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Implicit Iteration Methods for Variational Inequalities in Banach Spaces

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Abstract. In this paper, we introduce three new iteration methods, which are implicit and converge strongly, based on the steepest descent method with a strongly accretive and strictly pseudocontractive mapping and the modified Halpern's iterative scheme, for finding a solution of variational inequalities over the set of common fixed points of a nonexpansive semigroup on a real Banach space which has a uniformly Gâteaux differentiable norm.

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1. Introduction

Let E be a Banach space with the dual space E^*. For the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|.\|$. We write $\langle x, x^* \rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$.

A mapping J from E into E^*, satisfying the following condition

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 \text{ and } \|x^*\| = \|x\|\},$$

is called a normalized duality mapping of E. It is well known that if $x \neq 0$, then $J(tx) = tJ(x)$, for all $t > 0$ and $x \in E$, and $J(-x) = -J(x)$.

Let T be a nonexpansive mapping on a nonempty, closed and convex subset C of a Banach space E, i.e., $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. Denote the set of fixed points of T by $\mathrm{Fix}(T)$, i.e., $\mathrm{Fix}(T) = \{x \in C : x = T(x)\}$.

Let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on C, that is,

(1) for each $s > 0, T(s)$ is a nonexpansive mapping on C;
(2) $T(0)x = x$ for all $x \in C$;
(3) $T(s_1 + s_2) = T(s_1) \circ T(s_2)$ for all $s_1, s_2 > 0$;
(4) for each $x \in C$, the mapping $T(.)x$ from $(0, \infty)$ into C is continuous.

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Let $F : E \to E$ be an η-strongly accretive and γ-strictly pseudocontractive mapping, i.e., F satisfies, respectively, the following conditions:

$$\langle F(x) - F(y), j(x-y) \rangle \geq \eta \|x-y\|^2, \tag{1.1}$$

and

$$\langle F(x) - F(y), j(x-y) \rangle \leq \|x-y\|^2 - \gamma \|(I-F)x - (I-F)y\|^2, \tag{1.2}$$

for all $x, y \in E$ and some element $j(x-y) \in J(x-y)$, where I denotes the identity mapping of E, η and $\gamma \in (0,1)$ are some positive constants.

The problem, considered in this paper, is to present some new implicit iteration schemes for finding a point $p^* \in E$ such that

$$p^* \in \mathscr{F} : \quad \langle F(p^*), j(p^* - p) \rangle \leq 0 \quad \forall p \in \mathscr{F}, \tag{1.3}$$

where $\mathscr{F} := \cap_{s>0} \mathrm{Fix}(T(s))$ and $\{T(s) : s > 0\}$ is a nonexpansive semigroup on a uniformly convex Banach space E with a uniformly Gâteaux differentiable norm. Problem (1.3) is named a variational inequality, which was firstly studied by Stampacchia in [21]. In [14], Stampacchia and Lions extended the result of [21] and announced the full proofs of their results. Ever since, variational inequalities have been widely investigated, because it covers as diverse disciplines, as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance (see, e.g., [7, 8, 11, 13, 17, 26]).

Clearly, from (1.2), it follows that $\|F(x) - F(y)\| \leq L\|x-y\|$ with $L = 1 + 1/\gamma$ and, in this case, F is called L-Lipschitz continuous. If $L \in [0,1)$, then F is called contractive and if F satisfies (1.2) with $\gamma = 0$, then it is said to be pseudocontractive. It is easy to see that every nonexpansive mapping is pseudocontractive. The convergence of a parallel iterative algorithm for two finite families of uniformly L-Lipschitzian mappings was considered in [9].

In the case that $\{T(s) : s > 0\}$ is a nonexpansive semigroup on C, a closed and convex subset of a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, in [4], Chen and Song proposed the following implicit algorithm:

$$x_k = \gamma_k f(x_k) + (1-\gamma_k)\frac{1}{t_k}\int_0^{t_k} T(s)x_k ds, \tag{1.4}$$

where f is a contractive mapping on C and γ_k, t_k are two positive parameters of iteration. They proved the following result.

