{[(10-2)/4]+10/2+3,10\}=10$, which is a contradiction. Therefore $n>519$. Similarly, we have that $388=(520/2+[(520+2)/4]-2)-1\le t(521,G)-1\le t(521,S)\le 10$ by Table 4, a contradiction.

Let $p=7$. Clearly $e(191,7)=10$ and $191\in\pi(S)$ if $m\ge 8$ by Table 1. Therefore $191\in\pi(G)$. Since $e(191,3)=95$, it follows that $n\ge 95$ by Table 1. By (C), we know that $t(191,G)-1\le t(191,S)$. If $95\le n\le 189$, then $\eta(95)=95\ge(n+1)/2$ and so $t(191,G)\ge[(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $70=[(3\cdot 95+1)/4]-1\le[(3n+1)/4]-1\le t(191,G)-1\le t(191,S)\le\max\{[(10-2)/4]+10/2+3,10\}=10$, which is a contradiction. Therefore $n>189$. Similarly, we have that $140=(3\cdot 95-3)/2-1\le t(191,G)-1\le t(191,S)\le 10$ by Table 4, a contradiction.

Let $p=11$. Clearly $e(3221,11)=5$ and $3221\in\pi(S)$ if $m\ge 8$ by Table 1. Therefore $3221\in\pi(G)$. Since $e(3221,3)=644$, it follows that $n\ge 323$ by Table 1. By (C), we know that $t(3221,G)-1\le t(3221,S)$. If $323\le n\le 643$, then $\eta(644)=322\ge(n+1)/2$ and so $t(3221,G)\ge[(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $240=[(3\cdot 323+1)/4]-1\le[(3n+1)/4]-1\le t(3221,G)-1\le t(3221,S)\le 2\cdot 5$, which is a contradiction. Therefore $n>643$. Similarly, we have that $480=(644/2+[(644+2)/4]-2)-1\le t(3221,G)-1\le t(3221,S)\le 10$ by Table 4, a contradiction.

Let $p=13$. Clearly $e(2411,13)=10$ and $2411\in\pi(S)$ if $m\ge 8$ by Table 1. Therefore $2411\in\pi(G)$. Since $e(2411,3)=1205$, it follows that $n\ge 1205$ by Table 1. By (C), we know that $t(2411,G)-1\le t(2411,S)$. If $1205\le n\le 2409$, then $\eta(1205)=1205\ge(n+1)/2$ and so $t(2411,G)\ge[(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $903=[(3\cdot 1205+1)/4]-1\le[(3n+1)/4]-1\le t(2411,G)-1\le t(2411,S)\le\max\{[(10-2)/4]+10/2+3,10\}=10$, which is a contradiction. Therefore $n>2409$. Similarly, we have that $1805=(3\cdot 1205-3)/2-1\le t(2411,G)-1\le t(2411,S)\le 10$ by Table 4, a contradiction.

Let $p=41$. Clearly $e(4111,41)=10$ and $4111\in\pi(S)$ if $m\ge 8$ by Table 1. Therefore $4111\in\pi(G)$. Since $e(4111,3)=822$, it follows that $n\ge 412$ by Table 1. By (C), we know that $t(4111,G)-1\le t(4111,S)$. If $412\le n\le 821$, then $\eta(822)=411\ge(n+1)/2$ and so $t(4111,G)\ge[(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $308=[(3\cdot 412+1)/4]-1\le[(3n+1)/4]-1\le t(4111,G)-1\le t(4111,S)\le\max\{[(10-2)/4]+10/2+3,10\}=10$, which is a contradiction. Therefore $n>821$. Similarly, we have that $614=(822/2+[(822+2)/4]-2)-1\le t(4111,G)-1\le t(4111,S)\le 10$ by Table 4, a contradiction.

