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Oscillation Theorems for Second-Order Quasi-Linear Delay Dynamic Equations

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Abstract. This paper is concerned with the oscillation of a class of second-order quasi-linear delay dynamic equations on an arbitrary time scale. Three new oscillation theorems and two illustrative examples are presented that improve those known results in the literature.

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1. Introduction

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [12], is an area of mathematics that has recently received a lot of attention. Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal *et al.* [1] and the references cited therein. For an excellent introduction to the calculus on time scales; see Bohner and Peterson [3]. Further information on working with dynamic equations on time scales can be found in [4].

In the last few years, much attention is attracted by questions of the oscillation of different classes of dynamic equations on time scales, we refer the reader to [2–11, 13–18] and the references therein. In this paper, we are concerned with the oscillation of the second-order quasi-linear delay dynamic equation

$$\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \tag{1.1}$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is the quotient of odd positive integers, r and p are rd-continuous positive functions defined on $\mathbb{T}$, $\tau \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$, $\tau(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \infty$. Since we are interested in oscillatory behavior, we assume throughout this paper that the

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given time scale $\mathbb{T}$ is unbounded above and define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$.

By a solution of (1.1) we mean a nontrivial real-valued function x which has the properties $x \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $r\left(x^{\Delta}\right)^{\gamma} \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies equation (1.1) for all $t \in [t_0,\infty)_{\mathbb{T}}$. We consider only those solutions x of equation (1.1) which satisfy $\sup\{|x(t)| : t \in [T,\infty)_{\mathbb{T}}\} > 0$ for all $T \in [t_0,\infty)_{\mathbb{T}}$ and assume that equation (1.1) possesses such a solution. As usual, a solution of equation (1.1) is called oscillatory if it has arbitrarily large generalized zeros on $[t_0,\infty)_{\mathbb{T}}$; otherwise, it is called non-oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$. On any time scale we define the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} | s < t\}$, where $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$, $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf\mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup\mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$. Note that when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, and when $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t+1$. When $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t+h$, and when $\mathbb{T} = \{t : t = \rho^k, k \in \mathbb{N}_0, \rho > 1\}$, we have $\sigma(t) = \rho t$. When $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}_0\}$, we have $\sigma(t) = (\sqrt{t}+1)^2$, and when $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}\}$ where $\{t_n\}$ is the harmonic numbers that are defined by $t_0 = 0$ and $t_n = \sum_{k=1}^{n} 1/k$, $n \in \mathbb{N}_0$, we have $\sigma(t_n) = t_{n+1}$. When $\mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt{t^2+1}$, and when $\mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt[3]{t^3+1}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. The function f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. The derivative and the shift operator σ are related by the formula $f^{\sigma}(t) = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may actually be replaced by any Banach space), the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists. Let f be a real-valued function defined on an interval $[a,b]_{\mathbb{T}}$. Let f be Δ-differentiable function. Then f is increasing, decreasing, non-decreasing, and non-increasing if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a,b)_{\mathbb{T}}$, respectively. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t),$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}.$$

For $a,\ b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by $\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a)$. The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_a^t f(s)\Delta s.$$

Note that the integration formula on a discrete time scale is defined by $\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} f(t)\mu(t)$. In order to prove our main results, we will use the chain rule formula

$$(f^{\gamma})^{\Delta}(t) = \gamma f^{\Delta}(t)\int_0^1 \left[hf^{\sigma}(t) + (1-h)f(t)\right]^{\gamma-1}\mathrm{d}h, \quad \gamma > 0, \tag{1.2}$$

which is a simple consequence of Keller's chain rule (see Bohner and Peterson [3, Theorem 1.90]).

