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On the 2-Absorbing Ideals in Commutative Rings

[1]SH. PAYROVI AND [2]S. BABAEI
[1]Imam Khomeini International University, Postal Code: 34149-6818 Qazvin, Iran
[2]Department of Mathematics, Takestan Branch, Islamic Azad University, Takestan, Iran
[1]shpayrovi@ikiu.ac.ir, [2]sakine-babaei@yahoo.com

Abstract. Let R be a commutative ring with identity. In this article, we study a generalization of prime ideal. A proper ideal I of R is called a 2-absorbing ideal if whenever $abc \in I$ for $a,b,c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$. It is shown that if I is a 2-absorbing ideal of a Noetherian ring R, then R/I has some ideals J_n, where $1 \leq n \leq t$ and t is a positive integer, such that J_n possesses a prime filtration $F_{J_n}: \ 0 \subset R(x_1+I) \subset R(x_1+I) \oplus R(x_2+I) \subset \cdots \subset R(x_1+I) \oplus \cdots \oplus R(x_n+I) = J_n$ with $\mathrm{Ass}_R(J_n) = \{I :_R x_i \mid i = 1,\ldots,n\}$ and $|\mathrm{Ass}_R(J_n)| = n$. Also, a 2-Absorbing Avoidance Theorem is proved.

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1. Introduction

In this article, we study 2-absorbing ideals in commutative rings with non-zero identity, which are a generalization of prime ideals. The concept of 2-absorbing ideals was introduced and investigated in [1, 2]. A proper ideal I of a commutative ring R is called a 2-absorbing ideal if whenever $abc \in I$ for $a,b,c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$. The reader is referred to [1] and [2] for more results and examples on 2-absorbing ideals.

For any Noetherian module M over a commutative ring R, there exists a chain $F_M : 0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ in which the factors $M_i/M_{i-1} (i = 1,\ldots,n)$ are isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. These chains are useful devices in the study of Noetherian modules. Following the notation of [4], we call such a chain a prime filtration of M and we denote the set of prime ideals occurring as a factor in F_M by $\mathbb{P}(F_M)$ and the length of F_M by $l(F_M)$. It easily follows that $\mathrm{Ass}(M) \subseteq \mathbb{P}(F_M)$ and $|\mathrm{Ass}(M)| \leq n = l(F_M)$. In [4], A. Li studied finitely generated modules M over a Noetherian ring R for which there exists a prime filtration F_M such that $\mathrm{Ass}(M) = \mathbb{P}(F_M)$ and $|\mathrm{Ass}(M)| = l(F_M)$.

This article is devoted to the study of 2-absorbing ideals and construction of some prime filtrations F_M, for R-modules M, such that $\mathrm{Ass}(M) = \mathbb{P}(F_M)$ and $|\mathrm{Ass}(M)| = l(F_M)$.

Let I be a 2-absorbing ideal of a commutative ring R. In section 2, the basic properties of the ideals $I :_R x$ are studied. It is shown that $I :_R x$ is a 2-absorbing ideal of R, and $\{I :_R$

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$x \mid x \in R\}$ is a totally ordered set. Also, it is shown that if R is a Noetherian ring, then R/I has some ideals $J_n = R(x_1 + I) \oplus \cdots \oplus R(x_n + I)$, where $1 \leq n \leq t$ and t is a positive integer, which are the direct sum of cyclic R-modules. Furthermore, $\mathrm{Ass}(J_n) = \{I :_R x_1, \ldots, I :_R x_n\}$.

The Prime Avoidance Theorem [6, 3.61], states that: let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$, where $n \geq 2$, be ideals of R such that at most two of $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ are not prime. Let I be an ideal of R such that $I \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_n$. Then $I \subseteq \mathfrak{p}_i$ for some i with $1 \leq i \leq n$. In section 3, we prove a 2-Absorbing Avoidance Theorem. Precisely, we prove: let $I_1, I_2, \ldots, I_n$, where $n \geq 2$, be ideals of R such that at most two of $I_1, I_2, \ldots, I_n$ are not 2-absorbing and let $I_i \nsubseteq I_j :_R x$, for all $x \in \sqrt{I_j} \setminus I_j$ with $i \neq j$. Let I be an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$. Then $I \subseteq I_i$ for some i with $1 \leq i \leq n$.