Let $p=61$. Clearly $e(131,61)=5$ and $131\in\pi(S)$ if $m\ge 8$ by Table 1. Therefore $131\in\pi(G)$. Since $e(131,3)=65$, it follows that $n\ge 65$ by Table 1. By (C), we know that $t(131,G)-1\le t(131,S)$. If $65\le n\le 129$, then $\eta(65)=65\ge(n+1)/2$ and so $t(131,G)\ge[(3n+1)/4]$ by Table 5. Now by Corollary 3.1, we have that $48=[(3\cdot 65+1)/4]-1\le[(3n+1)/4]-1\le t(131,G)-1\le t(3221,S)\le 2\cdot 5$, which is a contradiction. Therefore

$n > 129$. Similarly, we have that $95 = (3\cdot 65-3)/2-1 \leq t(131,G)-1 \leq t(131,S) \leq 10$ by Table 4, a contradiction.

Let $n = 9$. Therefore $t(G) = [(3\cdot 9+1)/4] = 7$ and so $m \geq 7$. Thus $(p^{10}-1)\big||S|$ by Table 1. On the other hand, by (A), we have that $3 \neq p \in \pi(S) \subseteq \pi(G) = \pi(D_9(3)) = \{2,3,5,7,11,13,17,41,61,73,193,547,757,1093\}$ by Table 1. However, for each previous prime p, $p^{10}-1$ has a prime divisor which is not in $\pi(G)$, a contradiction.

Step 3. We prove that the simple group S is not isomorphic to a simple exceptional group of Lie type.

Let $n \geq 9$. Since $t(S) \geq t(G)-1 \geq [(3\cdot 9+1)/4]-1 = 6$, S is isomorphic to $E_7(q)$ or $E_8(q)$ by [15, Table 9].

Case 1. Let $S \cong E_7(q)$, where $q = p^{\alpha}$.

Since $t(S) = 8$ by Table 9 in [15], it follows that $[(3n+1)/4] \leq t(G) \leq t(S)+1 = 9$ and so $9 \leq n \leq 12$. By our assumption, $n = 9$ or 11.

Let $n = 9$. It is obvious that $(p^{12}-1)\big||S|$ by Table 1. On the other hand, by (A), we have that $p \in \pi(S) \subseteq \pi(G) = \pi(D_9(3)) = \{2,3,5,7,11,13,17,41,61,73,193,547,757,1093\}$. However, for each previous prime $p \neq 2,3$, $p^{12}-1$ has a prime divisor which is not in $\pi(G)$, a contradiction. Thus $p = 2$ or 3. Suppose $p = 2$. Since $q(q^{18}-1)\big||S|$, it follows that $19\big||S|$. Then $19 \in \pi(E_7(q)) \subseteq \pi(G)$, a contradiction. Hence $p = 3$. However, $19 \in \pi(3^{18}-1) \subseteq \pi(E_7(3^{\alpha})) \subseteq \pi(G)$, a contradiction.

Let $n = 11$. It is obvious that $(p^{18}-1)\big||S|$ by Table 1. On the other hand, by (A), we have that $p \in \pi(S) \subseteq \pi(G) = \pi(D_{11}(3)) = \{2,3,5,7,11,13,17,19,23,37,41,61,73,193,547,757,1093,1181,3851\}$. However, for each previous prime $p \neq 2,3$, $p^{18}-1$ has a prime divisor which is not in $\pi(G)$, a contradiction. Thus $p = 2$ or 3. Suppose $p = 2$. Then $127 \in \pi(E_7(q)) \subseteq \pi(G)$, a contradiction. Hence $p = 3$. On one hand, 2 is non-adjacent to r_{11} in $\Gamma(G)$ by Table 2 and 2 can only be non-adjacent to one of u_7, u_9, u_{14} or u_{18} in $\Gamma(S)$ by Table 7 in [15]. Since $\rho(2,G) \subseteq \rho(2,S)$, it follows that $e(r_{11},3^{\alpha}) \in \{7,9,14,18\}$. On the other hand, since it is an easy number-theoretic observation that $e(r_i,p^{\alpha}) = i/(\alpha,i)$ if r_i is a Zsigmondy prime of p^i-1 for a prime p, we have that $e(r_{11},3^{\alpha}) \in \{1,11\}$. This contradiction shows that this case does not occur.

