BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(4) (2013), 1195–1203*

Positive Solution for Fractional q-Difference Boundary Value Problems with ϕ-Laplacian Operator

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Abstract. In this paper, we investigate the existence of at least one positive solution for a class of fractional q-difference boundary value problems with ϕ-Laplacian operator. The arguments mainly rely on the upper and lower solutions method as well as the Schauder's fixed point theorem. Nonlinear term may be singular at $t = 0, 1$ or $u = 0$. Furthermore, two examples are presented to illustrate the main results.

2010 Mathematics Subject Classification: 39A13, 34B18, 34A08

Keywords and phrases: Fractional q-difference, boundary value problems, ϕ-Laplacian operator, positive solution, upper and lower solutions method.

1. Introduction

In recent years, boundary value problems involving nonlinear fractional q-difference equations have been addressed extensively by several researchers. There have been some papers dealing with the existence and multiplicity of solutions or positive solutions for boundary value problems involving nonlinear fractional q-difference equations by the use of some well-known fixed point theorems. For some recent developments on the subject, see [3, 4, 6, 7, 10] and the references therein. El-Shahed and Al-Askar [5] studied the existence of multiple positive solutions to the nonlinear q-fractional boundary value problems by using Guo-Krasnoselskii's fixed point theorem in a cone. Ma and Yang [11] considered the existence of solutions for multi-point boundary value problems of nonlinear fractional q-difference equations by means of the Banach contraction principle and Krasnoselskii's fixed point theorem.

The upper and lower solutions method is regarded as an excellent tool for investigating the existence results for certain boundary value problem. Many boundary value problems has been obtained based on the upper and lower solutions method combining with standard fixed point theorems, see for example [2, 9, 12, 14, 17] and references therein. Zhang and Liu [18] and Zhang [19] investigated the existence of positive solutions for singular fourth-order four-point and integral boundary value problem with p-Laplacian operator by using the upper and lower solutions method and fixed point theorem, respectively. By employing

Communicated by Norhashidah M. Ali.
Received: May 9, 2012; *Revised:* August 13, 2012.

the upper and lower solutions method, Wang and Xiang [16] discussed the existence of at least one positive solution for singular fractional boundary value problems with p-Laplacian operator. Mao *et al.* [13] studied the existence and uniqueness of positive solutions for a second order integral boundary value problem based on the method of lower and upper solutions and the maximal principle.

However, to the best of author's knowledge, few results exist in the literatures devoted to investigate ϕ-Laplacian fractional q-difference boundary value problems by applying the upper and lower solutions method. To fill this gap, in this paper, we consider the following fractional q-difference boundary value problem with ϕ-Laplacian operator

$$\begin{aligned} &D_q^{\beta}(\phi_\mu(D_q^{\alpha}u(t))) = f(t,u(t)), \quad 0<t<1,\\ &u(0)=u(1)=0, \quad D_q^{\alpha}u(0) = D_q^{\alpha}u(1) = 0, \end{aligned} \tag{1.1}$$

where $1<\alpha,\beta\leq 2$, and D_q^{α} is the fractional q-derivative of the Riemann-Liouville type, $\phi_\mu(s)=|s|^{\mu-2}s$, $\mu>1$, $(\phi_\mu)^{-1}=\phi_\nu$, $(1/\mu)+(1/\nu)=1$, and nonlinear term $f(t,u)$ may be singular at $t=0,1$ or $u=0$. By applying the upper and lower solutions method associated with the Schauder's fixed point theorem, the existence results of at least one positive solution for the above fractional q-difference boundary value problem with ϕ-Laplacian operator are established. At the end of this paper, we will give two examples to show the effectiveness of the main result.

2. Preliminaries

In this section, we present some necessary definitions and lemmas. For details, the readers can see [8] and references therein.

Let $q\in(0,1)$ and define

$$[a]_q = \frac{q^a-1}{q-1}, \quad a\in\mathbb{R}.$$

The q-analogue of the power $(a-b)^n$ with $n\in\mathbb{N}_0$ is

$$(a-b)^{(0)}=1, \quad (a-b)^{(n)}=\prod_{k=0}^{n-1}(a-bq^k), \quad n\in\mathbb{N}, \quad a,b\in\mathbb{R}.$$

