# Fixed Point Results in Orbitally Complete Partial Metric Spaces 

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#### Abstract

In this paper, we prove two fixed point theorems for maps that satisfy a contraction principle involving a rational expression in complete partial metric spaces.


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## 1. Introduction

In [13], Matthews introduced the notion of a partial metric space as part of a study on denotational semantics of data-flow networks, and obtained, among other results, a nice relationship between partial metric spaces and weightable quasi-metric spaces. He obtained the following analog of Banach fixed point theorem in complete partial metric spaces.
Theorem 1.1. Let $T$ be a self-mapping of a complete partial metric space $(X, p)$ such that there is a real number $h$ with $0 \leq h<1$, satisfying

$$
p(T x, T y) \leq h p(x, y)
$$

for all $x, y \in X$. Then $T$ has a unique fixed point.
O'Neill [17] defined the concept of dualistic partial metric, which is more general than partial metric. In [16], Oltra and Valero gave a Banach fixed point theorem on complete dualistic partial metric spaces. They also showed that the contractive condition in Banach fixed point theorem in complete dualistic partial metric spaces cannot be replaced by the contractive condition of Banach fixed point theorem for complete partial metric spaces. Later, Valero [20] generalized the main theorem of [16] using nonlinear contractive condition instead of Banach contractive condition. Altun et al. [3], Ilić et al. [6], Oltra et al. [15], Nashine [14] and Romaguera [18] also studied fixed point theorems in partial metric spaces. Recently, Karapınar and Erhan [7] introduced the notion of orbitally complete and orbitally

[^0]continuous in partial metric spaces and obtained fixed point results using these concepts (see also [1, 4], [8]-[12]).

Motivated by results above, we prove two fixed point theorems for a map satisfying contraction involving rational expressions. The first result is based on continuous maps under rational type contraction condition in complete partial metric spaces while the second result is based on an orbitally continuous map satisfying different rational type contractive condition on orbitally complete partial metric spaces.

## 2. Preliminaries

We introduce some notations and definitions that will be used in the following section.
The following definition was introduced in [13, 16, 20].
Definition 2.1. A partial metric on a nonempty set $X$ is a function $p: X \times X \rightarrow[0, \infty)$ such that
$\left(\mathrm{p}_{1}\right) x=y \Leftrightarrow p(x, x)=p(x, y)=p(y, y)$,
( $\mathrm{p}_{2}$ ) $p(x, x) \leq p(x, y)$,
$\left(\mathrm{p}_{3}\right) p(x, y)=p(y, x)$,
$\left(\mathrm{p}_{4}\right) p(x, y) \leq p(x, z)+p(z, y)-p(z, z)$,
for all $x, y, z \in X$.
A partial metric space (in short, PMS) is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$. It is clear that, if $p(x, y)=0$, then from $\left(p_{1}\right)$ and $\left(p_{2}\right) x=y$. But if $x=y$, then the value $p(x, y)$ may not be 0 .

If $p$ is a partial metric on $X$, then the function $d_{p}: X \times X \rightarrow[0, \infty)$ given by

$$
d_{p}(x, y)=2 p(x, y)-p(x, x)-p(y, y)
$$

is a metric on $X$.
Example 2.1. [13, 7, 1] Consider $X=[0, \infty)$ with $p(x, y)=\max \{x, y\}$. Then $(X, p)$ is a partial metric space. It is clear that $p$ is not a (usual) metric. Note that in this case $d_{p}(x, y)=|x-y|$.

Example 2.2. [6] Let $X=\{[a, b]: a, b, \in \mathbb{R}, a \leq b\}$ and define $p([a, b],[c, d])=\max \{b, d\}-$ $\min \{a, c\}$. Then $(X, p)$ is a partial metric space.

Example 2.3. [6] Let $X:=[0,1] \cup[2,3]$ and define $p: X \times X \rightarrow[0, \infty)$ by

$$
p(x, y)=\left\{\begin{array}{c}
\max \{x, y\} \text { if }\{x, y\} \cap[2,3] \neq \emptyset, \\
|x-y| \text { if }\{x, y\} \subset[0,1] .
\end{array}\right.
$$

Then ( $\mathrm{X}, \mathrm{p}$ ) is a complete partial metric space.
Other examples of PMS which are interesting from a computational point of view may be found in $[5,13]$.