Case 2. Let $S \cong E_8(q)$, where $q = p^{\alpha}$.

Since $t(S) = 12$ by [15, Table 9], it follows that $[(3n+1)/4] \leq t(G) \leq t(S)+1 = 13$ and so $9 \leq n \leq 18$. It is clear that $q(q^{30}-1)\big||E_8(q)|$ and so $31 \in \pi(E_8(q)) \subseteq \pi(G)$. On the other hand, $e(31,3) = 30$ implies that $n \geq 16$. Therefore $n = 16$ or 17. Since n is odd, $n = 17$. It is obvious that $(p^{30}-1)\big||S|$ by Table 1. By (A), $p \in \pi(S) \subseteq \pi(G) = \pi(D_{17}(3))$. However, for each previous prime $p \neq 3$, $p^{30}-1$ has a prime divisor which is not in $\pi(G)$, a contradiction. Thus $p = 3$ and $S \cong E_8(3^{\alpha})$. Clearly $\rho(2,G) = \{2,r_{17}\}$ and $\rho(2,E_8(3^{\alpha})) = \{2,u_{15},u_{20},u_{24},u_{30}\}$ by Table 7 in [15]. Since $\rho(2,G) \subseteq \rho(2,S)$, it follows that $e(r_{17},3^{\alpha}) \in \{15,20,24,30\}$. On the other hand, $e(r_{17},3^{\alpha}) = 17/(\alpha,17) \in \{1,17\}$. This contradiction shows that this case does not occur.

Step 4. We prove that the simple group S is not isomorphic to a sporadic simple group.

Since $n \geq 9$, it follows that $t(G) \geq [(3n+1)/4] \geq [(3\cdot 9+1)/4] = 7$. By (C), it follows that $t(S) \geq t(G) - 1 \geq 6$. By [15, Table 2], we have that $S \cong J_4$. However, $t(J_4) = 6$ implies that $t(G) \leq t(J_4) + 1 = 7$ and so $n = 9$. On the other hand, by (A), $43 \in \pi(J_4) \subseteq \pi(G) = \pi(D_9(3))$, a contradiction.

Up to now, it is proved that S can only be isomorphic to a classical simple group over a field of characteristic 3. Thus we begin to prove the following.

Step 5. We prove that the simple group S is isomorphic to $D_n(3)$.

Case 1. Let $S \cong A_{m-1}(q)$, where $q = 3^{\alpha}$.

Let $n \geq 9$. By (C), it is clear that $6 \leq [(3\cdot 9+1)/4] - 1 \leq [(3n+1)/4] - 1 \leq t(G) - 1 \leq t(S)$. Hence $t(S) = [(m+1)/2]$ by Table 3. Therefore $m \geq 11$. Moreover, $[(3n+1)/4] - 1 \leq [(m+1)/2]$ implies that $n < m$ if $n \geq 9$. By (B), $\{2, r_n\} = \rho(2,G) \subseteq \rho(2,S)$. On the other hand, $\rho(2,S) = \rho(2, A_{m-1}(q)) = \{2, u_m\}$ or $\{2, u_{m-1}\}$ or $\{2, u_{m-1}, u_m\}$ if $m \geq 11$ by Table 2. Thus $r_n = u_m$ or $r_n = u_{m-1}$.

If $r_n = u_m$, then $e(r_n, 3^{\alpha}) = n/(\alpha, n) = m$, which is impossible since $m > n$.