Theorem 1.1. [4] *Let C be a closed and convex subset of a uniformly convex Banach space E, whose norm is uniformly Gâteaux differentible and let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on C such that $\mathscr{F} := \cap_{s>0} \mathrm{Fix}(T(s)) \neq \emptyset$. Then, the sequence $\{x_k\}$, defined by* (1.4) *with the conditions $t_k \to \infty$ and $\gamma_k \to 0$ as $k \to \infty$, converges strongly to an element $p^* \in \mathscr{F}$, solving* (1.3) *with $F = I - f$.*

A special case of (1.4) has been considered by Shijoi and Takahashi in [20], as follows:

$$x_k = \gamma_k u + (1-\gamma_k)\frac{1}{t_k}\int_0^{t_k} T(s)x_k ds,$$

where u is a fixed element in C, $\{\gamma_k\} \subset (0,1)$ and $\{t_k\}$ is a real positive and divergent sequence. Next, in [22], Suzuki improved the Shijoi and Takahashi's result and proved the following theorem.

Theorem 1.2. *Let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on C, a nonempty, closed and convex subset of a Hilbert space H, such that*

$$\mathscr{F} := \cap_{s>0} \mathrm{Fix}(T(s)) \neq \emptyset,$$

and let $\{\gamma_k\}$ and $\{t_k\}$ be sequences of real numbers, satisfying

$$0 < \gamma_k < 1,\ t_k > 0,\ \lim_{k\to\infty} t_k = \lim_{k\to\infty} \frac{\gamma_k}{t_k} = 0. \tag{1.5}$$

Fix $u \in C$ and define a sequence $\{x_k\}$ in C by

$$x_k = \gamma_k u + (1-\gamma_k)T(t_k)x_k. \tag{1.6}$$

Then, $\{x_k\}$ converges strongly to the p^*, an element in $\mathscr{F}$ with minimal norm.

Further, in [12], He and Chen considered a more general scheme

$$x_k = \gamma_k f(x_k) + (1-\gamma_k)T(t_k)x_k \tag{1.7}$$

and obtained the following result.

Theorem 1.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Suppose that f is a contractive mapping on C with coefficient $\alpha \in (0,1)$, $\{T(s) : s > 0\}$ is a nonexpansive semigroup on C such that $\cap_{s>0} \mathrm{Fix}(T(s)) \neq \emptyset$. Assume that $\{\gamma_k\}$ and $\{t_k\}$ are two sequences of real numbers, satisfying* (1.5)*. Then, the sequence $\{x_k\}$, defined by* (1.7)*, converges strongly to the p^*, solving the following variational inequality*

$$\langle F(p^*), p^* - p\rangle \leq 0 \quad \forall p \in \mathscr{F}. \tag{1.8}$$

with $F = I - f$.

In [24], Xu established a Banach space version of (1.6). Recently, in [3], Chen and He studied the strong convergence of algorithm (1.7) in Banach spaces, and in [15], Li *et al.* extended the result in Hilbert spaces to that in a uniformly convex Banach space with an additional condition:

$$\lim_{s\to 0} \sup_{x\in K} \|T(s)x - x\| = 0, \tag{1.9}$$

for any bounded subset $K \subset C$ and an η-strongly accretive and γ-strictly pseudocontractive mapping f. In [1], Ceng *et al.* investigated (1.7) for the case that E is a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm, $\{T(s) : s > 0\}$ is a weakly uniformly asymptotically regular nonexpansive semigroup and that $t_k \to \infty$ as $k \to \infty$.

When $F = A - \gamma f$, where A is a strongly positive, linear and bounded mapping, defined on a Hilbert space H, in [16], Li *et al.* studied the following algorithm

$$x_k = \gamma_k \gamma f(x_k) + (I - \lambda_k A)\frac{1}{t_k}\int_0^{t_k} T(s)x_k ds \tag{1.10}$$

and proved the following result.