Each partial metric $p$ on $X$ generates a $T_{0}$ topology $\tau_{p}$ on $X$ which has the family open $p$-balls $\left\{B_{p}(x, \varepsilon): x \in X, \varepsilon>0\right\}$ as a base where $B_{p}(x, \varepsilon)=\{y \in X: p(x, y)<p(x, x)+\varepsilon\}$ for all $x \in X$ and $\varepsilon>0$.

Definition 2.2. [2] Let $(X, p)$ be a PMS. Then:
(1) A sequence $\left\{x_{n}\right\}$ in a PMS $(X, p)$ converges to a point $x \in X$ if and only if

$$
p(x, x)=\lim _{n \rightarrow \infty} p\left(x, x_{n}\right) .
$$

(2) A sequence $\left\{x_{n}\right\}$ in a PMS $(X, p)$ is called a Cauchy sequence if and only if

$$
\lim _{n, m \rightarrow \infty} p\left(x_{n}, x_{m}\right)
$$

exists (and is finite).
(3) A partial metric space $(X, p)$ is said to be complete if every Cauchy sequence $\left\{x_{n}\right\}$ in $X$ converges, with respect to $\tau_{p}$, to a point $x \in X$ such that $p(x, x)=$ $\lim _{n, m \rightarrow \infty} p\left(x_{n}, x_{m}\right)$.
(4) A mapping $T: X \rightarrow X$ is said to be continuous at $x_{0} \in X$, if for every $\varepsilon>0$, there exists $\delta>0$ such that $T\left(B_{p}\left(x_{0}, \delta\right)\right) \subset B_{p}\left(T x_{0}, \varepsilon\right)$.
Lemma 2.1. $[13,16]$ Let $(X, p)$ be a $P M S$.
(a) $\left\{x_{n}\right\}$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in the metric space $\left(X, d_{p}\right)$.
(b) A PMS $(X, p)$ is complete if and only if the metric space $\left(X, d_{p}\right)$ is complete. Furthermore,
$\lim _{n \rightarrow \infty} d_{p}\left(x_{n}, x\right)=0$ if and only if

$$
p(x, x)=\lim _{n \rightarrow \infty} p\left(x_{n}, x\right)=\lim _{n, m \rightarrow \infty} p\left(x_{n}, x_{m}\right) .
$$

Lemma 2.2. [1, 7] Assume $x_{n} \rightarrow z$ as $n \rightarrow \infty$ in a PMS $(X, p)$ such that $p(z, z)=0$. Then $\lim _{n \rightarrow \infty} p\left(x_{n}, y\right)=p(z, y)$ for every $y \in X$.

## 3. Main results

The first theorem of the paper is as follows:
Theorem 3.1. Let $(X, p)$ be a complete PMS and $T: X \rightarrow X$ be continuous mapping satisfying

$$
\begin{equation*}
p(T x, T y) \leq \alpha p(x, y)+\frac{\beta[1+p(x, T x)] p(y, T y)}{1+p(x, y)} \tag{3.1}
\end{equation*}
$$

for all distinct $x, y \in X$, where $\alpha, \beta$ are nonnegative real numbers with $\alpha+\beta<1$. Then $T$ has a fixed point $u$ in $X$. Moreover, $p(u, u)=p(T u, T u)=p(u, T u)=0$.
Proof. Let $x_{0} \in X$. Suppose $T x_{0} \neq x_{0}$. Define $x_{n}=T^{n} x_{0}$ and so $x_{n+1}=T x_{n}$. If there exists $n_{0} \in\{1,2, \cdots\}$ such that $p\left(x_{n_{0}}, x_{n_{0}-1}\right)=0$, then by $\left(p_{2}\right)$ we have $p\left(x_{n_{0}-1}, x_{n_{0}-1}\right)=$ $p\left(x_{n_{0}}, x_{n_{0}}\right)$. Thus by $\left(p_{1}\right)$, we get that $x_{n_{0}-1}=x_{n_{0}}=T x_{n_{0}-1}$. So we are done in this case. Now we suppose that

$$
p\left(x_{n}, x_{n+1}\right)>0
$$

for all $n \geq 1$.
We claim that for all $n \in \mathbb{N}$, we have

$$
\begin{equation*}
p\left(x_{n+1}, x_{n+2}\right) \leq h^{n+1} p\left(x_{0}, x_{1}\right), \text { where } 0<h<1 . \tag{3.2}
\end{equation*}
$$