In the following, we present the background details that motivate the contents of this paper. Agarwal *et al.* [2], Grace *et al.* [7,8], Hassan [10,11], Saker *et al.* [16,17], Zhang [18] have studied (1.1) for the case where $\tau(t) = t$, i.e.,

$$\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(t) = 0, \tag{1.3}$$

and they considered two cases

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty, \tag{1.4}$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} < \infty. \tag{1.5}$$

Some results on (1.3) are established for the case where (1.4) holds. However, there are few results known that apply to (1.3) when (1.5) holds. In [7,8,11,17,18], the authors obtained some oscillation criteria for (1.3) provided that (1.5) holds and

$$\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t} p(s)\left(\int_s^{\infty}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\right)^{\gamma}\Delta s\right]^{\frac{1}{\gamma}}\Delta t = \infty.$$

In this paper, we shall derive three new oscillation theorems for (1.1) under the case where (1.5) holds. The paper is organized as follows: In Section 2, we shall establish several new oscillation criteria for (1.1). In Section 3, we provide two examples to illustrate our main results.

2. The main results

In this section, by employing the Riccati transformation technique we establish several oscillation criteria for (1.1). For the sake of convenience in our proofs, we use the notation

$$(\delta^{\Delta}(t))_+ := \max\{0, \delta^{\Delta}(t)\}, \quad \theta(t,u) := \frac{\int_u^{\tau(t)}\Delta s/(r^{\frac{1}{\gamma}}(s))}{\int_u^t \Delta s/(r^{\frac{1}{\gamma}}(s))},$$

and present the following known result.

Theorem 2.1. [6, Theorem 2.1] *Assume that* (1.4) *holds. Further, assume that there exists a positive real-valued Δ-differentiable function δ such that for all sufficiently large T_*, and for $\tau(T) > T_*$,*

$$\limsup_{t\to\infty}\int_T^t\left[\theta^\gamma(s,T_*)\delta(s)p(s)-\frac{r(s)((\delta^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\right]\Delta s=\infty. \tag{2.1}$$

Then every solution of equation (1.1) *is oscillatory.*

Below, we denote $R(t) := \int_t^\infty (\Delta s)/(r^{1/\gamma}(s))$ and establish the following results. First, we establish an oscillation criterion for (1.1) when $\gamma \le 1$.

Theorem 2.2. *Let* (1.5) *hold and $\gamma \le 1$. Assume that there exists a positive real-valued Δ-differentiable function δ such that for all sufficiently large T_*, and for $\tau(T) > T_*$,* (2.1) *holds. If*

$$\limsup_{t\to\infty}\int_T^t\left[p(s)R^{\gamma\sigma}(s)-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{1}{R^\sigma(s)r^{\frac{1}{\gamma}}(s)}\right]\Delta s=\infty, \tag{2.2}$$

then equation (1.1) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (1.1). Without loss of generality we assume $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. In view of (1.1), we obtain that, for $t \in [t_0,\infty)_{\mathbb{T}}$,

$$(r(t)(x^\Delta(t))^\gamma)^\Delta = -p(t)x^\gamma(\tau(t)) < 0. \tag{2.3}$$

Hence $r(x^\Delta)^\gamma$ is strictly decreasing, and so there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x^\Delta(t) > 0$, or $x^\Delta(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Assume first that $x^\Delta(t) > 0$. From [6, Theorem 2.1], we can obtain a contradiction to (2.1). Assume now that $x^\Delta(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Define the function ω by

$$\omega(t) := \frac{r(t)(x^\Delta(t))^\gamma}{x^\gamma(t)}. \tag{2.4}$$

Then $\omega(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. By virtue of (2.3), we get

$$x^\Delta(s) \le \frac{r^{\frac{1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(s)}x^\Delta(t) \quad \text{for all} \quad s\in[t,\infty)_{\mathbb{T}}.$$

Integrating it from t to l, we have

$$x(l) \le x(t) + r^{\frac{1}{\gamma}}(t)x^\Delta(t)\int_t^l \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} \quad \text{for} \quad l\in[t,\infty)_{\mathbb{T}}.$$

Letting $l \to \infty$ in the last inequality, we get

$$x(t) + r^{\frac{1}{\gamma}}(t)x^\Delta(t)R(t) \ge 0 \quad \text{for} \quad t\in[t_1,\infty)_{\mathbb{T}}.$$

Thus, we obtain

$$r^{\frac{1}{\gamma}}(t)R(t)\frac{x^\Delta(t)}{x(t)} \ge -1. \tag{2.5}$$

Using (2.4) and (2.5), we have

$$-1 \le R^\gamma(t)\omega(t) \le 0. \tag{2.6}$$

Now, we obtain by (2.3) and (2.4) that

$$\omega^{\Delta}(t) \le -p(t) - \frac{r(t)(x^{\Delta}(t))^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}. \tag{2.7}$$

