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Oscillation Theorems for Second Order Nonlinear Differential Equations with Damping

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Abstract. Some oscillation criteria for solutions of a general ordinary differential equation of second order of the form

$$(r(t)\psi(x(t))\dot{x}(t))^{\cdot} + h(t)\dot{x}(t) + q(t)\phi(g(x(t)), r(t)\psi(x(t))\dot{x}(t)) = H(t, x(t), \dot{x}(t))$$

with alternating coefficients are discussed. Our results improve and extend some existing results in the literature. Some illustrative examples are given with its numerical solutions which are computed using Runge Kutta method of fourth order.

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1. Introduction

In this paper, we consider the second order nonlinear ordinary differential equation of the form

$$(r(t)\psi(x(t))\dot{x}(t))^{\cdot} + h(t)\dot{x}(t) + q(t)\phi(g(x(t)), r(t)\psi(x(t))\dot{x}(t)) = H(t, x(t), \dot{x}(t)) \tag{1.1}$$

where r, h and q are continuous functions on the interval $[t_0, \infty)$, $t_0 \geqslant 0$, $\psi \in C(\mathbb{R}, \mathbb{R}^+)$ and $r(t)$ is a positive function. g is a continuous function for $x \in (-\infty, \infty)$, continuously differentiable satisfies $xg(x) > 0$ and $\dot{g}(t)(x) \geqslant k > 0$ for all $x \neq 0$. The function ϕ is continuous function on $\mathbb{R} \times \mathbb{R}$ with $u\phi(u, v) > 0$ for all $u \neq 0$ and $\phi(\lambda u, \lambda v) = \lambda\phi(u, v)$ for any $\lambda \in (0, \infty)$ and H is a continuous function on $[t_0, \infty) \times \mathbb{R} \times \mathbb{R}$ with $H(t, x(t), \dot{x}(t))/g(x(t)) \leqslant p(t)$ for all $x \neq 0$ and $t \geqslant t_0$.

Throughout this paper, we restrict our attention only to the solutions of the differential equation (1.1) which exist on some ray $[t_0, \infty)$. Such solution of the equation (1.1) is said to be oscillatory if it has an infinite number zeros, and otherwise it is said to be non-oscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory, and otherwise it is called non-oscillatory.

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Equation (1.1) is said to be superlinear if

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{dx}{g(x)} < \infty \quad \text{for all}\, \varepsilon > 0.$$

The problem of finding oscillation criteria for second order ordinary differential equations has received a great attention of many authors, see for example [1–15].

Kamenev [7] studied the equation

$$\ddot{x}(t) + q(t)x(t) = 0 \tag{1.2}$$

and proved that the condition

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds = \infty \text{ for some integer } n \geqslant 3,$$

is sufficient for the oscillation of the equation (1.2). Yan [15] proved that if

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds < \infty \text{ for some integer } n \geqslant 3,$$

and there is a continuous function Ω on $[t_0,\infty)$ with

$$\int_{t_0}^{\infty} \Omega_+^2(s)ds = \infty,$$

where $\Omega_+(t) = \max\{\Omega(t), 0\}$, $t \geqslant t_0$ such that

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds \geqslant \Omega(T) \text{ for every } T \geqslant t_0,$$

then every solution of the equation (1.2) oscillates. Philos [11] improved Kamenev's result [7] as follows: He supposed that there exist continuous functions h, $H : D = \{(t,s) : t \geqslant s \geqslant t_0\} \to \mathbb{R}$ such that $H(t,t) = 0$ for $t \geqslant t_0$ and $H(t,s) > 0$ for $t > s \geqslant t_0$. H has a continuous and non-positive partial derivative on D with respect to the second variable such that

$$-\frac{\partial}{\partial s} H(t,s) = h(t,s)\sqrt{H(t,s)} \text{ for all } (t,s) \in D,$$

then, equation (1.2) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)q(s) - \frac{1}{4}h^2(t,s) \right) ds = \infty.$$

Also, Philos [11] extended and improved Yan's result [15] by proving that H and h as in above, moreover, supposed that

$$0 < \inf_{s\geqslant t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] \leqslant \infty,$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} h^2(t,s)ds \leqslant \infty$$

and assume that $\Omega(t)$ as in Yan's result [15] with

$$\int_{t_0}^{\infty} \Omega_+^2(s)ds = \infty.$$

Then, equation (1.2) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)q(s) - \frac{1}{4}h^2(t,s) \right) ds \geqslant \Omega(T) \text{ for every } T \geqslant t_0.$$

Lu and Meng [4] studied the following equation

$$(r(t)\dot{x}(t))^{\cdot} + h(t)\dot{x}(t) + q(t)g(x(t)) = 0 \tag{1.3}$$

and derived some oscillation criteria for equation (1.3).

