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Topological Structures on LA-Semigroups

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Abstract. In this short paper topological spaces using ideal theory on LA-semigroups are introduced. The formation of topological spaces guarantee preservation of finite intersection and arbitrary union between the set of ideals and the open subsets of resultant topologies.

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1. Introduction

An LA-semigroup (LA-semigroup) [2] is a groupoid S with left invertive law

(1.1) $$(ab)c = (cb)a, \text{ for all } a,\, b,\, c \in S.$$

Every LA-semigroup S satisfy the medial law [2]

(1.2) $$(ab)(cd) = (ac)(bd), \text{ for all } a,b,c,d \in S.$$

In every LA-semigroup with left identity the following laws [5] hold

(1.3) $$(ab)(cd) = (db)(ca), \text{ for all } a,b,c,d \in S.$$

(1.4) $$a(bc) = b(ac), \text{ for all } a,b,c,d \in S.$$

Many characteristics of LA-semigroups are similar to a commutative semigroup. Some of these are studied in [3] and [4].

The aim of this short paper is to show that in appropriate LA-semigroups, certain interesting sets of ideals are in fact closed under arbitrary union and finite intersection. This is accomplished via the introduction of new topological structures to this setting.

As in [1], a subset I of an LA-semigroup S is called a right (left) ideal if $IS \subseteq I$ $(SI \subseteq I)$, and is called an ideal if it is two sided ideal. If I is a left ideal of an LA-semigroup S with left identity then using (1.1) and (1.4), I^2 becomes an ideal of S. By a bi-ideal of an LA-semigroup S, we mean an LA-sub-semigroup B of S such that $(BS)B \subseteq B$. It is easy to see

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that each right ideal is a bi-ideal. If S has a left identity and B is a bi-ideal of S then, using the fact that $ab = (ba)e$ for any $\{a,b\} \subseteq S$, it follows that B^2 is a bi-ideal of S and that $B^2 \subseteq SB^2 = B^2S$ because

$$\begin{aligned}[(b_1b_2)s](b_3b_4) &= [(b_1b_2)b_3](sb_4) = [(b_1b_2)b_3][(b_4s)e] \\ &= [(b_1b_2)(b_4s)](b_3e) = \{[(b_4s)b_2]b_1\}(b_3e) \\ &= [(b_3e)b_1][(b_4s)b_2] \in B^2.\end{aligned}$$

Also

$$s(b_1b_2) = (es)[(b_2b_1)e] = [e(b_2b_1)](se) \in B^2S.$$

Then

$$(b_1b_2)s = (sb_2)b_1 = [(es)b_2](eb_1) = [(es)e](b_2b_1) \in SB^2.$$

If $E(B_S)$ denotes the set of all idempotents subsets of S with left identity e, then $E(B_S)$ forms a semilattice structure. Also, if $C = C^2$ then $(CS)C \in E(B_S)$. The intersection of any set of bi-ideals of an LA-semigroup S is either empty or a bi-ideal of S. Also the intersection of prime bi-ideals of an LA-semigroup S is a semiprime bi-ideal of S.

An element a_0 of an LA-semigroup S is called a left (right) zero if $a_0a = a_0(aa_0 = a_0)$ for all $a \in S$ and is called zero if $a_0a = aa_0 = a_0$, for all $a \in S$. We denote the zero element of S (if it contains one) by 0. Now if $0 \in S$, then $0s = s0 = 0$, for all s in S. Let us denote an LA-semigroup S with 0 by S^0.

Example 1.1. Let $S^0 = \{0,1,2,3\}$. Then S^0 under the binary operation "·" defined below is an LA-semigroup with 0.