In view of (1.2), we see that

$$(x^{\gamma}(t))^{\Delta} \le \gamma x^{\gamma-1}(t)x^{\Delta}(t), \quad \text{since} \quad \gamma \le 1.$$

Thus, inequality (2.7) yields

$$\omega^{\Delta}(t) \le -p(t) - \gamma \frac{r(t)(x^{\Delta}(t))^{\gamma+1}}{x(t)x^{\gamma}(\sigma(t))}. \tag{2.8}$$

On the other hand, we have by $x^{\Delta}(t) < 0$ that $x(t) \ge x^{\sigma}(t)$, and so

$$-\gamma \frac{r(t)(x^{\Delta}(t))^{\gamma+1}}{x(t)x^{\gamma}(\sigma(t))} \le -\gamma \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} \omega^{\frac{\gamma+1}{\gamma}}(t).$$

Hence by (2.8), we have

$$\omega^{\Delta}(t) + p(t) + \gamma r^{\frac{-1}{\gamma}}(t)\omega^{\frac{\gamma+1}{\gamma}}(t) \le 0 \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.9}$$

Multiplying (2.9) by $R^{\gamma\sigma}(t)$, we obtain

$$R^{\gamma\sigma}(t)\omega^{\Delta}(t) + p(t)R^{\gamma\sigma}(t) + \gamma R^{\gamma\sigma}(t)r^{\frac{-1}{\gamma}}(t)\omega^{\frac{\gamma+1}{\gamma}}(t) \le 0 \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Integrating it from t_1 to t, we get

$$\int_{t_1}^{t} R^{\gamma\sigma}(s)\omega^{\Delta}(s)\Delta s + \int_{t_1}^{t} p(s)R^{\gamma\sigma}(s)\Delta s + \gamma \int_{t_1}^{t} R^{\gamma\sigma}(s)r^{\frac{-1}{\gamma}}(s)\omega^{\frac{\gamma+1}{\gamma}}(s)\Delta s \le 0. \tag{2.10}$$

Integrating by parts, we have

$$\int_{t_1}^{t} R^{\gamma\sigma}(s)\omega^{\Delta}(s)\Delta s = R^{\gamma}(t)\omega(t) - R^{\gamma}(t_1)\omega(t_1) - \int_{t_1}^{t} (R^{\gamma}(s))^{\Delta}\omega(s)\Delta s. \tag{2.11}$$

From (1.2), we obtain

$$(R^{\gamma}(t))^{\Delta} = \gamma R^{\Delta}(t) \int_0^1 \left[hR^{\sigma}(t) + (1-h)R(t)\right]^{\gamma-1} \mathrm{d}h. \tag{2.12}$$

Noting that $R^{\Delta}(t) = -(1/r(t))^{1/\gamma} < 0$, we get by (2.12) and $\gamma \le 1$ that

$$-\int_{t_1}^{t} (R^{\gamma}(s))^{\Delta}\omega(s)\Delta s \ge \gamma \int_{t_1}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (R^{\sigma}(s))^{\gamma-1}\omega(s)\Delta s. \tag{2.13}$$

By virtue of (2.10), (2.11), and (2.13), we see that

$$\begin{aligned} &R^{\gamma}(t)\omega(t) - R^{\gamma}(t_1)\omega(t_1) + \int_{t_1}^{t} p(s)R^{\gamma\sigma}(s)\Delta s + \gamma \int_{t_1}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (R^{\sigma}(s))^{\gamma-1}\omega(s)\Delta s \\ &\quad + \gamma \int_{t_1}^{t} R^{\gamma\sigma}(s)r^{\frac{-1}{\gamma}}(s)\omega^{\frac{\gamma+1}{\gamma}}(s)\Delta s \le 0. \end{aligned} \tag{2.14}$$