The following are some basic facts about primary and 2-absorbing ideals in a commutative ring R.

Theorem 1.1. [6, Lemma 4.14] *Let Q be a $\mathfrak{p}$-primary ideal of R and let $x \in R$.*

(i) *If $x \in Q$, then $Q :_R x = R$.*
(ii) *If $x \notin Q$, then $Q :_R x$ is a $\mathfrak{p}$-primary ideal of R, so that, $\sqrt{Q :_R x} = \mathfrak{p}$.*
(iii) *If $x \notin \mathfrak{p}$, then $Q :_R x = Q$.*

Theorem 1.2. [2, Theorem 2.4] *Let I be a 2-absorbing ideal of R. Then one of the following statements must hold:*

(i) *$\sqrt{I} = \mathfrak{p}$ is a prime ideal of R such that $\mathfrak{p}^2 \subseteq I$.*
(ii) *$\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2$, $\mathfrak{p}_1\mathfrak{p}_2 \subseteq I$, and $(\sqrt{I})^2 \subseteq I$, where $\mathfrak{p}_1, \mathfrak{p}_2$ are the only distinct prime ideals of R that are minimal over I.*

Theorem 1.3. [2, Theorem 2.5] *Let I be a 2-absorbing ideal of R such that $\sqrt{I} = \mathfrak{p}$ is a prime ideal of R and suppose that $I \neq \mathfrak{p}$. Then for each $x \in \mathfrak{p} \setminus I$, $I :_R x$ is a prime ideal of R containing $\mathfrak{p}$. Furthermore, either $I :_R x \subseteq I :_R y$ or $I :_R y \subseteq I :_R x$, for every $x, y \in \mathfrak{p} \setminus I$.*

Theorem 1.4. [2, Theorem 2.6] *Let I be a 2-absorbing ideal of R such that $\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2$ where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are the only non-zero distinct prime ideals of R that are minimal over I. Then for each $x \in \sqrt{I} \setminus I$, $I :_R x$ is a prime ideal of R containing $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Furthermore, either $I :_R x \subseteq I :_R y$ or $I :_R y \subseteq I :_R x$, for every $x, y \in \sqrt{I} \setminus I$.*

Throughout this article, R denotes a commutative ring with non-zero identity and I is an ideal of R. Let $\sqrt{I} = \{r \in R \, : \text{ there exists } n \in \mathbb{N} \text{ with } r^n \in I\}$ denote the radical of I and let $I :_R x$ denote the ideal $\{r \in R : rx \in I\}$ of R. We say that $\mathfrak{p} \in Spec(R)$ is an associated prime ideal of an R-module M if there exists a non-zero element $m \in M$ such that $0 :_R m = \mathfrak{p}$. The set of associated prime ideals of M is denoted by $\mathrm{Ass}(M)$. For notations and terminologies not given in this article, the reader is referred to [6].

2. The results

Let I be a 2-absorbing ideal of R and $x \in R$. In the following, we study the ideals $I :_R x$, where $x \notin \sqrt{I}$. First of all, the following example shows that $\{I :_R x \mid I \subset I :_R x \subset \sqrt{I}, x \notin \sqrt{I}\}$ may be a non-empty set. Suppose that $R = \mathbb{Z}[x, y, z]$, where $\mathbb{Z}$ is the ring of integers, x, y, z are indeterminates, $I = (4, 2x, 2y, xy, xz, x^2)R$ and $\mathfrak{p} = (2, x)R$. Example 2.12, in [2], shows that I is a 2-absorbing ideal of R and $\sqrt{I} = \mathfrak{p}$. It is easy to see that $z \notin \sqrt{I}$, $x \in I :_R z \setminus I$, $2 \in \sqrt{I} \setminus I :_R z$ and $I \subset I :_R z \subset \sqrt{I}$.