Using (3.1) for $x_{n}$ and $x_{n+1}$ in place of $x$ and $y$ respectively, we get

$$
p\left(x_{n+1}, x_{n+2}\right)=p\left(T x_{n}, T x_{n+1}\right)
$$

$$
\begin{aligned}
& \leq \alpha p\left(x_{n}, x_{n+1}\right)+\frac{\beta p\left(x_{n+1}, T x_{n+1}\right)\left[1+p\left(x_{n}, T x_{n}\right)\right]}{1+p\left(x_{n}, x_{n+1}\right)} \\
& =\alpha p\left(x_{n}, x_{n+1}\right)+\frac{\beta p\left(x_{n+1}, x_{n+2}\right)\left[1+p\left(x_{n}, x_{n+1}\right)\right]}{1+p\left(x_{n}, x_{n+1}\right)} \\
& \leq \alpha p\left(x_{n}, x_{n+1}\right)+\beta p\left(x_{n+1}, x_{n+2}\right)
\end{aligned}
$$

which yields that

$$
p\left(x_{n+1}, x_{n+2}\right) \leq \frac{\alpha}{1-\beta} p\left(x_{n}, x_{n+1}\right) .
$$

Set $h=\alpha /(1-\beta)$. Thus we have

$$
\begin{equation*}
p\left(x_{n+1}, x_{n+2}\right) \leq h p\left(x_{n}, x_{n+1}\right) \leq h\left(h p\left(x_{n-1}, x_{n}\right)\right) \leq \cdots \leq h^{n+1} p\left(x_{0}, x_{1}\right) . \tag{3.3}
\end{equation*}
$$

We will show that $\left\{x_{n}\right\}$ is a Cauchy sequence. Without loss of generality assume that $n>m$. Then, using (3.3) and the triangle inequality for partial metrics $\left(p_{4}\right)$ we have

$$
\begin{aligned}
p\left(x_{n}, x_{n+m}\right) & \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+m}\right)-p\left(x_{n+1}, x_{n+1}\right) \\
& \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+m}\right) \\
& \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+2}\right)+p\left(x_{n+2}, x_{n+m}\right)-p\left(x_{n+2}, x_{n+2}\right) \\
& \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+2}\right)+p\left(x_{n+2}, x_{n+m}\right) .
\end{aligned}
$$

Inductively, we have

$$
\begin{aligned}
0 & \leq p\left(x_{n}, x_{n+m}\right) \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+2}\right)+\cdots+p\left(x_{n+m-1}, x_{n+m}\right) \\
& \leq\left(h^{n}+h^{n+1}+\cdots+h^{n+m-1}\right) p\left(x_{0}, T x_{0}\right) \\
& =h^{n}\left(1+h+\cdots+h^{m-1}\right) p\left(x_{0}, T x_{0}\right) \\
& \leq \frac{h^{n}}{1-h} p\left(x_{0}, T x_{0}\right) .
\end{aligned}
$$

Since $\alpha+\beta<1$ then $h<1$. Thus,

$$
\begin{equation*}
\lim _{m, n \rightarrow \infty} p\left(x_{n}, x_{m}\right)=0 . \tag{3.4}
\end{equation*}
$$

Therefore by $\left(p_{2}\right)$, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} p\left(x_{n}, x_{n}\right)=0 \text { and } \lim _{m \rightarrow \infty} p\left(x_{m}, x_{m}\right)=0 . \tag{3.5}
\end{equation*}
$$

Hence,

$$
d_{p}\left(x_{n}, x_{m}\right)=2 p\left(x_{n}, x_{m}\right)-p\left(x_{n}, x_{n}\right)-p\left(x_{m}, x_{m}\right) \rightarrow 0 \text { as } n \rightarrow \infty .
$$