Set $p := (\gamma+1)/\gamma$, $q := \gamma+1$,

$$A := -(\gamma+1)^{\frac{\gamma}{\gamma+1}} \left(\frac{R^{\gamma\sigma}(t)}{r^{\frac{1}{\gamma}}(t)}\right)^{\frac{\gamma}{\gamma+1}} \omega(t),$$

and

$$B := \frac{\gamma}{\gamma+1}(\gamma+1)^{\frac{1}{\gamma+1}}\left(\frac{1}{r^{\frac{1}{\gamma}}(t)}\right)^{\frac{1}{\gamma+1}}\frac{1}{(R^{\sigma}(t))^{\frac{1}{\gamma+1}}}.$$

Then, using the inequality

$$\frac{A^p}{p}+\frac{B^q}{q}\geq AB \quad \text{for} \quad \frac{1}{p}+\frac{1}{q}=1, \tag{2.15}$$

we have

$$\gamma R^{\gamma\sigma}(t)r^{\frac{-1}{\gamma}}(t)\omega^{\frac{\gamma+1}{\gamma}}(t)+\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{1}{R^{\sigma}(t)r^{\frac{1}{\gamma}}(t)}\geq -\gamma\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}(R^{\sigma}(t))^{\gamma-1}\omega(t). \tag{2.16}$$

Thus, we get by (2.14) and (2.16) that

$$R^{\gamma}(t)\omega(t)-R^{\gamma}(t_1)\omega(t_1)+\int_{t_1}^{t}\left[p(s)R^{\gamma\sigma}(s)-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{1}{R^{\sigma}(s)r^{\frac{1}{\gamma}}(s)}\right]\Delta s\leq 0.$$

Therefore, we obtain by (2.2) that

$$R^{\gamma}(t)\omega(t)\to-\infty \quad \text{as} \quad t\to\infty,$$

which contradicts (2.6). The proof is complete. ∎

Next, we establish an oscillation criterion for (1.1) when $\gamma\geq 1$.

Theorem 2.3. *Let* (1.5) *hold and* $\gamma \geq 1$. *Assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that for all sufficiently large* T_*, *and for* $\tau(T)>T_*$, (2.1) *holds. If*

$$\limsup_{t\to\infty}\int_{T}^{t}\left[p(s)R^{\gamma\sigma}(s)-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2}r^{\frac{1}{\gamma}}(s)}\right]\Delta s=\infty, \tag{2.17}$$

then equation (1.1) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (1.1). Without loss of generality we assume $x(t)>0$ and $x(\tau(t))>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. In view of (1.1), we obtain (2.3). Therefore, $r(x^{\Delta})^{\gamma}$ is strictly decreasing, and there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x^{\Delta}(t)>0$, or $x^{\Delta}(t)<0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Assume first that $x^{\Delta}(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. From [6, Theorem 2.1], we can obtain a contradiction to (2.1). Assume now that $x^{\Delta}(t)<0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Define ω as in (2.4). We have (2.6). Using (2.3) and (2.4), we obtain (2.7). In view of (1.2), we have

$$(x^{\gamma}(t))^{\Delta}\leq\gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t), \quad \text{since} \quad \gamma\geq 1.$$

Thus, we get

$$\omega^{\Delta}(t)\leq -p(t)-\gamma\frac{r(t)(x^{\Delta}(t))^{\gamma+1}}{x^{\gamma}(t)x(\sigma(t))}. \tag{2.18}$$

On the other hand, we have by $x^{\Delta}(t)<0$ that $x(t)\geq x^{\sigma}(t)$, and so

$$-\gamma\frac{r(t)(x^{\Delta}(t))^{\gamma+1}}{x^{\gamma}(t)x(\sigma(t))}\leq -\gamma\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\omega^{\frac{\gamma+1}{\gamma}}(t).$$

Hence by (2.18), we get (2.9). Then we obtain that (2.10) and (2.11) hold. By virtue of (1.2), we have (2.12). From (2.12), $\gamma \geq 1$, and $R^{\Delta}(t) = -(1/r(t))^{1/\gamma} < 0$, we see that

$$-\int_{t_1}^{t} (R^{\gamma}(s))^{\Delta}\omega(s)\Delta s \geq \gamma \int_{t_1}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} R^{\gamma-1}(s)\omega(s)\Delta s. \tag{2.19}$$