Theorem 2.1. *Let I be a 2-absorbing ideal of R and let $\mathfrak{p},\mathfrak{q}$ be distinct prime ideals of R.*

(i) *If $\sqrt{I}=\mathfrak{p}$, then $I:_R x$ is a 2-absorbing ideal of R with $\sqrt{I:_R x}=\mathfrak{p}$, for all $x\in R\setminus\mathfrak{p}$, and $\Sigma=\{I:_R x \mid x\in R\}$ is a totally ordered set.*

(ii) *If $\sqrt{I}=\mathfrak{p}\cap\mathfrak{q}$, then $I:_R x$ is a 2-absorbing ideal of R with $\sqrt{I:_R x}=\mathfrak{p}\cap\mathfrak{q}$, for all $x\in R\setminus\mathfrak{p}\cup\mathfrak{q}$, and $\Sigma=\{I:_R x \mid x\in R\setminus\mathfrak{p}\cup\mathfrak{q}\}$ is a totally ordered set.*

(iii) *If $\sqrt{I}=\mathfrak{p}\cap\mathfrak{q}$, then $I:_R x=\mathfrak{q}$, for all $x\in\mathfrak{p}\setminus\mathfrak{q}$, and $I:_R x=\mathfrak{p}$, for all $x\in\mathfrak{q}\setminus\mathfrak{p}$. Also $I:_R x$ is a prime ideal of R containing $\mathfrak{p}$ and $\mathfrak{q}$, for all $x\in\mathfrak{p}\cap\mathfrak{q}\setminus I$.*

Proof. (i) Let $x\in R\setminus\mathfrak{p}$ and let $a,b,c\in R$ be such that $abc\in I:_R x$. Then $abcx\in I$. So $ax\in I$ or $bcx\in I$ or $abc\in I$ since I is a 2-absorbing ideal of R. If either $ax\in I$ or $bcx\in I$, we are done. If $abc\in I$, then $ab\in I$ or $ac\in I$ or $bc\in I$, which implies $abx\in I$ or $acx\in I$ or $bcx\in I$. Hence, $I:_R x$ is a 2-absorbing ideal of R. It is easy to see that $I\subseteq I:_R x\subseteq\mathfrak{p}$, so that, $\sqrt{I:_R x}=\mathfrak{p}$.

For the second assertion, suppose that $x,y\in R\setminus\mathfrak{p}$. It is clear that $xy\in R\setminus\mathfrak{p}$. Also, it is clear that $I:_R x\subseteq I:_R xy$ and $I:_R y\subseteq I:_R xy$. Thus $(I:_R x)\cup(I:_R y)\subseteq I:_R xy$. To establish the reverse inclusion, let $z\in I:_R xy$. Then $xyz\in I$. It follows that either $xz\in I$ or $yz\in I$ since $xy\notin I$. Thus either $z\in I:_R x$ or $z\in I:_R y$. Hence, $I:_R xy\subseteq(I:_R x)\cup(I:_R y)$, and therefore $I:_R xy=(I:_R x)\cup(I:_R y)$. So it follows that either $I:_R xy=I:_R x$ or $I:_R xy=I:_R y$. Thus either $I:_R x\subseteq I:_R y$ or $I:_R y\subseteq I:_R x$. Therefore, $\Sigma'=\{I:_R x \mid x\in R\setminus\mathfrak{p}\}$ is a totally ordered set. On the other hand, for each $x\in\mathfrak{p}\setminus I$, $I:_R x$ is a prime ideal of R containing $\mathfrak{p}$ and $\Sigma''=\{I:_R x \mid x\in\mathfrak{p}\setminus I\}$ is a totally ordered set, by 1.3. Hence, $\Sigma=\{I:_R x \mid x\in R\}$ is a totally ordered set.

(ii) By a similar argument to that of (i), we can prove $I:_R x$ is a 2-absorbing ideal of R and $\sqrt{I:_R x}=\mathfrak{p}\cap\mathfrak{q}$, for each $x\notin\mathfrak{p}\cup\mathfrak{q}$. Also, it is easy to see $\Sigma=\{I:_R x \mid x\in R\setminus\mathfrak{p}\cup\mathfrak{q}\}$ is a totally ordered set.