Sun *et al.* [13] studied a second order nonlinear neutral functional differential equation and also derived some oscillation criteria for the same functional equation.

In this paper, we continue in this direction the study of oscillatory properties of equation (1.1). The purpose of this paper is to improve and extend the above mentioned results. Our results are more general than the previous results.

2. Main results

We state and prove here our oscillation theorems.

Theorem 2.1. *Suppose that*

(1) $a_1 \leqslant \psi(x) \leqslant a_2$, $a_1, a_2 > 0$ *for all* $x \in \mathbb{R}$,
(2) $h(t) \geqslant 0$ *for all* $t \geqslant t_0$,
(3) $q(t) > 0$ *for all* $t \geqslant t_0$.

Moreover, assume that there exist a differentiable function $\rho\colon [t_0,\infty) \to (0,\infty)$, $(\rho h)^{\cdot}(t) \leqslant 0$ *for* $t \geqslant t_0$ *and the continuous functions* h, $H : D = \{(t,s) : t \geqslant s \geqslant t_0\} \to \mathbb{R}$. *The function* H *has a continuous and non-positive partial derivative on* D *with respect to the second variable such that*

$$H(t,t) = 0 \text{ for } t \geqslant t_0 \text{ and } H(t,s) > 0 \text{ for } t > s \geqslant t_0,$$

$$-\frac{\partial}{\partial s}H(t,s) = h(t,s)\sqrt{H(t,s)} \text{ for all } (t,s) \in D.$$

If

(4) $\displaystyle\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \rho(s)r(s)\sigma^2(t,s)ds < \infty$, *where* $\sigma(t,s) = \left[h(t,s) - \frac{\dot{\rho}(t)}{\rho(t)}\sqrt{H(t,s)}\right]$.

(5) $\displaystyle\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)\rho(s)\left(C_0 q(s) - p(s)\right) ds = \infty$, *where* C_0 *is a positive constant* $p\colon [t_0,\infty) \to [0,\infty)$, *then, every solution of superlinear equation* (1.1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0$ on $[T,\infty)$ for some $T \geqslant t_0 \geqslant 0$. We define the function ω as

$$\omega(t) = \frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}\,, t \geqslant T.$$

From $\omega(t)$, Eq. (1.1), condition (1) and since $\phi(1,\frac{\omega(t)}{\rho(t)}) > 0$ then, there exists a positive constant C_0 such that $\phi(1,\omega(t)/\rho(t)) > C_0$ thus, we have

$$\dot{\omega}(t) \leqslant \rho(t)p(t) - \frac{\rho(t)h(t)\dot{x}(t)}{g(x(t))} - C_0\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{k}{a_2\rho(t)r(t)}\omega^2(t)\,, t \geqslant T.$$

Integrating the last inequality multiplied by $H(t,s)$ from T to t, we have

$$\begin{aligned}\int_T^t H(t,s)\rho(s)\left(C_0q(s)-p(s)\right)ds \leqslant{}& -\int_T^t H(t,s)\dot{\omega}(s)ds - \int_T^t \frac{H(t,s)\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds \\ &+\int_T^t \frac{\dot{\rho}(s)}{\rho(s)}H(t,s)\omega(s)ds - \int_T^t \frac{kH(t,s)}{a_2\rho(s)r(s)}\omega^2(s)ds\,, t \geqslant T. \end{aligned} \tag{2.1}$$

From the first integral in the R. H. S. for $t \geqslant T$, we have

$$\begin{aligned} -\int_T^t H(t,s)\dot{\omega}(s)ds &= H(t,T)\omega(T) - \int_T^t \left[-\frac{\partial}{\partial t}H(t,s)\right]\omega(s)ds \\ &= H(t,T)\omega(T) - \int_T^t h(t,s)\sqrt{H(t,s)}\omega(s)ds\,, t \geqslant T. \end{aligned} \tag{2.2}$$