It follows from (2.10), (2.11), and (2.19) that

$$\begin{aligned} &R^{\gamma}(t)\omega(t) - R^{\gamma}(t_1)\omega(t_1) + \int_{t_1}^{t} p(s)R^{\gamma\sigma}(s)\Delta s + \gamma \int_{t_1}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} R^{\gamma-1}(s)\omega(s)\Delta s \\ &\quad + \gamma \int_{t_1}^{t} R^{\gamma\sigma}(s) r^{\frac{-1}{\gamma}}(s)\omega^{\frac{\gamma+1}{\gamma}}(s)\Delta s \leq 0. \end{aligned} \tag{2.20}$$

Set $p := (\gamma+1)/\gamma$, $q := \gamma+1$,

$$A := -(\gamma+1)^{\frac{\gamma}{\gamma+1}} \left(\frac{R^{\gamma\sigma}(t)}{r^{\frac{1}{\gamma}}(t)}\right)^{\frac{\gamma}{\gamma+1}} \omega(t),$$

and

$$B := \frac{\gamma}{\gamma+1}(\gamma+1)^{\frac{1}{\gamma+1}} \left(\frac{1}{r^{\frac{1}{\gamma}}(t)}\right)^{\frac{1}{\gamma+1}} \frac{R^{\gamma-1}(t)}{(R^{\sigma}(t))^{\frac{\gamma^2}{\gamma+1}}}.$$

Then, by the inequality (2.15), we have

$$\gamma R^{\gamma\sigma}(t) r^{\frac{-1}{\gamma}}(t)\omega^{\frac{\gamma+1}{\gamma}}(t) + \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{R^{\gamma^2-1}(t)}{(R^{\sigma}(t))^{\gamma^2} r^{\frac{1}{\gamma}}(t)} \geq -\gamma\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} R^{\gamma-1}(t)\omega(t). \tag{2.21}$$

Hence by (2.20) and (2.21), we obtain

$$R^{\gamma}(t)\omega(t) - R^{\gamma}(t_1)\omega(t_1) + \int_{t_1}^{t} \left[p(s)R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2} r^{\frac{1}{\gamma}}(s)}\right]\Delta s \leq 0.$$

It follows from (2.17) that

$$R^{\gamma}(t)\omega(t) \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts (2.6). This completes the proof. ∎

Finally, we establish an oscillation criterion for (1.1) when $\gamma > 0$.

Theorem 2.4. *Let* (1.5) *hold and* $\gamma > 0$. *Assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that for all sufficiently large* T_*, *and for* $\tau(T) > T_*$, (2.1) *holds. If*

$$\int_{T}^{\infty} p(s)R^{\gamma+1}(\sigma(s))\Delta s = \infty, \tag{2.22}$$

then equation (1.1) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (1.1). Without loss of generality we assume $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. Similar to the proof of Theorem 2.2 or Theorem 2.3, we consider two cases. Assume first that $x^{\Delta}(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. By (2.1), this case is not true. Assume now that $x^{\Delta}(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. If $\gamma \leq 1$, proceeding as in the proof of Case 2 of Theorem 2.1, we obtain (2.6) and (2.9). Multiplying (2.9) by $R^{\gamma+1}(\sigma(t))$ and integrating it from t_1 to t, we get

$$\int_{t_1}^{t} R^{\gamma+1}(\sigma(s))\omega^{\Delta}(s)\Delta s+\int_{t_1}^{t} p(s)R^{\gamma+1}(\sigma(s))\Delta s+\gamma\int_{t_1}^{t} R^{\gamma+1}(\sigma(s))r^{\frac{-1}{\gamma}}(s)\omega^{\frac{\gamma+1}{\gamma}}(s)\Delta s\le 0.$$

Integrating by parts, we see that

$$\int_{t_1}^{t} R^{\gamma+1}(\sigma(s))\omega^{\Delta}(s)\Delta s=R^{\gamma+1}(t)\omega(t)-R^{\gamma+1}(t_1)\omega(t_1)-\int_{t_1}^{t}(R^{\gamma+1}(s))^{\Delta}\omega(s)\Delta s. \tag{2.23}$$