(iii) Assume that $x\in\mathfrak{p}\setminus\mathfrak{q}$. We show $I:_R x=\mathfrak{q}$. It is easy to see that $I:_R x\subseteq\mathfrak{q}$. Now, suppose that $z\in\mathfrak{q}$. Thus $xz\in\mathfrak{p}\mathfrak{q}$ and $\mathfrak{p}\mathfrak{q}\subseteq I$, by 1.2(ii). Hence, $xz\in I$ and $z\in I:_R x$. So $I:_R x=\mathfrak{q}$. By a similar argument, we can show that $I:_R x=\mathfrak{p}$ whenever $x\in\mathfrak{q}\setminus\mathfrak{p}$. The last claim follows by 1.4. ▮

Corollary 2.1. *Let I be a 2-absorbing ideal of R and let $\mathfrak{p},\mathfrak{q}$ be distinct prime ideals of R.*

(i) *If $\sqrt{I}=\mathfrak{p}$, then $\mathrm{Ass}(R/I)$ is a totally ordered set.*

(ii) *If $\sqrt{I}=\mathfrak{p}\cap\mathfrak{q}$, then $\mathrm{Ass}(R/I)$ is the union of two totally ordered sets.*

Proof. (i) Let $\mathfrak{q}'\in\mathrm{Ass}(R/I)$. Then there exists $x\in R\setminus I$ such that $\mathfrak{q}'=I:_R x$. If $x\notin\mathfrak{p}$, then $\mathfrak{q}'=\mathfrak{p}$, by 2.1(i). Otherwise, $\mathfrak{p}\subseteq I:_R x=\mathfrak{q}'$, by 1.3. Also, $\mathrm{Ass}(R/I)$ is a totally ordered set.

(ii) Let $\mathfrak{q}'\in\mathrm{Ass}(R/I)$. Then there exists $x\in R\setminus I$ such that $\mathfrak{q}'=I:_R x$. If $x\notin\mathfrak{p}\cap\mathfrak{q}$, then in view of 2.1 (ii) and (iii), either $\mathfrak{q}'=\mathfrak{p}$ or $\mathfrak{q}'=\mathfrak{q}$. If $x\in\mathfrak{p}\cap\mathfrak{q}$, then we have $\mathfrak{p}\subseteq\mathfrak{q}'$ and $\mathfrak{q}\subseteq\mathfrak{q}'$, by 1.4. Also, $\mathrm{Ass}(R/I)$ is the union of two totally ordered sets. ▮

The following theorem offers some R-modules M for which there exists a prime filtration F_M such that $\mathrm{Ass}(M)=\mathbb{P}(F_M)$ and $|\mathrm{Ass}(M)|=l(F_M)$.

Theorem 2.2. *Let R be a Noetherian ring and let I be a 2-absorbing ideal of R. Then there are $x_1,\ldots,x_n\in R$ with $1\leq n\leq t$, where t is a positive integer, and ideals $J_n=R(x_1+I)\oplus\cdots\oplus R(x_n+I)$ of R/I such that*

$$\mathrm{Ass}(J_n)=\{I:_R x_1,\ldots,I:_R x_n\}.$$

Proof. Let $x_1,\dots,x_m \in R$ be such that $\sqrt{I}/I = (x_1+I,\dots,x_m+I)$. Then in view of 1.3 and 1.4, we have a chain $I :_R x_1 \subseteq \cdots \subseteq I :_R x_m$ of prime ideals of R. Now, we can omit any superfluous terms to obtain a strictly ascending chain $I :_R x_1 \subset \cdots \subset I :_R x_t$, where $1 \le t \le m$ is a positive integer. Thus $x_1+I,\dots,x_t+I$ is an associated sequence of R/I, see [4] Definition 1.4. Hence, in view of Theorem 3.2 in [4], for every $1 \le n \le t$, the ideal J_n of R/I generated by $x_1+I,\dots,x_n+I$ is the direct sum $R(x_1+I)\oplus\cdots\oplus R(x_n+I)$. Also,

$$0 \subset R(x_1+I) \subset R(x_1+I)\oplus R(x_2+I) \subset \cdots \subset R(x_1+I)\oplus\cdots\oplus R(x_n+I) = J_n$$

is a prime filtration of submodules of J_n with $\oplus_{j=1}^{i} R(x_j+I)/\oplus_{j=1}^{i-1} R(x_j+I) \cong R/I :_R x_i$ and

$$\mathrm{Ass}(J_n) = \{I :_R x_1,\dots,I :_R x_n\}.$$

An ideal I of R is said to be irreducible precisely when I is proper and I cannot be expressed as the intersection of two strictly larger ideals of R. The following theorem shows the relationship between irreducible and 2-absorbing ideals.