If $r_n = u_{m-1}$, then $e(r_n, 3^{\alpha}) = n/(\alpha, n) = m-1$. Since $m > n \geq 9$, it follows that $n = m-1$. On the other hand, since $\pi(S) \subseteq \pi(G)$, we have that $\alpha(m-1) \leq 2n-2$ by Lemma 2.2 and so $\alpha = 1$. Therefore $(\alpha, m) = (1, n+1)$ and thus $S \cong A_n(3)$. Let $\rho = \{r_{2(n-1)}, r_{2(n-2)}, r_{2(n-3)}\}$. Then ρ is an independent subset of $\pi(G)$ by Table 3 if $n \geq 9$. By Lemma 2.5(b), at most one number of ρ divides the product $|K||\overline{G}/S| = |G|/|S|$ and so at least two numbers in it divide $|S|$, which is impossible by Table 1.

Case 2. Let $S \cong {}^2A_{m-1}(q)$, where $q = 3^{\alpha}$.

Let $n \geq 9$. By (C), it is clear that $6 \leq [(3\cdot 9+1)/4] - 1 \leq [(3n+1)/4] - 1 \leq t(G) - 1 \leq t(S)$. Hence $t(S) = [(m+1)/2]$ by Table 3. Therefore $m \geq 11$. Moreover, $[(3n+1)/4] - 1 \leq [(m+1)/2]$ implies that $n < m$ if $n \geq 9$. By (B), $\{2, r_n\} = \rho(2,G) \subseteq \rho(2,S)$. On the other hand, $\rho(2,S) = \rho(2, {}^2A_{m-1}(q)) \subseteq \{2, u_{2m}, u_{2m-2}, u_m, u_{m/2}\}$ if $m \geq 11$ by Table 2. Thus $r_n = u_{2m}, u_{m/2}, u_m$ or u_{2m-2}.

If $r_n = u_m, u_{2m}, u_{2m-2}$, then $e(r_n, 3^{\alpha}) = n/(\alpha, n) = m, 2m$ or $2m-2$, which is impossible since $m > n$.

If $r_n = u_{m/2}$, then $e(r_n, 3^{\alpha}) = n/(\alpha, n) = m/2$. Since $m > n \geq 9$, it follows that $n = m/2$. On the other hand, since $\pi(S) \subseteq \pi(G)$, we have that $(m\alpha)/2 \leq 2n-2$ by Lemma 2.2 and so $\alpha = 1$. Therefore $(\alpha, m) = (1, 2n)$ and thus $S \cong {}^2A_{2n-1}(3)$. Thus $r_{2n} \in \pi(S) \setminus \pi(G)$, a contradiction.

Case 3. Let $S \cong B_m(q)$ or $C_m(q)$, where $q = 3^{\alpha}$.

Let $n \geq 9$, it is clear that $6 \leq [(3\cdot 9+1)/4]-1 \leq [(3n+1)/4]-1 \leq t(G)-1 \leq t(S) \leq [(3m+5)/4]$. Therefore $m \geq 7$ and $[(3n+1)/4]-1 \leq [(3m+5)/4]$, which implies that $n-4 < m$. Clearly $\rho(2,B_m(q)) = \rho(2,C_m(q)) = \{2,u_m\}$ or $\{2,u_{2m}\}$ by Table 2. By (B), $r_n = u_m$ or u_{2m}.

If $r_n = u_{2m}$, then $e(r_n,3^\alpha) = n/(\alpha,n) = 2m$, which is impossible since n is odd.

If $r_n = u_m$, then $e(r_n,3^\alpha) = n/(\alpha,n) = m$. Since $m > n-4$, it follows that $n = m$. On the other hand, since $\pi(S) \subseteq \pi(G)$, we have that $\alpha m \leq 2n-2$ by Lemma 2.2 and so $\alpha = 1$. Therefore $(\alpha,m) = (1,n)$ and so $S \cong B_n(3)$ or $C_n(3)$. Thus $r_{2n} \in \pi(S) \setminus \pi(G)$, a contradiction.