Theorem 1.4. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Suppose that f is a contractive mapping on C with coefficient $\alpha \in (0,1)$, $\{T(s) : s > 0\}$ is a nonexpansive semigroup on C such that $\cap_{s>0} \mathrm{Fix}(T(s)) \neq \emptyset$, and A is a strongly positive, linear and bounded mapping with coefficient $\tilde{\gamma} > 0$. Let $\{\gamma_k\} \subset [0,1]$, $\{t_k\} \subset (0,\infty)$ satisfy the conditions $\gamma_k \to 0$ and $t_k \to \infty$, as $k \to \infty$. Then, for any $0 < \gamma < \tilde{\gamma}/\alpha$, there exists a* *unique element* $x_k \in C$*, solving* (1.10)*, and the sequence* $\{x_k\}$ *converges strongly to the* p^**, a unique solution of* (1.8)*.*

Very recently, in [25], Yao and Liou introduced the implicit algorithm,

$$x_t = P_C\left[t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\right], \quad t \in (0,1), \tag{1.11}$$

where P_C denotes the metric projection of H onto a closed and convex subset C in H, and proved the following result.

Theorem 1.5. *Let* C *be a nonempty, closed and convex subset of a Hilbert space* H*. Suppose that* $f : C \to H$ *is a contractive mapping (possibly non-self) with coefficient* $\alpha \in (0,1)$*,* $\{T(s) : s > 0\}$ *is a nonexpansive semigroup on* C *such that* $\mathscr{F} := \cap_{s>0}\mathrm{Fix}(T(s)) \neq \emptyset$*, and* A *is a strongly positive, linear and bounded mapping with coefficient* $\tilde{\gamma} > 0$*. Let* $\{\lambda_t\}_{0<t<1}$ *be a continuous net of positive and real numbers such that* $\lim_{t\to 0}\lambda_t = \infty$*. Then, for any* $0 < \gamma < \tilde{\gamma}/\alpha$ *and* $\beta \in [0,1)$*, there exists a unique element* $x_t \in C$*, solving* (1.11)*, and the net* $\{x_t\}$ *converges strongly to the* p^**, a unique solution of* (1.8) *with* $F = A - \gamma f$*, as* $t \to 0$*.*

At this time, in [5], Cho and Kang proved the following result.

Theorem 1.6. *Let* H *be a real Hilbert space and let* $\{T(s) : s > 0\}$ *be a nonexpansive semigroup on* H *such that* $\mathscr{F} := \cap_{s>0}\mathrm{Fix}(T(s)) \neq \emptyset$*. Let* f *be a contractive mapping on* H *with coefficient* $\alpha \in (0,1)$*, and let* A *be a strongly positive, linear and bounded mapping with coefficient* $\tilde{\gamma} > 0$*. Assume that* $0 < \gamma < \tilde{\gamma}/\alpha$*. Let* $\{\gamma_k\}$ *and* $\{t_k\}$ *be two sequences of real numbers, satisfying* (1.5)*. Define a sequence* $\{x_k\}$ *in the manner:*

$$x_k = \gamma_k\gamma f(x_k) + (1-\gamma_k A)T(t_k)x_k \quad \forall k \geq 1. \tag{1.12}$$

Then, $\{x_k\}$ *converges strongly to the* p^**, solving* (1.8) *with* $F = A - \gamma f$*.*

Clearly, all algorithms, listed above, are some different modifications of the explicit Halpern's iteration method (see [10]),

$$x_{k+1} = \gamma_k u + (1-\gamma_k)Tx_k,$$

for finding a fixed point for a nonexpansive mapping T on a closed and convex subset C in a Hilbert space. Qin *et al.* motivated by Halpern and many others to introduce an iterative method for an infinite family of nonexpansive mappings in the framework of Hilbert spaces (see [19]).

Recently, to solve (1.3) with $\mathscr{F} = \mathrm{Fix}(T)$, the set of fixed points of a continuous pseudocontractive mapping T on a Banach space E, in [2], Ceng *et al.* proposed a new implicit algorithm:

$$x_t = t(I - \mu_t F)x_t + (1-t)Tx_t. \tag{1.13}$$

They proved the following results.