Thus $\left\{x_{n}\right\}$ is a Cauchy sequence in $\left(X, d_{p}\right)$. Since $(X, p)$ is complete, by Lemma 2.1, the corresponding metric space $\left(X, d_{p}\right)$ is also complete. Therefore, the sequence $\left\{x_{n}\right\}$ converges in the metric space $\left(X, d_{p}\right)$, say $u \in X$ such that $\lim _{n \rightarrow \infty} d_{p}\left(x_{n}, u\right)=0$. Again from Lemma 2.1 and (3.5), we have

$$
\begin{equation*}
p(u, u)=\lim _{n \rightarrow \infty} p\left(x_{n}, u\right)=\lim _{n, m \rightarrow \infty} p\left(x_{n}, x_{m}\right)=0 . \tag{3.6}
\end{equation*}
$$

We have the facts that $\left\{x_{n+1}\right\}$ converges to $u$ in $(X, p)$ and $p(u, u)=0$. So by Lemma 2.2, we get that

$$
\lim _{n \rightarrow \infty} p\left(x_{n+1}, T u\right)=p(u, T u) .
$$

Assume that $p(u, T u)>0$. Since $T$ is continuous, for a given $\varepsilon>0$, there exists $\delta>0$ such that $T\left(B_{p}(u, \delta)\right) \subseteq B_{p}(T u, \varepsilon)$. Since $p(u, u)=\lim _{n \rightarrow \infty} p\left(x_{n}, u\right)=0$, then there exists $k \in \mathbb{N}$ such that $p\left(x_{n}, u\right)<p(u, u)+\delta$ for all $n \geq k$. Therefore, we have $\left\{x_{n}\right\} \subset B_{p}(u, \boldsymbol{\delta})$ for all $n \geq k$. Thus $T x_{n} \in T\left(B_{p}(u, \delta)\right) \subset B_{p}(T u, \varepsilon)$ and so $p\left(T x_{n}, T u\right)<p(T u, T u)+\varepsilon$ for all $n \geq k$. For any $\varepsilon>0$, we know

$$
-\varepsilon+p(T u, T u)<p(T u, T u) \leq p\left(x_{n+1}, T u\right)
$$

which yield that

$$
\left|p\left(x_{n+1}, T u\right)-p(T u, T u)\right|<\varepsilon .
$$

This shows that $p(T u, T u)=\lim _{n \rightarrow \infty} p\left(x_{n+1}, T u\right)$. By uniqueness of the limit in $\mathbb{R}$, we obtain

$$
\lim _{n \rightarrow \infty} p\left(x_{n+1}, T u\right)=p(u, T u)=p(T u, T u) .
$$

Using the inequality in (3.1) for $x=x_{n}$ and $y=u$ we have

$$
\begin{aligned}
p\left(x_{n+1}, T u\right)=p\left(T x_{n}, T u\right) & \leq \alpha p\left(x_{n}, u\right)+\frac{\beta\left[1+p\left(x_{n}, T x_{n}\right)\right] p(u, T u)}{1+p\left(x_{n}, u\right)} \\
& =\alpha p\left(x_{n}, u\right)+\frac{\beta\left[1+p\left(x_{n}, x_{n+1}\right)\right] p(u, T u)}{1+p\left(x_{n}, u\right)} .
\end{aligned}
$$

Taking the limit as $n \rightarrow \infty$ together with (3.6), we obtain

$$
p(u, T u) \leq \beta p(u, T u)
$$

which implies $p(u, T u)=0$ since $\beta<1$. Hence $p(T u, T u)=0=p(u, T u)$. By $\left(p_{1}\right)$, we conclude that $u=T u$. Therefore $u$ is a fixed point of $T$.

We construct the following example to demonstrate the validity of the hypotheses of Theorem 3.1:

Example 3.1. Let $X=[0,+\infty)$ endowed with the usual partial metric $p$ defined by $p$ : $X \times X \rightarrow[0,+\infty)$ with $p(x, y)=\max \{x, y\}$. The partial metric space $(X, p)$ is complete because ( $X, d_{p}$ ) is complete. Indeed, for any $x, y \in X$,

$$
d_{p}(x, y)=2 p(x, y)-p(x, x)-p(y, y)=2 \max \{x, y\}-(x+y)=|x-y|
$$

Thus, $\left(X, d_{p}\right)=([0,+\infty),|\cdot|)$ is the usual metric space, which is complete. Again, we define

$$
T(t)=t / 3, \text { if } t \geq 0
$$

By the Lemma 2.2 [19], the function $T$ is continuous on $(X, p)$. In particular, for any $y \leq x$, we have

$$
\begin{equation*}
p(T x, T y)=x / 3, p(x, y)=x, p(x, T x)=x, p(T y, y)=y . \tag{3.7}
\end{equation*}
$$

The inequality (3.1) is true for $\alpha+\beta=1 / 3$. Hence all the conditions of Theorem 3.1 are satisfied. Therefore, $u=0$ is fixed point of the mapping $T$.