More generally, if $\alpha\in\mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^{\alpha}\prod_{n=0}^{\infty}\frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that, if $b=0$ then $a^{(\alpha)}=a^{\alpha}$. The q-gamma function is defined by

$$\Gamma_q(x)=\frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x\in\mathbb{R}\setminus\{0,-1,-2,\dots\},$$

and satisfies $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_qf)(x)=\frac{f(x)-f(qx)}{(1-q)x}, \quad (D_qf)(0)=\lim_{x\to 0}(D_qf)(x),$$

and q-derivatives of higher order by

$$(D_q^0f)(x)=f(x) \quad \text{and} \quad (D_q^nf)(x)=D_q(D_q^{n-1}f)(x), \quad n\in\mathbb{N}.$$

The q-integral of a function f defined in the interval $[0,b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0,b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q (I_q^{n-1}) f(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [8]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^{\alpha}(t-s)^{(\alpha)}, \quad {}_tD_q (t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha)},$$

$$\left({}_xD_q \int_0^x f(x,t) d_q t \right)(x) = \int_0^x {}_xD_q f(x,t) d_q t + f(qx,x).$$

Denote that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$ [6].

Definition 2.1. [1] *Let $\alpha \ge 0$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0 f(x) = f(x)$ and*

$$(I_q^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, \ x \in [0,1].$$

Definition 2.2. [15] *The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by $D_q^0 f(x) = f(x)$ and*

$$(D_q^{\alpha} f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 2.1. [6] *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:*

$$(I_q^{\alpha} D_q^{\alpha} f)(x) = (D_q^{\alpha} I_q^{\alpha} f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

Lemma 2.2. [7] *Let $y \in C[0,1]$ and $1 < \alpha \le 2$, the unique solution of*

$$D_q^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \quad u(0) = u(1) = 0, \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G(t,qs) y(s) d_q s,$$

where

$$G(t,s)=\frac{1}{\Gamma_q(\alpha)}\begin{cases}(t(1-s))^{(\alpha-1)}-(t-s)^{(\alpha-1)}, & 0\le s\le t\le 1,\\ (t(1-s))^{(\alpha-1)}, & 0\le t\le s\le 1.\end{cases}\tag{2.2}$$

Lemma 2.3. *Let $y\in C[0,1]$, $1<\alpha,\beta\le 2$. The fractional q-difference boundary value problem*

$$D_q^{\beta}(\phi_\mu(D_q^{\alpha}u(t)))=y(t),\quad 0<t<1,\quad u(0)=u(1)=0,\quad D_q^{\alpha}u(0)=D_q^{\alpha}u(1)=0,$$

is given by

$$u(t)=\int_0^1 G(t,qs)\phi_\nu\left(\int_0^1 H(s,q\tau)y(\tau)d_q\tau\right)d_qs,$$

where $G(t,s)$ is defined by (2.2) and

$$H(t,s)=\frac{1}{\Gamma_q(\beta)}\begin{cases}(t(1-s))^{(\beta-1)}-(t-s)^{(\beta-1)}, & 0\le s\le t\le 1,\\ (t(1-s))^{(\beta-1)}, & 0\le t\le s\le 1.\end{cases}\tag{2.3}$$

Proof. The proof is similar to Lemma 2.2, we omit it here. ∎

Lemma 2.4. [7] *Let $1<\alpha,\beta\le 2$. Then functions $G(t,s)$ and $H(t,s)$ defined by (2.2) and (2.3) respectively, are continuous on $[0,1]\times[0,1]$ satisfying*

(a) *$G(t,qs)\ge 0$, $G(t,qs)\le G(qs,qs)$ and $G(t,qs)\ge t^{\alpha-1}G(1,qs)$ for all $t,s\in[0,1]$;*
(b) *$H(t,qs)\ge 0$, $H(t,qs)\le H(qs,qs)$ and $H(t,qs)\ge t^{\beta-1}H(1,qs)$ for all $t,s\in[0,1]$.*

From Lemmas 2.2 and 2.4, it is easy to obtain the following lemma.

Lemma 2.5. *Let $0\le y(t)\in C[0,1]$ and $1<\alpha\le 2$. Then the fractional q-difference boundary value problem (2.1) has a unique solution $u(t)\ge 0$, $t\in[0,1]$.*

Let $E=\{u : u,\phi_\mu(D_q^{\alpha}u)\in C^2[0,1]\}$. Now we introduce the following definitions about the upper and lower solutions of the fractional q-difference boundary value problem (1.1).