Theorem 1.7. *Let* E *be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose that* $T : E \to E$ *is a continuous pseudocontractive mapping and* $\mathscr{F} = \mathrm{Fix}(T) \neq \emptyset$*. Assume that* $F : E \to E$ *is* η*-strongly accretive and* λ*-strictly pseudocontractive with* $\eta + \lambda > 1$*. For each* $t \in (0,1)$*, choose a number* $\mu_t \in (0,1)$ *arbitrarily and let* $\{x_t\}$ *be defined by* (1.13)*. Then, as* $t \to 0^+$*,* x_t *converges strongly to the unique solution of* (1.3)*.*

Motivated by (1.7)-(1.13), in this paper, we obtain strong convergence theorems for three new implicit algorithms for solving problem (1.3). The first algorithm is a modification of (1.13) as follows:

$$x_k = \gamma_k(I-\lambda_k F)x_k + (1-\gamma_k)\frac{1}{t_k}\int_0^{t_k} T(s)x_k ds, \quad k \geq 1, \tag{1.14}$$

where the real numbers λ_k and γ_k are, respectively, in (0,1] and (0,1). The second algorithm is a modification of (1.7) and (1.13), generated by

$$x_k = \gamma_k(I-\lambda_k F)x_k + (1-\gamma_k)T(t_k)x_k, \quad k \geq 1. \tag{1.15}$$

The third iteration scheme is generated by

$$x_k = \frac{1}{t_k}\int_0^{t_k} T(s)(I-\lambda_k F)x_k ds, \quad k \geq 1, \tag{1.16}$$

where $\lambda_k \to 0$, as $k \to \infty$. In both algorithms (1.14) and (1.16), we assume that $0 < t_k \to \infty$ as $k \to \infty$. Meantimes, in (1.15), $0 < t_k \to 0$.

In Section 2, we give some preliminaries. In Section 3, we prove our main results, strong convergence of (1.14)-(1.16).

2. Preliminaries

Let E be a real normed linear space. Let $S_1(0) := \{x \in E : \|x\| = 1\}$. The space E is said to have a *Gâteaux differentiable* norm (or to be smooth) if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x, y \in S_1(0)$. The space E is said to have a *uniformly Gâteaux differentiable* norm if the limit is attained uniformly for $x \in S_1(0)$.

It is well known that if E is smooth, then the normalized duality mapping is single valued; and if the norm of E is uniformly Gâteaux differentiable, then the normalized duality mapping is norm to weak star uniformly continuous on every bounded subset of E (see [6]). In the sequel, we shall denote the single valued normalized duality mapping by j.

Recall that a Banach space E is said to be strictly convex, if for $x, y \in S_1(0)$ with $x \neq y$, then

$$\|(1-\lambda)x+\lambda y\| < 1 \quad \forall \lambda \in (0,1),$$

and uniformly convex, if for any ε, $0 < \varepsilon \leq 2$, the inequalities $\|x\| \leq 1, \| y\| \leq 1$, and $\|x-y\| \geq \varepsilon$ imply that there exists a $\delta = \delta(\varepsilon) \geq 0$ such that $\|(x+y)/2\| \leq 1-\delta$. It is well-known that every uniformly convex Banach space is reflexive and strictly convex.

Let μ be a continuous linear functional on l^∞ and let $(a_1, a_2, ...) \in l^\infty$. We write $\mu_k(a_k)$ instead of $\mu((a_1, a_2, ...))$. We recall that μ is a Banach limit when μ satisfies $\|\mu\| = \mu_k(1) = 1$ and $\mu_k(a_{k+1}) = \mu_k(a_k)$ for each $(a_1, a_2, ...) \in l^\infty$. For a Banach limit μ, we know that

$$\liminf_{k\to\infty} a_k \leq \mu_k(a_k) \leq \limsup_{k\to\infty} a_k$$

for all $(a_1, a_2, ...) \in l^\infty$. If $a = (a_1, a_2, ...) \in l^\infty$, $b = (b_1, b_2, ...) \in l^\infty$ and $a_k \to c$ (respectively, $a_k - b_k \to 0$), as $k \to \infty$, we have $\mu_k(a_k) = \mu(a) = c$ (respectively, $\mu_k(a_k) = \mu_k(b_k)$).