Since H has a continuous and non-positive partial derivative on D with respect to the second variable and ρh is non-increasing. The second integral in the R. H. S. is by using the Bonnet's theorem twice as follows: for $t \geqslant T$, there exists $a_t \in [T,t]$ such that

$$\int_T^t \frac{H(t,s)\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds = H(t,T)\int_T^{a_t} \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds$$

and $b_t \in [T,a_t]$ such that

$$\begin{aligned} H(t,T)\int_T^{a_t} \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds &= H(t,T)\rho(T)h(T)\int_T^{b_t} \frac{\dot{x}(s)}{g(x(s))}ds \\ &= H(t,T)\rho(T)h(T)\int_{x(T)}^{x(b_t)} \frac{du}{g(u)}. \end{aligned}$$

Since H and $\rho(t)$ are positive functions, by condition (2) and the equation (1.1) is superlinear, we have

$$\int_{x(T)}^{x(b_t)} \frac{du}{g(u)} < \begin{cases} 0, \text{if} & x(b_t) < x(T); \\ \int_{x(T)}^{\infty} \frac{du}{g(u)}, \text{if} & x(b_t) \geqslant (T). \end{cases}$$

Thus, it follows that

$$\int_T^t \frac{H(t,s)\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds = H(t,T)\int_T^{a_t} \frac{\rho(s)h(s)\dot{x}(s)}{g(x(s))}ds \geqslant A_1 H(t,T)\rho(T)h(T), \tag{2.3}$$

where $A_1 = \inf \int_{x(T)}^{x(b_t)} \frac{du}{g(u)}$.

Thus, from (2.2) and (2.3), the inequality (2.1) becomes

$$\begin{aligned}
&\int_T^t H(t,s)\rho(s)\left(C_0 q(s) - p(s)\right) ds \\
&\leqslant H(t,T)\omega(T) - A_1 H(t,T)\rho(T)h(T) \\
&- \int_T^t \left[\frac{kH(t,s)}{a_2\rho(s)r(s)}\omega^2(s) + \left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)\sqrt{H(t,s)}\omega(s)\right] ds.
\end{aligned}$$

Since $A_1 H(t,T)\rho(T)h(T) \geqslant 0$ and for $t \geqslant T$, we have

$$\begin{aligned}
&\int_T^t H(t,s)\rho(s)\left(C_0 q(s) - p(s)\right) ds \\
&\leqslant H(t,T)\omega(T) - \int_T^t \left[\frac{kH(t,s)}{a_2\rho(s)r(s)}\omega^2(s) + \sigma(t,s)\sqrt{H(t,s)}\omega(s)\right] ds.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\int_T^t H(t,s)\rho(s)\left(C_0 q(s) - p(s)\right) ds &\leqslant H(t,T)\omega(T) + \int_T^t \frac{a_2\rho(s)r(s)}{4k}\sigma^2(t,s)ds \\
&- \int_T^t \left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}}\omega(s) + \frac{1}{2}\sqrt{\frac{a_2\rho(s)r(s)}{k}}\sigma(t,s)\right]^2 ds. \qquad (2.4)
\end{aligned}$$

Then, for $t \geqslant T$, we have

$$\int_T^t H(t,s)\rho(s)\left(C_0 q(s) - p(s)\right) ds \leqslant H(t,T)\omega(T) + \frac{a_2}{4k}\int_T^t \rho(s)r(s)\sigma^2(t,s)ds, \ t \geqslant T.$$

Dividing the last inequality by $H(t,T)$, taking the limit superior as $t \to \infty$ and by condition (4), we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)\left(C_0 q(s) - p(s)\right) ds \\
&\leqslant \omega(T) + \frac{a_2}{4k}\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \rho(s)r(s)\sigma^2(t,s)ds < \infty,
\end{aligned}$$

which contradicts to the condition (5). Hence, the proof is completed.

∎

Theorem 2.2. *Suppose, in addition to the conditions (1), (2), (3) and (4) hold that there exist continuous functions h, H are defined as in Theorem 2.1 and*

(6) $0 < \inf_{s \geqslant t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] \leqslant \infty.$

If there exists a continuous function Ω on $[t_0,\infty)$ such that

(7)

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)(C_0 q(s) - p(s)) - \frac{a_2}{4k} r(s)\rho(s)\sigma^2(t,s) \right] ds \geqslant \Omega(T) \, for \, T \geqslant t_0,$$

where $\sigma(t,s) = \left[h(t,s) - \frac{\dot\rho(t)}{\rho(t)} \sqrt{H(t,s)} \right]$, k is a positive constant and a differentiable function, $\rho : [t_0,\infty) \to (0,\infty)$,

(8) $\int_T^\infty \frac{\Omega_+^2(s)}{\rho(s)r(s)} ds = \infty$, *where $\Omega_+(t) = \max\{\Omega(t), 0\}$, then every solution of superlinear equation* (1.1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0$ on $[T,\infty)$ for some $T \geqslant t_0 \geqslant 0$.