Definition 2.3. *A function $\varphi(t)$ is called a lower solution of fractional q-difference boundary value problem (1.1), if $\varphi(t)\in E$ and $\varphi(t)$ satisfies*

$$D_q^{\beta}(\phi_\mu(D_q^{\alpha}\varphi(t)))\le f(t,\varphi(t)),\quad 0<t<1,$$
$$\varphi(0)\le 0,\quad \varphi(1)\le 0,\quad D_q^{\alpha}\varphi(0)\ge 0,\quad D_q^{\alpha}\varphi(1)\ge 0.$$

Definition 2.4. *A function $\psi(t)$ is called an upper solution of fractional q-difference boundary value problem (1.1), if $\psi(t)\in E$ and $\psi(t)$ satisfies*

$$D_q^{\beta}(\phi_\mu(D_q^{\alpha}\psi(t)))\ge f(t,\psi(t)),\quad 0<t<1,$$
$$\psi(0)\ge 0,\quad \psi(1)\ge 0,\quad D_q^{\alpha}\psi(0)\le 0,\quad D_q^{\alpha}\psi(1)\le 0.$$

3. Main results

For the sake of simplicity, we make the following assumptions throughout this paper.

(H_1) $f(t,u)\in C[(0,1)\times(0,+\infty),[0,+\infty)]$ and $f(t,u)$ is nonincreasing relative to u;
(H_2) For any constant $\rho>0$, $f(t,\rho)\not\equiv 0$ and $0<\int_0^1 H(qs,qs)f(s,\rho s^{\alpha-1})d_qs<+\infty$.

We define $P = \{u \in C[0,1] :$ there exists a positive number λ_u such that $u(t) \geq \lambda_u t^{\alpha-1}$, $t \in [0,1]\}$. Obviously, $e(t) = t^{\alpha-1} \in P$. Therefore, P is not empty. And define an operator T by

$$Tu(t) = \int_0^1 G(t,qs)\phi_\nu\left(\int_0^1 H(s,q\tau)f(\tau,u(\tau))d_q\tau\right)d_qs, \quad \forall u \in P.$$

Theorem 3.1. *Suppose that conditions* (H_1)-(H_2) *are satisfied, then the boundary value problem* (1.1) *has at least one positive solution* $u(t)$*, which satisfies* $u(t) \geq \kappa t^{\alpha-1}$ *for some* $\kappa > 0$.

Proof. We will divide our proof into four steps.

Step 1. We show that T is well defined on P and $T(P) \subseteq P$.

Firstly, combining Lemma 2.4 and conditions (H_1)-(H_2), for any $u \in P$, by the definition of P, there exists $\lambda_u > 0$, such that

$$\int_0^1 H(s,q\tau)f(\tau,u(\tau))d_q\tau \leq \int_0^1 H(s,q\tau)f(\tau,\lambda_u\tau^{\alpha-1})d_q\tau < +\infty.$$

Therefore,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,qs)\phi_\nu\left(\int_0^1 H(s,q\tau)f(\tau,u(\tau))d_q\tau\right)d_qs \\ &\leq \int_0^1 G(qs,qs)d_qs \cdot \phi_\nu\left(\int_0^1 H(q\tau,q\tau)f(\tau,\lambda_u\tau^{\alpha-1})d_q\tau\right) < +\infty. \end{aligned}$$

Secondly, it follows from Lemma 2.4 that

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,qs)\phi_\nu\left(\int_0^1 H(s,q\tau)f(\tau,u(\tau))d_q\tau\right)d_qs \\ &\geq t^{\alpha-1}\int_0^1 G(1,qs)\phi_\nu\left(\int_0^1 H(s,q\tau)f(\tau,u(\tau))d_q\tau\right)d_qs = \lambda_{Tu}t^{\alpha-1}, \quad \forall t \in [0,1]. \end{aligned}$$

Consequently, It follows from the above that T is well defined and $T(P) \subseteq P$.

At the same time, by direct computations, we can obtain

$$\begin{gathered} D_q^\beta(\phi_\mu(D_q^\alpha(Tu)(t))) = f(t,u(t)), \quad 0 < t < 1, \\ (Tu)(0) = (Tu)(1) = 0, \quad D_q^\alpha(Tu)(0) = D_q^\alpha(Tu)(1) = 0. \end{gathered} \tag{3.1}$$

Let

$$m(t) = \min\{e(t),(Te)(t)\}, \quad n(t) = \max\{e(t),(Te)(t)\}.$$

Obviously $m(t)$ and $n(t)$ make sense and $m(t) \leq n(t)$.

Step 2. We will prove that the functions $\varphi(t) = Tn(t)$, $\psi(t) = Tm(t)$ are a couple of lower and upper solutions of the fractional q-difference boundary value problem (1.1), respectively.