By (1.2), we obtain

$$(R^{\gamma+1}(t))^{\Delta}=(\gamma+1)R^{\Delta}(t)\int_0^1\left[hR^{\sigma}(t)+(1-h)R(t)\right]^{\gamma}\mathrm{d}h.$$

Noting that $R^{\Delta}(t)=-(1/r(t))^{1/\gamma}<0$, we have

$$-\int_{t_1}^{t}(R^{\gamma+1}(s))^{\Delta}\omega(s)\Delta s\ge(\gamma+1)\int_{t_1}^{t}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}R^{\gamma}(s)\omega(s)\Delta s.$$

Thus, from (2.23), we get

$$\begin{aligned}&R^{\gamma+1}(t)\omega(t)-R^{\gamma+1}(t_1)\omega(t_1)+\int_{t_1}^{t}p(s)R^{\gamma+1}(\sigma(s))\Delta s\\&+(\gamma+1)\int_{t_1}^{t}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}R^{\gamma}(s)\omega(s)\Delta s+\gamma\int_{t_1}^{t}R^{\gamma+1}(\sigma(s))r^{\frac{-1}{\gamma}}(s)\omega^{\frac{\gamma+1}{\gamma}}(s)\Delta s\le 0.\end{aligned} \tag{2.24}$$

It follows from (2.6) that

$$-R^{\gamma+1}(t)\omega(t)\le R(t)<\infty \quad \text{as} \quad t\to\infty,$$

and

$$-\int_{t_1}^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}R^{\gamma}(s)\omega(s)\Delta s\le\int_{t_1}^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s<\infty.$$

Noting that $R^{\sigma}(t)/R(t)\le 1$, we obtain

$$\begin{aligned}\int_{t_1}^{\infty}R^{\gamma+1}(\sigma(s))r^{\frac{-1}{\gamma}}(s)\omega^{\frac{\gamma+1}{\gamma}}(s)\Delta s&=\int_{t_1}^{\infty}r^{\frac{-1}{\gamma}}(s)\left(\frac{R^{\sigma}(s)}{R(s)}\right)^{\gamma+1}(R^{\gamma}(s)\omega(s))^{\frac{\gamma+1}{\gamma}}\Delta s\\&\le\int_{t_1}^{\infty}r^{\frac{-1}{\gamma}}(s)\Delta s<\infty.\end{aligned}$$

Thus, from (2.24), we get

$$\int_{t_1}^{\infty}p(s)R^{\gamma+1}(\sigma(s))\Delta s<\infty,$$

which contradicts (2.22). When $\gamma\ge 1$, the proof is similar to the case where $\gamma\le 1$, and so we omit the details. The proof is complete. ∎

3. Examples

For some applications, we give the following examples.

Example 3.1. Consider the second-order differential equation

$$\left(t^2x'(t)\right)' + p_0x(t) = 0 \quad \text{for} \quad t \geq 1, \tag{3.1}$$

where $p_0 > 0$ is a constant. Letting $\delta(t) = 1$ and using Theorem 2.2, we see that (3.1) is oscillatory, if $p_0 > 1/4$. However, results of [7,8,11,17,18] cannot be applied to (3.1), since

$$\int_1^\infty \frac{1}{r^{\frac{1}{\gamma}}(v)} \left[\int_{t_1}^v p(u)R^\gamma(u)\mathrm{d}u\right]^{\frac{1}{\gamma}} \mathrm{d}v < \infty.$$

Example 3.2. Consider the second-order delay dynamic equation

$$\left((t\sigma(t))^\gamma(x^\Delta(t))^\gamma\right)^\Delta + \frac{\sigma^{\gamma+1}(t)}{t}x^\gamma(\tau(t)) = 0 \quad \text{for} \quad t \in [1,\infty)_{\mathbb{T}}, \tag{3.2}$$

where $\gamma > 0$ is a ratio of two positive odd integers. Note that $R(t) = 1/t$. Let $\delta(t) = 1$. It is easy to see that (3.2) is oscillatory by using Theorem 2.4.

Remark 3.1. Our results improve some of the results in the papers [7,8,11,17,18].

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