·	0	1	2	3
0	0	0	0	0
1	0	2	3	1
2	0	1	2	3
3	0	3	1	2

Proposition 1.1. *If T is a left ideal and B is a bi-ideal of an LA-semigroup S with left identity, then BT and T^2B are bi-ideals of S.*

Proof. Using (1.2), we get

$$\begin{aligned}&((BT)S)(BT) = ((BT)B)(ST) \subseteq ((BS)B)T \subseteq BT, \\ &\text{and } (BT)(BT) = (BB)(TT) \subseteq BT.\end{aligned}$$

Hence BT is a bi-ideal of S. By using (1.2), we obtain

$$\begin{aligned}((T^2B)S)(T^2B) &= ((T^2S)(BS))(T^2B) \subseteq (T^2(BS))(T^2B) \\ &= (T^2T^2)((BS)B) \subseteq T^2B, \text{ and} \\ (T^2B)(T^2B) &= (T^2T^2)(BB) \subseteq T^2B.\end{aligned}$$

Hence T^2B is a bi-ideal of S. ∎

Proposition 1.2. *The product of two bi-ideals of an LA-semigroup S with left identity is a bi-ideal of S.*

Proof. Using (1.2), we get

$$((B_1B_2)S)(B_1B_2) = ((B_1B_2)(SS))(B_1B_2) = ((B_1S)(B_2S))(B_1B_2)$$
$$= ((B_1S)B_1)((B_2S)B_2) \subseteq B_1B_2.$$ ∎

The above proposition leads to easy generalizations. That is, if B_1, B_2, B_3,... and B_n are bi-ideals of an LA-semigroup S with left identity, then

$$(...((B_1B_2)B_3)...)B_n \text{ and } (...((B_1^2B_2^2)B_3^2)...)B_n^2$$

are bi-ideals of S. Consequently, the set $\complement(S_B)$ of bi-ideals, forms an LA-semigroup.

If S is an LA-semigroup with left identity e then $\langle a\rangle_L = Sa$, $\langle a\rangle_R = aS$ and $\langle a\rangle_S = (Sa)S$ are bi-ideals of S. It is then easy to show that $\langle ab\rangle_L = \langle a\rangle_L\langle b\rangle_L$, $\langle ab\rangle_R = \langle a\rangle_R\langle b\rangle_R$, and $\langle ab\rangle_R = \langle b\rangle_L\langle a\rangle$. This implies that $\langle a\rangle_R\langle b\rangle_R = \langle b\rangle_L\langle a\rangle_L$ and $\langle a\rangle_L\langle b\rangle_L = \langle b\rangle_R\langle a\rangle_R$. Also $\langle a\rangle_L\langle b\rangle_R = \langle b\rangle_L\langle a\rangle_R$, $\langle a^2\rangle_L = [\langle a\rangle_L]^2$, $\langle a^2\rangle_R = [\langle a\rangle_R]^2$, $\langle a^2\rangle_L = \langle a^2\rangle_R$ and $\langle a\rangle_L = \langle a\rangle_R$ provided a is an idempotent. Consequently, $\langle a^2\rangle_L = \langle a^2\rangle_R$ implying further that $\langle a\rangle_R a^2 = a^2\langle a\rangle_L$.

Lemma 1.1. *If B is an idempotent bi-ideal of an LA-semigroup S with left identity, then B is an ideal of S.*

Proof. Using (1.1),

$$BS = (BB)S = (SB)B = (SB^2)B = (B^2S)B = (BS)B,$$

imply that every right ideal in S with left identity is left. ∎

Lemma 1.2. *If B is a proper bi-ideal of an LA-semigroup S with left identity e, then $e \notin B$.*

Proof. Let $e \in B$. Then $sb = (es)b \in B$ together with (1.1), imply that $s = (ee)s = (se)e \in (SB)B \subseteq B$. ∎

Proposition 1.3. *If A, B are bi-ideals of an LA-semigroup S with left identity, then the following assertions are equivalent.*

(i) *Every bi-ideal of S is idempotent,*
(ii) *$A \cap B = AB$, and*
(iii) *the ideals of S form a semilattice $(L_S, \wedge)$ where $A \wedge B = AB$.*

Proof. (i)⇒ (ii): Using Lemma 1.1, it is easy to deduce that $AB \subseteq A \cap B$. Since $A \cap B \subseteq A$, B implies that $(A \cap B)^2 \subseteq AB$, and so $A \cap B \subseteq AB$.

(ii)⇒(iii): $A \wedge B = AB = A \cap B = B \cap A = B \wedge A$ and $A \wedge A = AA = A \cap A = A$. Associativity follows similarly. Hence $(L_S, \wedge)$ is a semilattice.