We will make use the following well-known results.

Lemma 2.1. [18] *Let E be a real-normed linear space. Then, the following inequality holds*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \quad \forall x,y \in E,\ \forall j(x+y) \in J(x+y).$$

Lemma 2.2. [2] *Let E and $F : E \to E$ be a real smooth Banach space and an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta + \gamma > 1$, respectively. Then, for any $\lambda \in (0,1)$, $I - \lambda F$ is contractive with constant $1 - \lambda\tau$, where $\tau = 1 - \sqrt{(1-\eta)/\gamma} \in (0,1)$.*

Lemma 2.3. [4] *Let C be a nonempty, bounded, closed and convex subset of a uniformly convex Banach space E and let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on C. Then, for any $r > 0$ and $h > 0$,*

$$\lim_{t\to\infty} \sup_{y\in C\cap B_r} \left\| T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right) - \frac{1}{t}\int_0^t T(s)yds \right\| = 0,$$

where $B_r = \{x \in E : \|x\| \leq r\}$.

Lemma 2.4. [23] *Let C be a closed and convex subset of a Banach space E whose norm is uniformly Gâteaux differentiable. Let $\{x_k\}$ be a bounded subset of E, let z be an element of C and μ be a Banach limit. Then,*

$$\mu_k\|x_k - z\|^2 = \min_{u\in C} \mu_k\|x_k - u\|^2$$

if and only if $\mu_k\langle u - z, j(x_k - z)\rangle \leq 0$ for all $u \in C$.

3. Main results

Now, we are in a position to prove the following results.

Theorem 3.1. *Let F be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta + \gamma > 1$ and let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on E, which is a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm, such that $\mathscr{F} := \cap_{s>0}\mathrm{Fix}(T(s)) \neq \emptyset$. Then, the sequence $\{x_k\}$, defined by* (1.14) *with $\gamma_k \in (0,1)$, $\lambda_k \in (0,1]$ and $t_k > 0$ such that $\gamma_k \to 0$ and $t_k \to \infty$, as $k \to \infty$, converges strongly to a unique element p^*, solving* (1.3)*.*

Proof. Consider the mapping

$$T_k x = \gamma_k(I - \lambda_k F)x + (1-\gamma_k)\frac{1}{t_k}\int_0^{t_k} T(s)xds,$$

for all $k \geq 1$ and $x \in E$. Then, by Lemma 2.2, we have

$$\begin{aligned}
&\|T_k x - T_k y\| \\
&= \left\| \gamma_k(I - \lambda_k F)x + (1-\gamma_k)\frac{1}{t_k}\int_0^{t_k} T(s)xds - \left[\gamma_k(I-\lambda_k F)y + (1-\gamma_k)\frac{1}{t_k}\int_0^{t_k} T(s)yds\right] \right\| \\
&= \left\| \gamma_k[(I-\lambda_k F)x - (I-\lambda_k F)y] + (1-\gamma_k)\frac{1}{t_k}\int_0^{t_k}(T(s)x - T(s)y)ds \right\| \\
&\leq \gamma_k(1-\lambda_k\tau)\|x-y\| + (1-\gamma_k)\|x-y\| = (1-\gamma_k\lambda_k\tau)\|x-y\|
\end{aligned}$$

with $\gamma_k\lambda_k\tau \in (0,1)$. So, T_k is a contraction in E. By Banach's Contraction Principle, there exists a unique element $x_k \in E$ such that $x_k = T_k x_k$ for all $k \geq 1$.