Case 4. Let $S \cong {}^2D_m(q)$, where $q = 3^\alpha$.

Let $n \geq 9$. By (C), it is clear that $6 \leq [(3\cdot 9+1)/4]-1 \leq [(3n+1)/4]-1 \leq t(G)-1 \leq t(S) \leq [(3m+4)/4]$. Therefore $m \geq 7$ and $[(3n+1)/4]-1 \leq [(3m+4)/4]$, which implies that $n-4 < m$. By Table 2, $\rho(2,S) \subseteq \{2,u_{2m-2},u_{2m}\}$. By (B), $r_n = u_{2m-2}$ or u_{2m}. Thus $e(r_n,3^\alpha) = n/(\alpha,n) = 2m$ or $2m-2$, which is impossible since n is odd.

Case 5. Let $S \cong D_m(q)$, where $q = 3^\alpha$.

Let $n \geq 9$. By (C), it is clear that $6 \leq [(3\cdot 9+1)/4]-1 \leq [(3n+1)/4]-1 = t(G)-1 \leq t(S) \leq [(3m+1)/4]$. Therefore $m \geq 8$ and $[(3n+1)/4]-1 \leq [(3m+1)/4]$, which implies that $n-3 < m$. By Table 2, $\rho(2,S) \subseteq \{2,u_{2m-2},u_m,u_{m-1}\}$. By (B), $r_n \in \{u_{2m-2},u_m,u_{m-1}\}$.

If $r_n = u_{2m-2}$, then $e(r_n,3^\alpha) = n/(\alpha,n) = 2m-2$, which is impossible since n is odd.

If $r_n = u_{m-1}$, then $e(r_n,3^\alpha) = n/(\alpha,n) = m-1$. Since $m > n-3$, it follows that $n = m-1$. On the other hand, since $\pi(S) \subseteq \pi(G)$, we have that $\alpha(m-1) \leq 2n-2$ by Lemma 2.2 and so $\alpha = 1$. Therefore $(\alpha,m) = (1,n+1)$ and so $S \cong D_{n+1}(3)$. By (A), this is impossible.

Therefore $r_n = u_m$ and so $e(r_n,3^\alpha) = n/(\alpha,n) = m$. Since $m > n-3$, it follows that $n = m$. On the other hand, since $\pi(S) \subseteq \pi(G)$, we have that $\alpha m \leq 2n-2$ by Lemma 2.2 and so $\alpha = 1$. Therefore $(\alpha,m) = (1,n)$ and so $S \cong D_n(3)$, as desired. ∎

Theorem 3.2. *Let $n = 5$ or 7. If G is a finite group such that $\nabla(G) = \nabla(D_n(3))$, then $G \cong D_n(3)$.*

Proof. Since the proofs are similar, we only consider the case $n = 7$. By Tables 4 and 6 in [16], we have that $t(D_7(3)) = 6$ and $t(2,D_7(3)) \geq 2$. By Corollary 2.2, $\Gamma(G) = \Gamma(D_7(3))$ and $|G| = |D_7(3)| = 2^{22}\cdot 3^{42}\cdot 5^3\cdot 7^2\cdot 11^2\cdot 13^2\cdot 41\cdot 61\cdot 73\cdot 1093$. Thus $t(G) = 6$ and $t(2,G) \geq 2$. It follows from Corollary 2.1 that there is a finite nonabelian simple group S such that $S \lesssim G/K \lesssim \mathrm{Aut}(S)$ for the maximal normal solvable subgroup K of G and $t(S) \geq t(G)-1 \geq 5$. By Tables 8-9 in [15], we can immediately obtain that $S \cong A^\varepsilon_{m-1}(q), B_m(q), C_m(q), D^\varepsilon_m(q), E_6(q), F_4(q), E_7(q), E_8(q), {}^2E_6(q), {}^2G_2(q)$ or ${}^2F_4(q)$ for some suitable prime power q. Since the proofs of all cases are similar, we only give two of them as follows.