(iii)⇒(i):

$$A = A \wedge A = AA.$$ ∎

A bi-ideal B of an LA-semigroup S is called a *prime* bi-ideal if $B_1B_2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ for every bi-ideal B_1 and B_2 of S. The set of bi-ideals of S is totally ordered under inclusion if for all bi-ideals I, J either $I \subseteq J$ or $J \subseteq I$.

Theorem 1.1. *Let S be an LA-semigroup with left identity e. Then every bi-ideal of S is prime if and only if every bi-ideal of S is idempotent and the set of bi-ideals of S is totally ordered under inclusion.*

Proof. Let B be a bi-ideal of S. By Proposition 1.2, B^2 is prime. This implies that $B \subseteq B^2$ and hence B is idempotent. Therefore, if B_1 and B_2 are bi-ideals of S, then by Proposition 1.3, $B_1 \cap B_2$ is a bi-ideal of S and therefore by the hypothesis is prime. By Lemma 1.1, $B_1B_2 \subseteq B_1 \cap B_2$ and therefore either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. That is, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

Conversely, let B_1, B_2 and B be bi-ideals of S with $B_1B_2 \subseteq B$. Assume that $B_1 \subseteq B_2$. Since B_1 is idempotent, $B_1 = B_1B_1 \subseteq B_1B_2 \subseteq B$ implies that $B_1 \subseteq B$. Similarly, $B_2 \subseteq B_1$ implies that $B_2 \subseteq B$. Hence B is prime. ∎

An element a of an LA-semigroup S is called *intra-regular* if there exist elements x, $y \in S$ such that $a = (xa^2)y$. An LA-semigroup S is called *intra-regular* if every element of S is intra-regular.

Example 1.2. Let $S = \{1, 2, 3, 4, 5\}$ be an LA-semigroup, with left identity 4, defined by the following multiplication table.

·	1	2	3	4	5
1	4	5	1	2	3
2	3	4	5	1	2
3	2	3	4	5	1
4	1	2	3	4	5
5	5	1	2	3	4

Clearly $(S, \cdot)$ is intra-regular because $(2 \cdot 1^2) \cdot 3 = 1$, $(1 \cdot 2^2) \cdot 5 = 2$, $(2 \cdot 3^2) \cdot 5 = 3$, $(4 \cdot 4^2) \cdot 4 = 4$ and $(3 \cdot 5^2) \cdot 1 = 5$.

Lemma 1.3. *If B_1 and B_2 are bi-ideals of an intra-regular LA-semigroup S with left identity, then $B_1 \cup B_2$ is a bi-ideal of S.*

Proof.

$$\begin{aligned}[(B_1 \cup B_2)S](B_1 \cup B_2) &= (B_1S \cup B_2S)(B_1 \cup B_2)\\ &= (B_1S)(B_1 \cup B_2) \cup B_2S(B_1 \cup B_2)\\ &= (B_1S)B_1 \cup (B_1S)B_2 \cup (B_2S)B_1 \cup (B_2S)B_2\\ &\subseteq B_1 \cup (B_1S)B_2 \cup (B_2S)B_1 \cup B_2.\end{aligned}$$

Let $(bs)a \in (B_1S)B_2$, where $b \in B_1$, $s \in S$ and $a \in B_2$. Since S is intra-regular, therefore for $a \in S$ there exist $x, y \in S$ such that $(xa^2)y$. Using (1.4), (1.1), (1.3) and (1.2), we obtain

$$\begin{aligned}(bs)a = (bs)((xa^2)y) &= (xa^2)((bs)y) = (x(aa))((bs)y)\\ &= (a(xa))((bs)y) = [((bs)y)(xa)]a\\ &= [((bs)y)(x((xa^2)y))]a = [((bs)y)((xa^2)(xy))]a\\ &= [(xa^2)(((bs)y)(xy))]a = [((xy)((bs)y))(a^2x)]a\\ &= [a^2(((xy)((bs)y))x)]a \in (B_2S)B_2 \subseteq B_2.\end{aligned}$$

Similarly, we can show that $(B_2S)B_1 \subseteq B_1$. Therefore $[(B_1 \cup B_2)S](B_1 \cup B_2) \subseteq B_1 \cup B_2$. Hence $B_1 \cup B_2$ is a bi-ideal of S. ∎

A bi-ideal B of an LA-semigroup S is called a *strongly irreducible* bi-ideal if $B_1 \cap B_2 \subseteq B$ implies that either $B_1 \subseteq B$ or $B_2 \subseteq B$ for every bi-ideal B_1 and B_2 of S.