Since $Te \in P$, it follows that there a positive number λ_{Te} such that $Te(t) \geq \lambda_{Te}e(t)$. Therefore, $m(t) = \min\{1,\lambda_{Te}\}e(t) = \lambda_1 e(t)$. This implies $m(t) \in P$ and $n(t) \in P$. From (H_1), we know that T is nonincreasing relative to u. Furthermore, $Tm(t)$ and $Tn(t)$ make sense and

$$Tn(t) \leq Tm(t) \leq T(\lambda_1 e)(t), \quad t \in [0,1].$$

Therefore, $\varphi(t) \le \psi(t)$. With the aid of the decreasing property of the operator T, it follows that

$$Tn(t) \le Te(t) \le n(t), \quad Tm(t) \ge Te(t) \ge m(t), \quad t \in [0,1]. \tag{3.2}$$

By Step 1, we know $\varphi(t), \psi(t) \in P$. And it follows from (3.1) and (3.2), we obtain

$$\begin{gathered} D_q^\beta(\phi_\mu(D_q^\alpha \varphi(t))) - f(t,\varphi(t)) \le D_q^\beta(\phi_\mu(D_q^\alpha (Tn)(t))) - f(t,n(t)) = 0, \\ \varphi(0) = \varphi(1) = 0, \quad D_q^\alpha \varphi(0) = D_q^\alpha \varphi(1) = 0, \\ D_q^\beta(\phi_\mu(D_q^\alpha \psi(t))) - f(t,\psi(t)) \ge D_q^\beta(\phi_\mu(D_q^\alpha (Tm)(t))) - f(t,m(t)) = 0, \\ \psi(0) = \varphi(1) = 0, \quad D_q^\alpha \psi(0) = D_q^\alpha \psi(1) = 0, \end{gathered}$$

that is, $\varphi(t)$ and $\psi(t)$ are a couple of lower and upper solutions of fractional q-difference boundary value problem (1.1), respectively.

Step 3. We will show that the fractional q-difference boundary value problem

$$\begin{gathered} D_q^\beta(\phi_\mu(D_q^\alpha u(t))) = g(t,u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = 0, \quad D_q^\alpha u(0) = D_q^\alpha u(1) = 0, \end{gathered} \tag{3.3}$$

has a positive solution, where

$$g(t,u(t)) = \begin{cases} f(t,\varphi(t)), & \text{if } u(t) < \varphi(t), \\ f(t,u(t)), & \text{if } \varphi(t) \le u(t) \le \psi(t), \\ f(t,\psi(t)), & \text{if } u(t) > \psi(t). \end{cases}$$

To see this, we consider the operator $A : C[0,1] \to C[0,1]$ defined as follows:

$$Au(t) = \int_0^1 G(t,qs)\phi_\nu\left(\int_0^1 H(s,q\tau)g(\tau,u(\tau))d_q\tau\right)d_qs,$$

where $G(t,s)$ is defined as (2.2), $H(t,s)$ is defined as (2.3). It is clear that $Au \ge 0$, for all $u \in P$, and a fixed point of the operator A is a solution of the boundary value problem (3.3).

Since $\varphi(t) \in P$, there exists a positive number λ_φ such that $\varphi(t) \ge \lambda_\varphi t^{\alpha-1}$, $t \in [0,1]$. It follows from (H_2) that

$$\begin{aligned} \int_0^1 H(q\tau,q\tau)g(\tau,u(\tau))d_q\tau &\le \int_0^1 H(q\tau,q\tau)f(\tau,u(\tau))d_q\tau \\ &\le \int_0^1 H(q\tau,q\tau)f(\tau,\lambda_\varphi\tau^{\alpha-1})d_q\tau < +\infty. \end{aligned}$$

Consequently, for all $u(t) \in C[0,1]$, we have

$$\begin{aligned} Au(t) &= \int_0^1 G(t,qs)\phi_\nu\left(\int_0^1 H(s,q\tau)g(\tau,u(\tau))d_q\tau\right)d_qs \\ &\le \int_0^1 G(qs,qs)\phi_\nu\left(\int_0^1 H(s,q\tau)g(\tau,u(\tau))d_q\tau\right)d_qs \\ &\le \int_0^1 G(qs,qs)d_qs \cdot \phi_\nu\left(\int_0^1 H(q\tau,q\tau)f(\tau,\varphi(\tau))d_q\tau\right) < +\infty, \end{aligned}$$

which implies that the operator A is uniformly bounded.