Theorem 2.3. *Let I be an irreducible ideal of R and let $\mathfrak{p},\mathfrak{q}$ be distinct prime ideals of R.*

(i) *If $\sqrt{I} = \mathfrak{p}$, then I is 2-absorbing if and only if $\mathfrak{p}^2 \subseteq I$ and $I :_R x = I :_R x^2$, for all $x \in R\setminus\mathfrak{p}$.*

(ii) *If $\sqrt{I} = \mathfrak{p}\cap\mathfrak{q}$, then I is 2-absorbing if and only if $\mathfrak{p}\mathfrak{q} \subseteq I$ and $I :_R x = I :_R x^2$, for all $x \in R\setminus\mathfrak{p}\cap\mathfrak{q}$.*

Proof. (i) ($\Rightarrow$) By 1.2(i), we have $\mathfrak{p}^2 \subseteq I$. Assume that $x \in R\setminus\mathfrak{p}$. We have to show that $I :_R x = I :_R x^2$. It is clear that $I :_R x \subseteq I :_R x^2$. For the reverse inclusion, let $y \in I :_R x^2$. So $x^2y \in I$. This implies that either $xy \in I$ or $x^2 \in I$ since I is 2-absorbing. If $xy \in I$ we are done. Otherwise, $x^2 \in I$ which implies that $x \in \mathfrak{p}$ and this is a contradiction.

($\Leftarrow$) Assume that $x,y,z \in R$, $xyz \in I$ and $xy \notin I$. We show that either $xz \in I$ or $yz \in I$. From $xy \notin I$ it follows that $x \notin \mathfrak{p}$ or $y \notin \mathfrak{p}$. Otherwise, $x \in \mathfrak{p}$ and $y \in \mathfrak{p}$. Thus $xy \in \mathfrak{p}^2 \subseteq I$. But this is a contradiction because $xy \notin I$. So that, by assumption, either $I :_R x = I :_R x^2$ or $I :_R y = I :_R y^2$. Suppose that $I :_R x = I :_R x^2$. To establish the claim, suppose, on the contrary, that $xz \notin I$ and $yz \notin I$. We look for a contradiction. Let $a \in (I+xz)\cap(I+yz)$. Then there are $a_1,a_2 \in I$ and $r_1,r_2 \in R$ such that $a = a_1 + r_1xz = a_2 + r_2yz$. Thus $ax = a_1x + r_1x^2z = a_2x + r_2xyz \in I$. So that $r_1x^2z \in I$, and therefore $r_1xz \in I$ since $I :_R x = I :_R x^2$. Hence, $a = a_1 + r_1xz \in I$. This shows that $(I+xz)\cap(I+yz) \subseteq I$, and then $(I+xz)\cap(I+yz) = I$. But this is a contradiction since I is irreducible. Thus we have shown that either $xz \in I$ or $yz \in I$, as claimed.

(ii) This can be proved, by using 1.2(ii), in a very similar manner to the way in which (i) was proved.

3. 2-Absorbing Avoidance Theorem

The first major result of this section (Theorem 3.2) is a 2-Absorbing Avoidance Theorem. We will need the following theorem.