In the next theorem, we drop the continuity of the map $T$ and completeness of $X$. We prove our second fixed point result for orbitally continuous map over orbitally complete partial metric spaces. To this end, we recall the notion of orbitally continuous maps and orbitally complete metric spaces which are defined by Karapınar and Erhan [7].

Definition 3.1. [7] Let $(X, p)$ be a PMS. A map $T: X \rightarrow X$ is called orbitally continuous if

$$
\lim _{i \rightarrow \infty} p\left(T^{n_{i}} x, z\right)=p(z, z)
$$

implies

$$
\lim _{i \rightarrow \infty} p\left(T T^{n_{i}} x, T z\right)=p(T z, T z)
$$

for each $x \in X$.
Definition 3.2. [7] A PMS $(X, p)$ is called orbitally complete if every Cauchy sequence $\left\{T^{n_{i}} x\right\}_{i=1}^{\infty}$ converges in $(X, p)$, that is, if

$$
\lim _{i, j \rightarrow \infty} p\left(T^{n_{i}} x, T^{n_{j}} x\right)=\lim _{i \rightarrow \infty} p\left(T^{n_{i}} x, z\right)=p(z, z) .
$$

The second theorem of the paper is as follows:
Theorem 3.2. Let $(X, p)$ be a orbitally complete PMS and let $T: X \rightarrow X$ be an orbitally continuous mapping that satisfies

$$
\begin{equation*}
p(T x, T y) \leq \alpha p(x, y)+\frac{\beta p(x, T x) p(y, T y)}{1+p(x, y)} \tag{3.8}
\end{equation*}
$$

for all distinct $x, y \in X$, where $\alpha, \beta$ are nonnegative real numbers with $\alpha+\beta<1$. Then $T$ has a fixed point $z$ in $X$. Moreover, $p(z, z)=p(T z, T z)=p(z, T z)=0$.

Proof. Let $x_{0} \in X$. Suppose $T x_{0} \neq x_{0}$. Define $x_{n}=T^{n} x_{0}$ and so $x_{n+1}=T x_{n}$. If there exists $n_{0} \in\{1,2, \cdots\}$ such that $p\left(x_{n_{0}}, x_{n_{0}-1}\right)=0$, then by $\left(p_{2}\right)$ we have $p\left(x_{n_{0}-1}, x_{n_{0}-1}\right)=$ $p\left(x_{n_{0}}, x_{n_{0}}\right)$. Thus by $\left(p_{1}\right)$, we get that $x_{n_{0}-1}=x_{n_{0}}=T x_{n_{0}-1}$. We are done in this case. Suppose that

$$
p\left(x_{n}, x_{n+1}\right)>0
$$

for all $n \geq 0$.
We claim that for all $n \in \mathbb{N}$, we have

$$
\begin{equation*}
p\left(x_{n+1}, x_{n+2}\right) \leq h^{n+1} p\left(x_{0}, x_{1}\right), \tag{3.9}
\end{equation*}
$$

for some $h<1$. Using (3.8) for $x_{n}$ and $x_{n+1}$ in place of $x$ and $y$ respectively, we get

$$
\begin{aligned}
p\left(x_{n+1}, x_{n+2}\right) & =p\left(T x_{n}, T x_{n+1}\right) \\
& \leq \alpha p\left(x_{n}, x_{n+1}\right)+\frac{\beta p\left(x_{n+1}, T x_{n+1}\right) p\left(x_{n}, T x_{n}\right)}{1+p\left(x_{n}, x_{n+1}\right)} \\
& =\alpha p\left(x_{n}, x_{n+1}\right)+\frac{\beta p\left(x_{n+1}, x_{n+2}\right) p\left(x_{n}, x_{n+1}\right)}{1+p\left(x_{n}, x_{n+1}\right)} \\
& \leq \alpha p\left(x_{n}, x_{n+1}\right)+\beta p\left(x_{n+1}, x_{n+2}\right), \text { since } p\left(x_{n}, x_{n+1}\right) \leq 1+p\left(x_{n}, x_{n+1}\right) .
\end{aligned}
$$