By putting

$$z_k = \frac{1}{t_k}\int_0^{t_k} T(s)x_k ds,$$

and noting that $\|z_k - p\| \le \|x_k - p\|$ for any fixed $p \in \mathscr{F}$, we have

$$\begin{aligned}\|x_k - p\|^2 &= \|\gamma_k(I-\lambda_k F)x_k + (1-\gamma_k)z_k - p\|^2\\ &= \gamma_k\langle \lambda_k(I-F)x_k + (1-\lambda_k)x_k - p, j(x_k-p)\rangle + (1-\gamma_k)\langle z_k - p, j(x_k-p)\rangle\\ &\le \gamma_k\lambda_k\langle (I-F)x_k - p, j(x_k-p)\rangle + \gamma_k(1-\lambda_k)\|x_k-p\|^2 + (1-\gamma_k)\|x_k-p\|^2\\ &\le \gamma_k\lambda_k\langle (I-F)x_k - p, j(x_k-p)\rangle + (1-\gamma_k\lambda_k)\|x_k-p\|^2\\ &= \gamma_k\lambda_k\langle (I-F)x_k - (I-F)p - F(p), j(x_k-p)\rangle + (1-\gamma_k\lambda_k)\|x_k-p\|^2.\end{aligned}$$

Therefore, by Lemma 2.2, we have

$$\|x_k-p\|^2 \le (1-\tau)\|x_k-p\|^2 - \langle F(p), j(x_k-p)\rangle$$

and hence

$$\|x_k-p\|^2 \le \tau^{-1}\langle F(p), j(p-x_k)\rangle. \tag{3.1}$$

Consequently, $\|x_k - p\| \le \tau^{-1}\|F(p)\|$. It means that $\{x_k\}$ is bounded. So, are the sequences $\{z_k\}$ and $\{F(x_k)\}$. Further,

$$\|x_k - z_k\| = \|\gamma_k(x_k - z_k) - \gamma_k\lambda_k F(x_k)\| \le \gamma_k\|x_k - z_k\| + \gamma_k\lambda_k\|F(x_k)\|,$$

which implies that

$$\|x_k - z_k\| \le \frac{\gamma_k\lambda_k}{1-\gamma_k}\|F(x_k)\|.$$

Since $\gamma_k \to 0, \lambda_k \in (0,1]$ and $\{F(x_k)\}$ is bounded,

$$\lim_{k\to\infty}\|x_k - z_k\| = 0. \tag{3.2}$$

Next, we show that

$$\lim_{k\to\infty}\|x_k - T(h)x_k\| = 0 \quad \forall h > 0. \tag{3.3}$$

Consider the set

$$D = \{z \in E : \|z - p\| \le \tau^{-1}\|F(p)\|\}.$$

Clearly, D is a nonempty, closed, convex and $T(h)$-invariant subset of E. So, by Lemma 2.3,

$$\|z_k - T(h)z_k\| \to 0, \tag{3.4}$$

as $k \to \infty$. This fact together with (3.2) implies (3.3).

Now, for a Banach limit μ, we can define a mapping $\varphi : E \to \mathbb{R}$ by

$$\varphi(u) = \mu_k\|x_k - u\|^2 \quad \forall u \in E.$$

We see that $\varphi(u) \to \infty$ as $\|u\| \to \infty$, φ is continuous and convex, so as E is reflexive, there exists $\tilde{p} \in E$ such that $\varphi(\tilde{p}) = \min_{u\in E}\varphi(u)$. Moreover, the element $\tilde{p}$ is unique (see, [26]). From (3.3) it follows that

$$\varphi(T(h)\tilde{p}) = \mu_k\|x_k - T(h)\tilde{p}\|^2 = \mu_k\|T(h)x_k - T(h)\tilde{p}\|^2 \le \mu_k\|x_k - \tilde{p}\|^2 = \varphi(\tilde{p})$$

which implies that $T(h)\tilde{p} = \tilde{p}$, that is $\tilde{p} \in \mathscr{F}$. From Lemma 2.4, we know that $\tilde{p}$ is a minimizer of $\varphi(u)$ on E, if and only if

$$\mu_k\langle u - \tilde{p}, j(x_k - \tilde{p})\rangle \leq 0 \quad \forall u \in E. \tag{3.5}$$