Theorem 3.1. *Let $I_1,I_2,\dots,I_n$ ($n \ge 2$) be ideals of R such that at most two of $I_1,I_2,\dots,I_n$ are not 2-absorbing. If I is an ideal of R and $I \subseteq I_1\cup I_2\cup\cdots\cup I_n$, then $\sqrt{I} \subseteq \sqrt{I_i}$ for some i with $1 \le i \le n$.*

Proof. We may assume that I_i is a 2-absorbing ideal of R, for all $i > 2$. Let $2 < k \leq n$ be such that $\sqrt{I_i} = \mathfrak{p}_i$, for all i with $2 < i \leq k$, and $\sqrt{I_i} = \mathfrak{p}_{i,1} \cap \mathfrak{p}_{i,2}$, for all i with $k+1 \leq i \leq n$, where $\mathfrak{p}_i, \mathfrak{p}_{i,j}$, for $j = 1,2$, are prime ideals of R. Then

$$\sqrt{I} \subseteq \sqrt{I_1} \cup \sqrt{I_2} \cup \mathfrak{p}_3 \cup \cdots \cup \mathfrak{p}_k \cup (\mathfrak{p}_{k+1,1} \cap \mathfrak{p}_{k+1,2}) \cup \cdots \cup (\mathfrak{p}_{n,1} \cap \mathfrak{p}_{n,2})$$

and so

$$\sqrt{I} \subseteq \sqrt{I_1} \cup \sqrt{I_2} \cup \mathfrak{p}_3 \cup \cdots \cup \mathfrak{p}_k \cup \mathfrak{p}_{k+1,j_{k+1}} \cup \cdots \cup \mathfrak{p}_{n,j_n},$$

where $j_{k+1}, \ldots, j_n \in \{1,2\}$. By the Prime Avoidance Theorem [6, 3.61], we have $\sqrt{I} \subseteq \sqrt{I_1}$ or $\sqrt{I} \subseteq \sqrt{I_2}$ or $\sqrt{I} \subseteq \mathfrak{p}_i$, with $3 \leq i \leq k$, or $\sqrt{I} \subseteq \mathfrak{p}_{i,j_s}$, where $k+1 \leq i \leq n$ and $j_s \in \{1,2\}$. If $\sqrt{I} \subseteq \sqrt{I_1}$ or $\sqrt{I} \subseteq \sqrt{I_2}$ or $\sqrt{I} \subseteq \mathfrak{p}_i$, for some $3 \leq i \leq k$, we are done. If $\sqrt{I} \subseteq \cup_{i=k+1}^{n} \mathfrak{p}_{i,j}$, then we may assume that $\sqrt{I} \subseteq \cap_{i=k+1}^{s} \mathfrak{p}_{i,1}$ and $\sqrt{I} \not\subseteq \cup_{i=s+1}^{n} \mathfrak{p}_{i,1}$, where $k+1 \leq s \leq n$. Now, $\sqrt{I} \subseteq \mathfrak{p}_{k+1,2} \cup \cdots \cup \mathfrak{p}_{s,2} \cup \mathfrak{p}_{s+1,1} \cup \ldots \cup \mathfrak{p}_{n,1}$ yields $\sqrt{I} \subseteq \mathfrak{p}_{j,2}$, for some $k+1 \leq j \leq s$. Thus $\sqrt{I} \subseteq \mathfrak{p}_{j,1} \cap \mathfrak{p}_{j,2} = \sqrt{I_j}$, and this completes the proof. ∎

Corollary 3.1. *Let $I_1, I_2, \ldots, I_n$ be 2-absorbing ideals of R, and suppose that I is an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$. Then $I^2 \subseteq I_i$, for some i with $1 \leq i \leq n$.*

Proof. The claim follows by 3.1 and 1.2. ∎

Let $I, I_1, I_2, \ldots, I_n$ be ideals of R. Following [3], we call a covering $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ efficient if no I_k is superfluous. Analogously, we shall say that $I = I_1 \cup I_2 \cup \cdots \cup I_n$ is an efficient union if none of the I_k may be excluded. Any cover or union consisting of ideals of R can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.