If $S \cong {}^2D_m(q)$, where q is a power of a prime $p \in \pi(S)$, then $m \geq 6$ and so $(q^8-1)\,\big|\,|S|$. Clearly $p \in \pi(S) \subseteq \pi(G) = \{2,3,5,7,11,13,41,61,73,1093\}$. However, for each previous prime $p \neq 3$, p^8-1 has a prime divisor which is not in $\pi(G)$, a contradiction. Thus $p = 3$.

Since $\{2, r_7\} = \rho(2, G) \subseteq \rho(2, S)$ by Corollary 2.1(b), it follows that $r_7 = u_{2m}$ or u_{2m-2} by Table 2. Thus, $e(r_7, 3^\alpha) = 7/(\alpha, 7) = 2m$ or $2m-2$, which is impossible.

If $S \cong D_m(q)$, where q is a power of a prime $p \in \pi(S)$, then $m \geq 6$ and so $(q^8 - 1) \big| |S|$. Clearly $p \in \pi(S) \subseteq \pi(G)$. However, for each previous prime $p \neq 3$, $p^8 - 1$ has a prime divisor which is not in $\pi(G)$, a contradiction. Thus $p = 3$. Since $\{2, r_7\} = \rho(2, G) \subseteq \rho(2, S)$ by Corollary 2.1(b), it follows that $r_7 = u_m$ or u_{m-1} by Table 2. Thus, $e(r_7, 3^\alpha) = 7/(\alpha, 7) = m$ or $m-1$. Since $m \geq 6$, we have that $(\alpha, m) = (1, 7)$ or $(1, 8)$. Clearly the latter case will not occur and so $S \cong D_7(3)$. Since $S \lesssim G/K \lesssim Aut(G)$, it follows that $K = 1$ and so $G \cong S \cong D_7(5)$, as desired. ∎

In [8], authors have proved that $D_n(3)$, where $n \in \{p, p+1\}$ for a prime $p > 3$, is quasirecognizable by its prime graph. As a corollary of the above results, we obtain the following result, which shows that *AAM*'s conjecture is valid for $D_n(3)$ for some n.

Corollary 3.2. *Let $n \geq 5$ be an odd integer or $n = p+1$ for a prime $p > 3$. If G is a finite group such that $\nabla(G) = \nabla(D_n(3))$, then $G \cong D_n(3)$.*

Proof. If $n = p+1$ for a prime $p > 3$, then $D_n(3)$ is quasirecognizable by its prime graph by [8]. Thus $\nabla(G) = \nabla(D_n(3))$ implies $G \cong D_n(3)$ by Lemma 2.7. Furthermore, by Theorem 3.2, both $D_5(3)$ and $D_7(3)$ are recognizable by their noncommuting graphs. We now need only to consider the remaining case that $n \geq 9$ is odd. Since $\nabla(G) = \nabla(D_n(3))$, it follows that $|G| = |D_n(3)|$ and $\Gamma(G) = \Gamma(D_n(3))$ by Corollary 2.2. Since $\Gamma(G) = \Gamma(D_n(3))$, by Theorem 3.1, $D_n(3) \lesssim G/K \lesssim Aut(D_n(3))$, where K is the maximal normal solvable subgroup of G. Since $|G| = |D_n(3)|$, we have that $K = 1$ and so $G \cong D_n(3)$, as desired. ∎

Let G and H be finite groups. Since $\pi_e(G) = \pi_e(H)$ always implies that $\Gamma(G) = \Gamma(H)$, we can also obtain the following corollary by Theorem 3.1 and by [8], part of which was proved in [19].

Corollary 3.3. *Let $n \geq 5$ be an odd integer or $n = p+1$ for a prime $p > 3$. If G is a finite group such that $\pi_e(G) = \pi_e(D_n(3))$ and $|G| = |D_n(3)|$, then $G \cong D_n(3)$.*

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