It implies that

$$
p\left(x_{n+1}, x_{n+2}\right) \leq \frac{\alpha}{1-\beta} p\left(x_{n}, x_{n+1}\right) .
$$

Set $h=\alpha /(1-\beta)$. Since $\alpha+\beta<1$, then $h<1$. Thus we have

$$
\begin{equation*}
p\left(x_{n+1}, x_{n+2}\right) \leq h p\left(x_{n}, x_{n+1}\right) \leq h^{2} p\left(x_{n-1}, x_{n}\right) \leq \cdots \leq h^{n+1} p\left(x_{0}, x_{1}\right) . \tag{3.10}
\end{equation*}
$$

We will show that $\left\{x_{n}\right\}$ is a Cauchy sequence. Without loss of generality assume that $n>m$. Then, using (3.10) and the triangle inequality for partial metrics $\left(p_{4}\right)$ we have

$$
\begin{aligned}
p\left(x_{n}, x_{n+m}\right) & \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+m}\right)-p\left(x_{n+1} n, x_{n+1}\right) \\
& \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+m}\right) \\
& \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+2}\right)+p\left(x_{n+2}, x_{n+m}\right)-p\left(x_{n+2}, x_{n+2}\right) \\
& \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+2}\right)+p\left(x_{n+2}, x_{n+m}\right) .
\end{aligned}
$$

Inductively, we have

$$
\begin{aligned}
0 & \leq p\left(x_{n}, x_{n+m}\right) \leq p\left(x_{n}, x_{n+1}\right)+p\left(x_{n+1}, x_{n+2}\right)+\cdots+p\left(x_{n+m-1}, x_{n+m}\right) \\
& \leq\left(h^{n}+h^{n+1}+\cdots+h^{n+m-1}\right) p\left(x_{0}, T x_{0}\right) \\
& =h^{n}\left(1+h+\cdots+h^{m-1}\right) p\left(x_{0}, T x_{0}\right) \\
& \leq \frac{h^{n}}{1-h} p\left(x_{0}, T x_{0}\right)
\end{aligned}
$$

where $h=\alpha /(1-\beta)<1$ since $\alpha+\beta<1$. Thus,

$$
\begin{equation*}
\lim _{m, n \rightarrow \infty} p\left(x_{n}, x_{m}\right)=0 \tag{3.11}
\end{equation*}
$$

Regarding ( $\mathrm{p}_{2}$ ), we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} p\left(x_{n}, x_{n}\right)=0 \text { and } \lim _{m \rightarrow \infty} p\left(x_{m}, x_{m}\right)=0 \tag{3.12}
\end{equation*}
$$

Hence,

$$
d_{p}\left(x_{n}, x_{m}\right)=2 p\left(x_{n}, x_{m}\right)-p\left(x_{n}, x_{n}\right)-p\left(x_{m}, x_{m}\right) \rightarrow 0 \text { as } n \rightarrow \infty .
$$

So, we conclude that $\left\{x_{n}\right\}=\left\{T^{n} x_{0}\right\}$ is a Cauchy sequence in $\left(X, d_{p}\right)$. Since $(X, p)$ is orbitally complete then the sequence $\left\{T^{n} x_{0}\right\}$ converges in the metric space $\left(X, d_{p}\right)$, say $\lim _{n \rightarrow \infty} d_{p}\left(T^{n} x_{0}, z\right)=0$. Again from Lemma 2.1, (3.11) and (3.12) we have

$$
\begin{equation*}
p(z, z)=\lim _{n \rightarrow \infty} p\left(T^{n} x_{0}, z\right)=\lim _{n, m \rightarrow \infty} p\left(T^{n} x_{0}, T^{m} x_{0}\right)=0 \tag{3.13}
\end{equation*}
$$

Suppose that $p(z, T z)>0$. Since $T$ is orbitally continuous, by (3.13), we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} p\left(T^{n} x_{0}, z\right)=p(z, z) \Rightarrow \lim _{n \rightarrow \infty} p\left(T T^{n} x_{0}, T z\right)=p(T z, T z) \tag{3.14}
\end{equation*}
$$