Theorem 1.2. *The set Đ of all bi-ideals of an intra-regular LA semi-group S^0 with* 0 *and left identity, is closed under finite intersection and arbitrary union.*

Proof. Let Ω be the set of all strongly irreducible proper bi-ideals of S^0. Then $\Gamma(\Omega) = \{O_B : B \in Đ\}$, forms a topology on the set Ω, where $O_B = \{J \in \Omega; B \nsubseteq J\}$ and ϕ : bi-ideal$(S^0) \longrightarrow \Gamma(\Omega)$ preserves finite intersection and arbitrary union between the set of bi-ideals of S^0 and open subsets of Ω. As $\{0\}$ is a bi-ideal of S^0, and 0 belongs to every bi-ideal of S^0, therefore $O_{\{0\}} = \{J \in \Omega, \{0\} \nsubseteq J\} = \{\ \}$. Also $O_{S^0} = \{J \in \Omega, S \nsubseteq J\} = \Omega$ which is the first axiom for the topology. If $\{O_{B_\alpha} : \alpha \in I\} \subseteq \Gamma(\Omega)$, then $\cup O_{B_\alpha} = \{J \in \Omega, B_\alpha \nsubseteq J$, for some $\alpha \in I\} = \{J \in \Omega, < \cup B_\alpha > \nsubseteq J\} = O_{\cup B_\alpha}$, where $< \cup B_\alpha >$ is a bi-ideal of S^0 generated by $\cup B_\alpha$ and by Lemma 1.3, $\cup B_\alpha$ is a bi-ideal. Let O_{B_1} and $O_{B_2} \in \Gamma(\Omega)$. If $J \in O_{B_1} \cap O_{B_2}$, then $J \in \Omega$ and $B_1 \nsubseteq J$, $B_2 \nsubseteq J$. Suppose $B_1 \cap B_2 \subseteq J$. This implies that either $B_1 \subseteq J$ or $B_2 \subseteq J$, implying a contradiction. Hence $B_1 \cap B_2 \nsubseteq J$ which further implies that $J \in O_{B_1 \cap B_2}$. Thus $O_{B_1} \cap O_{B_2} \subseteq O_{B_1 \cap B_2}$. Now if $J \in O_{B_1 \cap B_2}$, then $J \in \Omega$ and $B_1 \cap B_2 \nsubseteq J$. Thus $J \in O_{B_1}$ and $J \in O_{B_2}$. Therefore $J \in O_{B_1} \cap O_{B_2}$, which implies that $O_{B_1 \cap B_2} \subseteq O_{B_1} \cap O_{B_2}$. Hence $\Gamma(\Omega)$ is the topology on Ω. Define ϕ : bi-ideal$(S^0) \longrightarrow \Gamma(\Omega)$ by $\phi(B) = O_B$. Then it is easy to see that ϕ preserves finite intersection and arbitrary union. ∎

An ideal P of an LA-semigroup S is called a *strongly irreducible* ideal if $A \cap B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$ for all ideals A and B in S.

Let P_{S^0} denote the set of proper strongly irreducible ideals of an LA-semigroup S^0. For an ideal I of S^0 define the set $\Theta_I = \{\ J \in P_{S^0} : I \nsubseteq J\}$ and $\Gamma(P_{S^0}) = \{\Theta_I, I$ is an ideal of $S^0\}$.

Theorem 1.3. *The set* $\Gamma(P_{S^0})$ *constitute a topology on the set* P_{S^0}.

Proof. Let Θ_{I_1}, $\Theta_{I_2} \in \Gamma(P_{S^0})$. If $J \in \Theta_{I_1} \cap \Theta_{I_2}$, then $J \in P_{S^0}$ and $I_1 \nsubseteq J$ and $I_2 \nsubseteq J$. Let $I_1 \cap I_2 \subseteq J$ which implies that either $I_1 \subseteq J$ or $I_2 \subseteq J$; implying a contradiction. Hence $J \in \Theta_{I_1 \cap I_2}$. Similarly $\Theta_{I_1 \cap I_2} \subseteq \Theta_{I_1} \cap \Theta_{I_2}$. The rest of the proof follows immediately from the proof of Theorem 1.2. ∎

The assignment $I \longrightarrow \Theta_I$ preserves finite intersection and arbitrary union between the ideal(S^0) and their corresponding open subsets of Θ_I.