On the other hand, since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. So, for fixed $s\in[0,1]$ and for any $\varepsilon>0$, there exists a constant $\delta>0$, such that any $t_1,t_2\in[0,1]$ and $|t_1-t_2|<\delta$,

$$|G(t_1,qs)-G(t_2,qs)|<\frac{\varepsilon}{\phi_\nu\left(\int_0^1 H(q\tau,q\tau)f(\tau,\lambda_\varphi\tau^{\alpha-1})d_q\tau\right)}.$$

Then, for all $u(t)\in C[0,1]$, we have

$$\begin{aligned}|Au(t_1)-Au(t_2)| &= \int_0^1 |G(t_1,qs)-G(t_2,qs)|\phi_\nu\left(\int_0^1 H(s,q\tau)g(\tau,u(\tau))d_q\tau\right)d_qs\\ &\le \int_0^1 |G(t_1,qs)-G(t_2,qs)|\phi_\nu\left(\int_0^1 H(s,q\tau)f(\tau,\varphi(\tau))d_q\tau\right)d_qs\\ &\le \int_0^1 |G(t_1,qs)-G(t_2,qs)|d_qs\cdot\phi_\nu\left(\int_0^1 H(q\tau,q\tau)f(\tau,\varphi(\tau))d_q\tau\right)<\varepsilon,\end{aligned}$$

that is to say, A is equicontinuous. Thus, from the Arzela-Ascoli Theorem, we know that A is a compact operator. by using the Schauder's fixed point theorem, the operator A has a fixed point; i.e., the fractional q-difference boundary value problem (3.3) has a positive solution.

Step 4. We will prove that the boundary value problem (1.1) has at least one positive solution. Suppose that $u(t)$ is a solution of (3.3), we only need to prove that $\varphi(t)\le u(t)\le\psi(t)$, $t\in[0,1]$. Now we claim that $\varphi(t)\le u(t)\le\psi(t)$, $t\in[0,1]$. From this it follows that

$$\begin{aligned}&D_q^\beta(\phi_\mu(D_q^\alpha u(t)))=f(t,u(t)),\quad 0<t<1,\\ &u(0)=u(1)=0,\quad D_q^\alpha u(0)=D_q^\alpha u(1)=0.\end{aligned}\tag{3.4}$$

Suppose by contradiction that $u(t)\ge\psi(t)$. According to the definition of g, we have

$$g(t,u(t))=f(t,\psi(t)),\quad 0<t<1.$$

Consequently, we obtain

$$D_q^\beta(\phi_\mu(D_q^\alpha u(t)))=f(t,\psi(t)),\quad 0<t<1.\tag{3.5}$$

On the other hand, since ψ is an upper solution to (1.1), we obviously have

$$D_q^\beta(\phi_\mu(D_q^\alpha\psi(t)))\ge f(t,\psi(t)),\quad 0<t<1.\tag{3.6}$$

Let $z(t)=\phi_\mu(D_q^\alpha\psi(t))-\phi_\mu(D_q^\alpha u(t))$, $0<t<1$. From (3.5) and (3.6), we have

$$D_q^\beta(\phi_\mu(D_q^\alpha\psi(t)))-D_q^\beta(\phi_\mu(D_q^\alpha u(t)))\ge f(t,\psi(t))-f(t,\psi(t))=0,\quad t\in[0,1]$$

and $z(0)=0$, $z(1)=0$. Thus, by Lemma 2.5, we have $z(t)\le0$, $t\in[0,1]$, which implies that

$$\phi_\mu(D_q^\alpha\psi(t))\le\phi_\mu(D_q^\alpha u(t)),\quad t\in[0,1].$$

Since ϕ_μ is monotone increasing, we obtain $D_q^\alpha\psi(t)\le D_q^\alpha u(t)$, i.e., $D_q^\alpha(\psi-u)(t)\le0$. Combining Lemma 2.5, we have $(\psi-u)(t)\ge0$. Therefore, $\psi(t)\ge u(t)$, $t\in[0,1]$, a contradiction to the assumption that $u(t)>\psi(t)$. Hence, $u(t)>\psi(t)$ is impossible.