By the triangle inequality ( $p_{4}$ ), we have

$$
\begin{aligned}
p(z, T z) & \leq p\left(z, T^{n+1} x_{0}\right)+p\left(T^{n+1} x_{0}, T z\right)-p\left(T^{n+1} x_{0}, T^{n+1} x_{0}\right) \\
& \leq p\left(z, x_{n+2}\right)+p\left(T^{n+1} x_{0}, T z\right) .
\end{aligned}
$$

Letting $n \rightarrow+\infty$ and using Lemma 2.2 with (3.14) we obtain

$$
\begin{aligned}
p(z, T z) & \leq \lim _{n \rightarrow+\infty} p\left(z, x_{n+2}\right)+\lim _{n \rightarrow+\infty} p\left(T^{n+1} x_{0}, T z\right) \\
& =p(T z, T z)
\end{aligned}
$$

Thus, we have $p(z, T z) \leq p(T z, T z)$. But from $\left(p_{2}\right)$, we also have $p(T z, T z) \leq p(z, T z)$. Hence

$$
\begin{equation*}
p(z, T z)=p(T z, T z) \tag{3.15}
\end{equation*}
$$

Using (3.8) for $x=T z$ and $y=z$,

$$
\begin{equation*}
p\left(T^{2} z, T z\right) \leq \alpha p(T z, z)+\frac{\beta p\left(T z, T^{2} z\right) p(z, T z)}{1+p(T z, z)} \leq(\alpha+\beta) p(T z, z) \tag{3.16}
\end{equation*}
$$

Notice that due to $\left(p_{2}\right)$ we have

$$
\begin{equation*}
p(T z, T z) \leq p\left(T^{2} z, T z\right) \tag{3.17}
\end{equation*}
$$

Combining (3.13), (3.16) and (3.17), we get

$$
p(T z, z)=p(T z, T z) \leq p\left(T^{2} z, T z\right) \leq\left(\frac{\alpha}{1-\beta}\right) p(T z, z)
$$

which is possible only if $p(T z, T z)=0($ since $\alpha+\beta<1)$ and so

$$
p(T z, T z)=p(z, T z)=p(z, z)=0 .
$$

By $\left(p_{1}\right)$, we conclude that $z=T z$. Therefore $z$ is a fixed point of $T$.
Now we present an example to show the validity of the hypotheses of Theorem 3.2:
Example 3.2. Let $X=[0,+\infty)$ endowed with the usual partial metric $p$ defined by $p$ : $X \times X \rightarrow[0,+\infty)$ with $p(x, y)=\max \{x, y\}$. The partial metric space $(X, p)$ is complete because ( $X, d_{p}$ ) is complete. Indeed, for any $x, y \in X$,

$$
d_{p}(x, y)=2 p(x, y)-p(x, x)-p(y, y)=2 \max \{x, y\}-(x+y)=|x-y| .
$$

Thus, $\left(X, d_{p}\right)=([0,+\infty),|\cdot|)$ is the usual metric space, which is complete. Again, we define

$$
T(t)=\left\{\begin{aligned}
\frac{t}{2} & \text { if } 0 \leq t<1 \\
\frac{t^{2}}{1+t} & \text { if } t \geq 1 .
\end{aligned}\right.
$$

Take $\alpha=1 / 2$ and $\beta \in[0,1 / 2)$ so that $\alpha+\beta<1$.
Let us denote the left-hand and right-hand side of inequality (3.8) by $L$ and $R$, respectively. We take $y \leq x$. Then there are two possibilities. If $x \in[0,1)$ (and so $y \in[0,1)$ ), then

$$
L=p(T x, T y)=\max \left\{\frac{x}{2}, \frac{y}{2}\right\}=\frac{x}{2} \leq R,
$$

holds true. If $x \geq 1$, then

$$
p(T x, T y)=\frac{x}{1+x}, p(x, y)=x, p(x, T x)=x, p(T y, y)=y .
$$

Now, if $1 \leq x$, then $L \leq R$ can be easily checked. Hence, in all possible cases the condition (3.8) holds. Hence all the conditions of the Theorem 3.2 are satisfied. Therefore, the sequence $\left\{T^{n} x\right\}=\{x /(1+n x)\}$ converges to the fixed point $z=0$ of the mapping $T$ for every $x \in X$.

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