Dividing (2.4) by $H(t,T)$ and taking the limit superior as $t \to \infty$, we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)(C_0 q(s) - p(s)) - \frac{a_2}{4k} \rho(s) r(s) \sigma^2(t,s) \right] ds \\ &\leqslant \omega(T) - \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}} \omega(s) + \frac{1}{2} \sqrt{\frac{a_2\rho(s)r(s)}{k}} \sigma(t,s) \right]^2 ds \\ &\leqslant \omega(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}} \omega(s) + \frac{1}{2} \sqrt{\frac{a_2\rho(s)r(s)}{k}} \sigma(t,s) \right]^2 ds. \end{aligned}$$

By condition (7), we get

$$\omega(T) \geqslant \Omega(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}} \omega(s) + \frac{1}{2} \sqrt{\frac{a_2\rho(s)r(s)}{k}} \sigma(t,s) \right]^2 ds.$$

This shows that

$$\omega(T) \geqslant \Omega(T) \text{ for every } t \geqslant T, \tag{2.5}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}} \omega(s) + \frac{1}{2} \sqrt{\frac{a_2\rho(s)r(s)}{k}} \sigma(t,s) \right]^2 ds < \infty.$$

Hence,

$$\infty > \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\frac{kH(t,s)}{a_2\rho(s)r(s)}} \omega(s) + \frac{1}{2} \sqrt{\frac{a_2\rho(s)r(s)}{k}} \sigma(t,s) \right]^2 ds$$

$$(2.6)\qquad \geqslant \liminf_{t\to\infty}\left[\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{kH(t,s)}{a_2\rho(s)r(s)}\omega^2(s)ds+\frac{1}{H(t,t_0)}\int_{t_0}^{t}\sigma(t,s)\sqrt{H(t,s)}\omega(s)ds\right].$$

Define

$$U(t)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{kH(t,s)}{\rho(s)r(s)}\omega^2(s)ds,\ t\geqslant t_0$$

and

$$V(t)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}\sigma(t,s)\sqrt{H(t,s)}\omega(s)ds,\ t\geqslant t_0.$$

Then, (2.6) becomes

$$(2.7)\qquad \liminf_{t\to\infty}[U(t)+V(t)]<\infty.$$

Now, suppose that

$$(2.8)\qquad \int_{t_0}^{\infty}\frac{\omega^2(s)}{\rho(s)r(s)}ds=\infty.$$

Then, by condition (6) we can easily see that

$$(2.9)\qquad \lim_{t\to\infty}U(t)=\infty.$$

Let us consider a sequence $\{T_n\}_{n=1,2,3,\ldots}$ in $[t_0,\infty)$ with $\lim_{n\to\infty}T_n=\infty$ and such that

$$\lim_{n\to\infty}[U(T_n)+V(T_n)]=\liminf_{t\to\infty}[U(t)+V(t)].$$

By inequality (2.7) there exists a constant N such that

$$(2.10)\qquad U(T_n)+V(T_n)\leqslant N,\ n=1,2,3,\ldots.$$

From inequality (2.9), we have

$$(2.11)\qquad \lim_{n\to\infty}U(T_n)=\infty.$$

Hence inequality (2.10) gives

$$(2.12)\qquad \lim_{n\to\infty}V(T_n)=-\infty.$$

By taking into account inequality (2.11), from inequality (2.10), we obtain

$$1+\frac{V(T_n)}{U(T_n)}\leqslant\frac{N}{U(T_n)}<\frac{1}{2},$$

provided that n is sufficiently large. Thus

$$\frac{V(T_n)}{U(T_n)}<-\frac{1}{2},$$

which by inequality (2.12) and inequality (2.11), we have

$$(2.13)\qquad \lim_{n\to\infty}\frac{V^2(T_n)}{U(T_n)}=\infty.$$

On the other hand by Schwarz's inequality, we have

$$V^2(T_n) = \frac{1}{H^2(T_n,t_0)}\left[\int_{t_0}^{T_n} \sigma(T_n,s)\sqrt{H(T_n,s)}\omega(s)ds\right]^2$$
$$\leqslant \left[\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{a_2\rho(s)r(s)}{k}\sigma^2(T_n,s)ds\right]\left[\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{kH(T_n,s)}{a_2\rho(s)r(s)}\omega^2(s)ds\right]$$
$$= \frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{a_2\rho(s)r(s)}{k}\sigma^2(T_n,s)ds \times U(T_n).$$