Similarly, suppose by contradiction that $u(t)\le\varphi(t)$. According to the definition of g, we have

$$g(t,u(t))=f(t,\varphi(t)),\quad 0<t<1.$$

Consequently, we obtain

$$D_q^{\beta}(\phi_\mu(D_q^{\alpha}u(t))) = f(t,\varphi(t)), \quad 0<t<1. \tag{3.7}$$

On the other hand, since ψ is an upper solution to (1.1), we obviously have

$$D_q^{\beta}(\phi_\mu(D_q^{\alpha}\varphi(t))) \le f(t,\varphi(t)), \quad 0<t<1. \tag{3.8}$$

Let $z(t) = \phi_\mu(D_q^{\alpha}u(t)) - \phi_\mu(D_q^{\alpha}\varphi(t))$, $0<t<1$. From (3.7) and (3.8), we have

$$D_q^{\beta}(\phi_\mu(D_q^{\alpha}u(t))) - D_q^{\beta}(\phi_\mu(D_q^{\alpha}\varphi(t))) \ge f(t,\varphi(t)) - f(t,\varphi(t)) = 0, \quad t\in[0,1]$$

and $z(0)=0$, $z(1)=0$. Thus, by Lemma 2.5, we have $z(t)\le 0$, $t\in[0,1]$, which implies that

$$\phi_\mu(D_q^{\alpha}u(t)) \le \phi_\mu(D_q^{\alpha}\varphi(t)), \quad t\in[0,1].$$

Since ϕ_μ is monotone increasing, we obtain $D_q^{\alpha}u(t)\le D_q^{\alpha}\varphi(t)$, i.e., $D_q^{\alpha}(u-\varphi)(t)\le 0$. Combining Lemma 2.5, we have $(u-\varphi)(t)\ge 0$. Therefore, $u(t)\ge\varphi(t)$, $t\in[0,1]$, a contradiction to the assumption that $u(t)<\varphi(t)$. Hence, $u(t)<\varphi(t)$ is impossible.

Consequently, we have that $\varphi(t)\le u(t)\le\psi(t)$, $t\in[0,1]$, that is, $u(t)$ is a positive solution of the boundary value problem (1.1). Furthermore, $\varphi(t)\in P$ implies that there exists a positive constant κ such that $u(t)\ge\varphi(t)\ge\kappa t^{\alpha-1}$, $t\in[0,1]$. Thus, we have finished the proof of Theorem 3.1. ∎

Theorem 3.2. *If $f(t,u)\in C([0,1]\times[0,+\infty),[0,+\infty))$ is decreasing in u and $f(t,\rho)\not\equiv 0$ for any $\rho>0$, then the boundary value problem (1.1) has at least one positive solution $u(t)$, which satisfies $u(t)\ge\kappa t^{\alpha-1}$ for some $\kappa>0$.*

Proof. The proof is similar to Theorem 3.1, we omit it here. ∎

4. Two examples

Example 4.1. Consider the fractional q-difference boundary value problem

$$\begin{aligned} &D_{1/2}^{4/3}(\phi_\mu(D_{1/2}^{3/2}u(t))) = \frac{2(1+\sqrt[3]{t})}{\sqrt{tu(t)}}, \quad 0<t<1,\\ &u(0)=u(1)=0, \quad D_{1/2}^{3/2}u(0)=D_{1/2}^{3/2}u(1)=0. \end{aligned} \tag{4.1}$$

It is easy to check that (H_1) holds. For any $\rho>0$, $f(t,\rho)\not\equiv 0$ and

$$\int_0^1 H(qs,qs)f(s,\rho s^{\alpha-1})d_qs = \frac{1}{\sqrt{\rho}}\int_0^1 H(qs,qs)\frac{2(1+\sqrt[3]{s})}{s^{3/4}}d_qs \approx \frac{1.86460}{\sqrt{\rho}} < +\infty,$$

which implies that (H_2) holds. Theorem 3.1 implies that the boundary value problem (4.1) has at least one positive solution.

Example 4.2. Consider the fractional q-difference boundary value problem

$$\begin{aligned} &D_{1/2}^{4/3}(\phi_\mu(D_{1/2}^{3/2}u(t))) = t^2 + \frac{1}{\sqrt{u(t)+4}}, \quad 0<t<1,\\ &u(0)=u(1)=0, \quad D_{1/2}^{3/2}u(0)=D_{1/2}^{3/2}u(1)=0. \end{aligned} \tag{4.2}$$

It is easy to check that $f(t,u):[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous and decreasing in u and $f(t,\rho)\not\equiv 0$ for any $\rho>0$. Theorem 3.2 implies that the boundary value problem (4.2) has at least one positive solution.

Acknowledgement. The author sincerely thanks the editor and reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript.

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