Taking $u = (I - F)(\tilde{p})$ in (3.5), we obtain that

$$\mu_k\langle F(\tilde{p}), j(\tilde{p} - x_k)\rangle \leq 0. \tag{3.6}$$

Using (3.1) and (3.6), we obtain that $\mu_k\|x_k - \tilde{p}\|^2 = 0$. Hence, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ which strongly converges to $\tilde{p}$ as $i \to \infty$. Again, from (3.1) and the norm to weak star continuous property of the normalized duality mapping j on bounded subsets of E, we obtain that

$$\langle F(p), j(\tilde{p} - p)\rangle \leq 0 \quad \forall p \in \mathscr{F}. \tag{3.7}$$

Since p and $\tilde{p}$ belong to $\mathscr{F}$, a closed and convex subset, by replacing p in (3.7) by $sp + (1-s)\tilde{p}$ for $s \in (0,1)$, using the well-known property $j(s(\tilde{p} - p)) = sj(\tilde{p} - p)$ for $s > 0$, dividing by s and taking $s \to 0$, we obtain

$$\langle F(\tilde{p}), j(\tilde{p} - p)\rangle \leq 0 \quad \forall p \in \mathscr{F}.$$

The uniqueness of p^* in (1.3) guarantees that $\tilde{p} = p^*$. So, all the sequence $\{x_k\}$ converges strongly to p^* as $k \to \infty$. This completes the proof. ∎

Theorem 3.2. *Let F be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta + \gamma > 1$ and let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on E, which is a real reflexive Banach space with a uniformly Gâteaux differentiable norm, such that $\mathscr{F} := \cap_{s>0} \mathrm{Fix}(T(s)) \neq \emptyset$ and condition* (1.9) *is satisfied for any bounded subset K of E. Then, the sequence $\{x_k\}$, defined by* (1.15) *with $\lambda_k \in (0,1]$, $\gamma_k \in (0,1)$ and $t_k > 0$, satisfying* (1.5)*, converges strongly to a unique element p^*, solving* (1.3).

Proof. Consider the mapping

$$T_k x = \gamma_k(I - \lambda_k F)x + (1 - \gamma_k)T(t_k)x,$$

for all $k \geq 1$ and $x \in E$. Then, as in the proof of Theorem 3.1, there exists a unique x_k, satisfying (1.15), the sequence $\{x_k\}$ is bounded and satisfies (3.1). Since E is reflexive and $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_j}\} \subset \{x_k\}$, that converges weakly to some element $\tilde{p} \in E$.

Now, we prove that $\tilde{p} = T(t)\tilde{p}$ for a fixed $t > 0$. It is easy to see that

$$\begin{aligned}
\|x_{k_j} - T(t)x_{k_j}\| &\leq \sum_{l=0}^{[t/t_{k_j}]-1} \|T(lt_{k_j})x_{k_j} - T((l+1)t_{k_j})x_{k_j}\| + \|T(t)x_{k_j} - T([t/t_{k_j}]t_{k_j})x_{k_j}\| \\
&\leq [t/t_{k_j}]\|x_{k_j} - T(t_{k_j})x_{k_j}\| + \|T(t - [t/t_{k_j}]t_{k_j})x_{k_j} - x_{k_j}\| \\
&\leq \frac{\gamma_{k_j}}{t_{k_j}} t\|(I - \lambda_{k_j}F)x_{k_j} - T(t_{k_j})x_{k_j}\| + \sup\{\|T(s)x_{k_j} - x_{k_j}\| : 0 \leq s \leq t_{k_j}\}.
\end{aligned}$$

This fact together with the boundedness of $\{x_k\}$ and $\{F(x_k)\}$, $t_{k_j}, \gamma_{k_j}/t_{k_j} \to 0$ and (1.9) implies that

$$\limsup_{j\to\infty} \|x_{k_j} - T(t)x_{k_j}\| = 0.$$

Further, by the argument as in the proof of Theorem 3.1, we obtain the conclusion. This completes the proof. ∎

Theorem 3.3. *Let F be an η-strongly accretive and γ-strictly pseudocontractive mapping with $\eta+\gamma>1$ and let $\{T(s): s>0\}$ be a nonexpansive semigroup on E, which is a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm, such that $\mathscr{F} := \cap_{s>0} \mathrm{Fix}(T(s)) \neq \emptyset$. Then, the sequence $\{x_k\}$, defined by* (1.16) *with $\lambda_k \in (0,1]$ and $t_k > 0$ such that $\lambda_k \to 0$ and $t_k \to \infty$, as $k \to \infty$, converges strongly to an element p^*, solving* (1.3).