Thus, we have

$$\frac{V^2(T_n)}{U(T_n)} \leqslant \frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \frac{a_2\rho(s)r(s)}{k}\sigma^2(T_n,s)ds \text{ for large } n.$$

By inequality (2.13), we have

$$\frac{a_2}{k}\lim_{n\to\infty}\frac{1}{H(T_n,t_0)}\int_{t_0}^{T_n} \rho(s)r(s)\sigma^2(T_n,s)ds = \infty.$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} \rho(s)r(s)\sigma^2(t,s)ds = \infty,$$

which contradicts to the condition (4), Thus inequality (2.8) fails and hence

$$\int_{t_0}^{\infty} \frac{\omega^2(s)}{\rho(s)r(s)}ds < \infty.$$

Hence from inequality (2.5), we have

$$\int_{t_0}^{\infty} \frac{\Omega_+^2(s)}{\rho(s)r(s)}ds \leqslant \int_{t_0}^{\infty} \frac{\omega^2(s)}{\rho(s)r(s)}ds < \infty,$$

which, contradicts to the condition (8), hence the proof is completed. ∎

Example 2.1. Consider the following differential equation

$$\left(\frac{(x^2(t)+2)}{t^6(x^2(t)+1)}\dot{x}(t)\right)^{\cdot} + \frac{\dot{x}(t)}{t^2} + \frac{1}{t^3}\left(x^7(t) + \frac{x^{133}(t)}{9x^{126}(t)+6\left(\frac{(x^2(t)+2)}{t^6(x^2(t)+1)}\dot{x}(t)\right)^{18}}\right)$$
$$= -\frac{x^9(t)\sin^2(\dot{x}(t))}{(x^2(t)+1)}, \quad t > 0.$$

Here we note $r(t) = 1/t^6$, $\psi(x) = (x^2(t)+2)/(x^2(t)+1)$ for all $x \in \mathbb{R}$, $h(t) = 1/t^2$, $q(t) = 1/t^3$ and $g(x) = x^7$.

$$\phi(u,v) = u + \frac{u^{19}}{9u^{18}+6v^{18}} \quad \text{and} \quad \frac{H(t,x(t),\dot{x}(t))}{g(x(t))} = -\frac{x^2(t)\sin^2(\dot{x}(t))}{(x^2(t)+1)} \leqslant 0 = p(t)$$

for all $t > 0$ and $x \in \mathbb{R}$.

Let $H(t,s) = (t-s)^2 > 0$ for all $t > s \geqslant t_0$, thus $\frac{\partial}{\partial s}H(t,s) = -2(t-s) = -h(t,s)\sqrt{H(t,s)}$ for all $t \geqslant t_0$. Taking $\rho(t) = 6$ such that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \rho(s)r(s)\sigma^2(t,s)ds \\
&= \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \rho(s)r(s)\left(h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right)^2 ds \\
&= \limsup_{t\to\infty} \frac{24}{(t-T)^2} \int_T^t \frac{1}{s^2}ds = 0 < \infty,
\end{aligned}$$

$$\inf_{s\geqslant t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] = \inf_{s\geqslant t_0}\left[\liminf_{t\to\infty}\frac{(t-s)^2}{(t-t_0)^2}\right] = 1 \text{ thus, } 0 < \inf_{s\geqslant t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] < \infty$$

and

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)(C_0q(s)-p(s)) - \frac{a_2}{4k}r(s)\rho(s)\sigma^2(t,s)\right] ds \\
&= \limsup_{t\to\infty} \frac{1}{(t-T)^2} \int_T^t \left[6C_0\frac{(t-s)^2}{s^3} - \frac{12}{ks^6}\right] ds \\
&= \frac{3C_0}{T^2} > \frac{3C_0}{4T^2}.
\end{aligned}$$