Let P be a left ideal of an LA-semigroup S. Then P is called *quasi-prime* if for left ideals A, B of S such that $AB \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$.

Theorem 1.4. *If S is an LA-semigroup S with left identity e, then a left ideal P of S is quasi-prime if and only if $(Sa)b \subseteq P$ implies that either $a \in P$ or $b \in P$.*

Proof. Let P be a left ideal of an LA-semigroup S with left identity e. If $(Sa)b \subseteq P$ then

$$S((Sa)b) \subseteq SP \subseteq P, \text{ that is}$$
$$S((Sa)b) = (Sa)(Sb).$$

Hence, either $a \in P$ or $b \in P$.

Conversely, assume that $AB \subseteq P$ where A and B are left ideals of S such that $A \nsubseteq P$. Then there exists $x \in A$ such that $x \notin P$. Now using the hypothesis we get $(Sx)y \subseteq (SA)B \subseteq AB \subseteq P$ for all $y \in B$. Since $x \notin P$, so by hypothesis, $y \in P$ for all $y \in B$, we obtain $B \subseteq P$. This shows that P is quasi-prime. ∎

An LA-semigroup S is said to be an *anti-rectangular* if $a = (ba)b$, for all a,b in S. It is straight forward to see that $S = S^2$.

Proposition 1.4. *If A and B are ideals of an anti-rectangular LA-semigroup S, then AB is an ideal.*

Proof. Using (1.2), we get

$$(AB)S = (AB)(SS) = (AS)(BS) \subseteq AB, \text{ and} S(AB) = (SS)(AB) = (SA)(SB) \subseteq AB$$

which shows that AB is an ideal. ∎

Consequently, if I_1, I_2, I_3,... and I_n are ideals of S, then

$$(...((I_1 I_2) I_3)...I_n) \quad \text{and} \quad (...((I_1^2 I_2^2) I_3^2)...I_n^2)$$

are ideals of S and the set S_I of ideals of S form an anti-rectangular LA-semigroup.

Lemma 1.4. *Any subset of an anti-rectangular LA-semigroup S is left ideal if and only if it is right.*

Proof. Let I be a right ideal of S, then using (1.1), we get, $si = ((xs)x)i = (ix)(xs) \in I$.

Conversely, suppose that I be a left prime ideal of S, then using (1.1), we get, $is = ((yi)y)s = (sy)(yi) \in I$. ∎

Therefore $SI = IS$. From above lemma we remark that each quasi prime ideal in an anti-rectangular LA-semigroup is in fact prime.

Lemma 1.5. *If I is an ideal of an anti-rectangular LA-semigroup S then, $H(a) = \{x \in S : (xa)x = a, \text{for } a \in I\} \subseteq I$.*

Proof. If $y \in H(a)$, then $y = (ya)y \in (SI)S \subseteq I$. Hence $H(a) \subseteq I$.

Also $H(a) = \{x \in S : (xa)x = x, \text{ for } a \in I\} \subseteq I$. ∎

An ideal I of an LA-semigroup S is called an *idempotent* if $I^2 = I$. An LA-semigroup S is said to be *fully idempotent* if every ideal of S is idempotent.

Proposition 1.5. *If S is an anti-rectangular LA-semigroup and A, B are ideals of S, then the following assertions are equivalent.*

(i) *S is fully idempotent,*
(ii) *$A \cap B = AB$, and*
(iii) *the ideals of S form a semilattice $(L_S, \wedge)$ where $A \wedge B = AB$.*

It follows easily from Proposition 1.4.

The set of ideals of S is totally ordered under inclusion if for all ideals I, J either $I \subseteq J$ or $J \subseteq I$. It is denoted by ideal(S).

Theorem 1.5. *Every ideal of an anti-rectangular LA-semigroup S is prime if and only it is idempotent and ideal(S) is totally ordered under inclusion.*

It follows easily from Theorem 1.1.

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