Theorem 3.2. *(2-Absorbing Avoidance Theorem) Let $I_1, I_2, \ldots, I_n$ $(n \geq 2)$ be ideals of R such that at most two of $I_1, I_2, \cdots, I_n$ are not 2-absorbing and let $I_i \not\subseteq I_j :_R x$, for all $x \in \sqrt{I_j} \setminus I_j$ with $i \neq j$. Let I be an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$. Then $I \subseteq I_i$, for some i with $1 \leq i \leq n$.*

Proof. We suppose that $I \not\subseteq I_j$ for all j with $1 \leq j \leq n$ and look for a contradiction. Our assumption means that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ is an efficient covering of ideals of R. Hence, $I = \cup_{i=1}^{n}(I_i \cap I)$ is an efficient union. Therefore, $(\bigcap_{i \neq k} I_i) \cap I \subseteq I_k \cap I$, by Lemma 2.1 in [5]. On the other hand, at most two of the I_i are not 2-absorbing. We can, and do, assume that they have been indexed in such a way that I_i is a 2-absorbing ideal, for all $i > 2$. Now, a similar argument to that of 3.1 shows that either $I \subseteq I_1 \cup I_2$ or $I \subseteq \sqrt{I_i}$, for some i with $2 < i \leq n$. In the former case, we have a contradiction since by assumption $I \not\subseteq I_j$, for all j with $1 \leq j \leq n$; the second possibility leads to the following contradiction. Assume that $I \subseteq \sqrt{I_j}$, for some $2 < j \leq n$. Thus there exists $x \in \sqrt{I_j} \setminus I_j$ such that $x \in I \setminus I_j$. Moreover, there are $r_i \in I_i \setminus (I_j :_R x)$ for all $i \neq j$, by hypothesis. Let $r = \prod_{i \neq j} r_i$. Then $rx \in (\bigcap_{i \neq j} I_i \cap I)$. Also, $rx \notin I_j \cap I$, for otherwise, $r \in I_j :_R x$, and this is a contradiction since $I_j :_R x$ is a prime ideal of R by 1.3 and 1.4. Therefore, there is a j with $1 \leq j \leq n$ such that $I \subseteq I_j$ and the proof is completed. ∎

Corollary 3.2. *Let $I_1, I_2, \ldots, I_n$ $(n \geq 2)$ be ideals of R such that at most two of $I_1, I_2, \ldots, I_n$ are not 2-absorbing and let $I_i \not\subseteq I_j :_R x$, for all $x \in \sqrt{I_j} \setminus I_j$ with $i \neq j$. Let I be an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$. Then $(\cup_{i=1}^{n} I_i) :_R I = \cup_{i=1}^{n}(I_i :_R I)$.*

There is a refinement of the 2-Absorbing Avoidance Theorem that is sometimes extremely useful.

Theorem 3.3. *Let $I_1, I_2, \dots, I_n$ be ideals of R such that at most one of $I_1, I_2, \dots, I_n$ are not 2-absorbing and let $I_i \not\subseteq I_j :_R x$, for all $x \in \sqrt{I_j} \setminus I_j$ whenever $i \neq j$. Let I be an ideal of R and let $e \in R$ be such that $I + e \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$. Then $I + e \subseteq I_i$ for some i with $1 \leq i \leq n$.*

Proof. Without destroying our assumption, we may assume that I_1 is not 2-absorbing. By using Theorem 12 in [3], and by a similar argument to that of 3.1, we can show either $I + e \subseteq I_1$ or $I + e \subseteq \sqrt{I_j}$, for some $2 \leq j \leq n$. If $I + e \subseteq I_1$, we are done. If $I + e \subseteq \sqrt{I_j}$ and $\sqrt{I_j} = I_j$, then the claim follows. Assume that $I + e \subseteq \sqrt{I_j}$ and $\sqrt{I_j} \neq I_j$. Thus $I \subseteq \sqrt{I_j}$ since $e \in \sqrt{I_j}$. If $I \not\subseteq I_j$, then there exists $x \in I \setminus I_j$, and by assumption, there are $r_i \in I_i \setminus I_j :_R x$, for all $i \neq j$. If $I + e \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ is an efficient covering of I, then $I \cap (\cap_{i \neq j} I_i) \subseteq I \cap I_j$, which implies that $\prod_{i \neq j} r_i \in I_j :_R x$, and therefore $r_i \in I_j :_R x$, for some $i \neq j$, which is a contradiction. Thus the covering is not efficient, and so $I + e \subseteq I_i$ for some i with $1 \leq i \leq n$. ∎

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