Set $\Omega(t) = \frac{3C_0}{4T^2}$, then $\Omega_+(t) = \frac{3C_0}{4T^2}$ and

$$\int_T^\infty \frac{\Omega_+^2(s)}{\rho(s)r(s)}ds = \frac{3C_0^2}{32}\int_T^\infty s^2 ds = \infty.$$

All conditions of Theorem 2.2 are satisfied, thus, the given equation is oscillatory. We also compute the numerical solutions of the given differential equation using the Runge Kutta method of fourth order (RK4). We have

$$\ddot{x}(t) = f(t,x(t),\dot{x}(t)) = -\frac{x^9(t)\sin^2(\dot{x}(t))}{x^2(t)+1} - \left(x^7(t) + \frac{x^{133}(t)}{9x^{126}(t)+6\dot{x}^{18}(t)}\right)$$

with initial conditions $x(1) = -1, \dot{x}(1) = 0.5$ on the chosen interval $[1,100]$, the functions $\psi(x) \equiv 1$ and $h(t) \equiv 0$ and finding values the functions r, q and f where we consider $H(t,x,\dot{x}) = f(t)l(x,\dot{x})$ at $t = 1$, $n = 500$ and $h = 0.198$.

Table 1. Numerical solution of ODE 1

k	t_k	$x(t_k)$
1	1	-1
2	1.198	-0.882
3	1.396	-0.7431
.	.	.
9	2.584	0.1373
10	2.782	0.2844
11	2.98	0.4314
.	.	
26	5.95	-0.1151
27	6.148	-0.2624
28	6.346	-0.4097
.	.	.
43	9.316	0.0959
44	9.514	0.2435
45	9.712	0.391

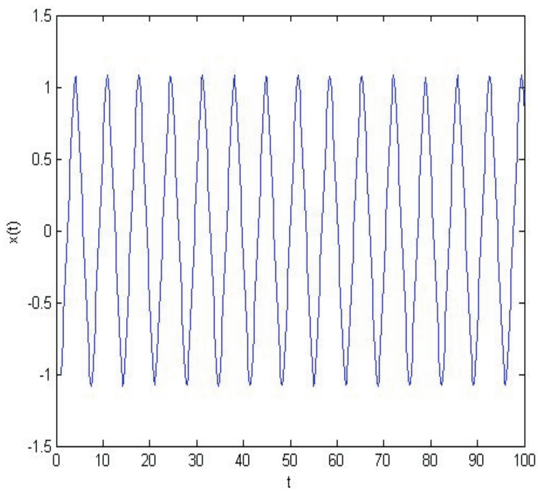

Figure 1. Solution curve of ODE 1

Remark 2.1. Theorem 2.1 and Theorem 2.2 extend and improve results of Kamenev [7], results of Philos [11] and results of Yan [15] who studied the equation (1.1) as $r(t) \equiv 1$, $\psi(x(t)) \equiv 1$, $h(t) \equiv 0$, $\phi(g(x(t)), r(t)\psi(x(t))\dot{x}(t)) \equiv x(t)$ and $H(t, x(t), \dot{x}(t)) \equiv 0$. Also their results [7, 11, 15] cannot be applied to the differential equation in Example 2.1.

We need the following lemma which will significantly simplify the proof of our next theorem.

Let $D = \{(t,s) : t \geqslant s \geqslant t_0\}$, we say that a function $H \in C(D,\mathbb{R})$ belongs to the class W if

(1) $H(t,t) = 0$ for $t \geqslant t_0$ and $H(t,s) > 0$ when $t \neq s$;
(2) $H(t,s)$ has partial derivatives on D such that

$$\frac{\partial}{\partial t}H(t,s) = h_1(t,s)\sqrt{H(t,s)},$$

$\dfrac{\partial}{\partial s}H(t,s) = -h_2(t,s)\sqrt{H(t,s)}$ for all $(t,s) \in D$, and some $h_1,\ h_2 \in L^1_{loc}(D,\mathbb{R})$.