Proof. Consider the mapping

$$\tilde{T}_k x = \frac{1}{t_k}\int_0^{t_k} T(s)(I-\lambda_k F)x ds \quad \forall x \in E.$$

From Lemma 2.2, it follows

$$\begin{aligned}\|\tilde{T}_k x - \tilde{T}_k y\| &= \frac{1}{t_k}\left\|\int_0^{t_k} T(s)[(I-\lambda_k F)x-(I-\lambda_k F)y]ds\right\| \\ &\leq \|(I-\lambda_k F)x-(I-\lambda_k F)y\| \leq (1-\lambda_k\tau)\|x-y\| \quad \forall x,y \in E.\end{aligned}$$

So, $\tilde{T}_k$ is a contraction in E. By Banach's Contraction Principle, there exists a unique element $x_k \in E$, satisfying (1.16).

Next, we show that $\{x_k\}$ is bounded. Indeed, for a point $p \in \mathscr{F}$, we have, by Lemma 2.2,

$$\begin{aligned}\|x_k - p\| &= \left\|\frac{1}{t_k}\int_0^{t_k} T(s)(I-\lambda_k F)x_k ds - \frac{1}{t_k}\int_0^{t_k} T(s)p ds\right\| \leq \|(I-\lambda_k F)x_k - p\| \\ &= \|(I-\lambda_k F)x_k - (I-\lambda_k F)p - \lambda_k F(p)\| \leq (1-\lambda_k\tau)\|x_k-p\| + \lambda_k\|F(p)\|.\end{aligned}$$

Therefore, $\|x_k - p\| \leq \|F(p)\|/\tau$, that implies the boundedness of $\{x_k\}$. So, is the sequence $\{F(x_k)\}$. Consider the set $C = \{z \in E : \|z-p\| \leq \|F(p)\|/\tau\}$. As in the proof of Theorem 3.1, we obtain (3.4). On the other hand,

$$\begin{aligned}\|x_k - z_k\| &= \left\|\frac{1}{t_k}\int_0^{t_k} T(s)(I-\lambda_k F)x_k ds - \frac{1}{t_k}\int_0^{t_k} T(s)x_k ds\right\| \\ &= \left\|\frac{1}{t_k}\int_0^{t_k}[T(s)(I-\lambda_k F)x_k - T(s)x_k]ds\right\| \\ &\leq \frac{1}{t_k}\int_0^{t_k}\|(I-\lambda_k F)x_k - x_k\|ds = \lambda_k\|F(x_k)\| \to 0,\end{aligned}$$

because $\lambda_k \to 0$, as $k \to \infty$, and hence (3.3) holds. Next, by the convexity of $\|\cdot\|^2$ and Lemmas 2.1 and 2.2, for any $p \in \mathscr{F}$, we have

$$\begin{aligned}\|x_k - p\|^2 &\leq \|(I-\lambda_k F)x_k - p\|^2 = \|(I-\lambda_k F)x_k - (I-\lambda_k F)p - \lambda_k F(p)\|^2 \\ &\leq (1-\lambda_k\tau)\|x_k-p\|^2 - 2\lambda_k\langle F(p), j(x_k - p - \lambda_k F(x_k))\rangle.\end{aligned}$$

So,

$$\|x_k - p\|^2 \leq \frac{2}{\tau}\langle F(p), j(p-x_k)\rangle + \frac{2}{\tau}\langle F(p), j(p-x_k+\lambda_k F(x_k)) - j(p-x_k)\rangle. \tag{3.8}$$

By using (3.3), (3.8) instead of (3.1), the normalized duality mapping is norm to weak star uniformly continuous on every bounded subset of E, and repeating the rest proof of Theorem 3.1, we obtain the conclusion. This completes the proof. ∎

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