Lemma 2.1. *Let* $A_0,\ A_1,\ A_2 \in C([t_0,\infty),\mathbb{R})$ *with* $A_2 > 0$ *and* $Z \in C^1([t_0,\infty),\mathbb{R})$. *If there exist* $(a,b) \subset [t_0,\infty)$ *and* $c \in (a,b)$ *such that*

$$Z' \leqslant -A_0(s) + A_1(s)Z - A_2(s)Z^2,\ s \in (a,b), \tag{2.14}$$

then

$$\frac{1}{H(c,a)}\int_a^c \left[H(s,a)\rho(s)A_0(s) - \frac{1}{4A_2(s)}\eta_1^2(s,a)\right]ds$$
$$+\frac{1}{H(b,c)}\int_c^b \left[H(b,s)\rho(s)A_0(s) - \frac{1}{4A_2(s)}\eta_2^2(b,s)\right]ds \leqslant 0 \tag{2.15}$$

for all $H \in W$ *and where*

$$\eta_1(s,a) = \left[h_1(s,a) - A_1(s)\sqrt{H(s,a)}\right]$$

and

$$\eta_2(b,s) = \left[h_2(b,s) - A_1(s)\sqrt{H(b,s)}\right].$$

The proof of this lemma is similar to that of Lu and Meng [4] and hence will be omitted.

Theorem 2.3. *Suppose in addition to the condition (3) holds that* $\psi(x) \equiv 1$ *for* $x \in \mathbb{R}$ *and assume that there exist* $c \in (a,b) \subset (T,\infty)$ *and* $H \in W$ *such that*

(9)

$$\frac{1}{H(c,a)}\int_a^c \left[H(s,a)\rho(s)(C_0q(s) - p(s)) - \frac{k}{4\rho(s)r(s)}\eta_1^2(s,a)\right]ds$$
$$+\frac{1}{H(b,c)}\int_c^b \left[H(b,s)\rho(s)(C_0q(s) - p(s)) - \frac{k}{4\rho(s)r(s)}\eta_2^2(b,s)\right]ds > 0,$$

where

$$\eta_1(t,a) = \left[h_1(t,a) - \left(\frac{\dot\rho(t)}{\rho(t)} - \frac{h(t)}{r(t)}\right)\sqrt{H(t,a)}\right],$$
$$\eta_2(b,t) = \left[h_2(b,t) - \left(\frac{\dot\rho(t)}{\rho(t)} - \frac{h(t)}{r(t)}\right)\sqrt{H(b,t)}\right]$$

and the function ρ *is defined as in Theorem 2.1. Then, every solution of equation* (1.1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.1) such that $x(t) > 0$ on $[T,\infty)$ for some $T \geqslant t_0 \geqslant 0$. We define the function ω as

$$\omega(t) = \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))},t \geqslant T.$$

This and (1.1), we obtain

$$\dot{\omega}(t) \leqslant \rho(t)p(t) - \frac{h(t)}{r(t)}\omega(t) - \rho(t)q(t)\phi(1,v_1(t)) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \frac{k}{\rho(t)r(t)}\omega^2(t),t \geqslant T,$$

where $v_1(t) = \frac{\omega(t)}{\rho(t)}$.

Since $\phi(1,v_1(t)) > 0$ then, there exists C_0 such that $\phi(1,v_1(t)) \geqslant C_0$, we have

$$\dot{\omega}(t) \leqslant -\rho(t)(C_0q(t) - p(t)) + \left(\frac{\dot{\rho}(t)}{\rho(t)} - \frac{h(t)}{r(t)}\right)\omega(t) - \frac{k}{\rho(t)r(t)}\omega^2(t),t \geqslant T.$$

From last inequality and by Lemma 2.1, we conclude that for any $c \in (a,b)$ and $H \in W$

$$\frac{1}{H(c,a)}\int_a^c \left[H(s,a)\rho(s)(C_0q(s) - p(s)) - \frac{k}{4\rho(s)r(s)}\eta_1^2(s,a)\right] ds$$
$$+ \frac{1}{H(b,c)}\int_c^b \left[H(b,s)\rho(s)(C_0q(s) - p(s)) - \frac{k}{4\rho(s)r(s)}\eta_2^2(b,s)\right] ds \leqslant 0,$$

where $A_0(t) = \rho(t)(C_0q(t) - p(t))$, $A_1(t) = \left(\frac{\dot{\rho}(t)}{\rho(t)} - \frac{h(t)}{r(t)}\right)$ and $A_2(t) = \frac{k}{\rho(t)r(t)}$.

This contradicts (9). Thus, the equation (1.1) is oscillatory. ∎

Remark 2.2. Theorem 2.3 is an extension of result of Lu and Meng [4].

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