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Hopf Hypersurfaces with η-Parallel Shape Operator in Complex Two-Plane Grassmannians

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Abstract. We consider a new notion of η-parallel shape operator in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and give a non-existence theorem for a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with η-parallel shape operator.

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1. Introduction

We denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It is the unique compact irreducible Riemannian manifold with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. Namely, $G_2(\mathbb{C}^{m+2})$ is a unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold. Accordingly, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometric conditions for real hypersurfaces M that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M (see [3–5]). Here the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$ and $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

By using two invariant conditions mentioned above and the result in Alekseevskii [1], Berndt and Suh [4] proved the following:

Theorem 1.1. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

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(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Furthermore, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in Section 3 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. When the distribution $\mathfrak{D}$ of a hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, we say M is a *$\mathfrak{D}$-invariant hypersurface*.

Using Theorem 1.1, many geometers have given characterizations for hypersurfaces in $G_2(\mathbb{C}^{m+2})$ under certain assumption for various geometrical conditions instead of the above two invariant conditions, for example, shape operator, normal (or structure) Jacobi operator, structure tensor, and so on. Actually, Lee and Suh [12] gave a characterization of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

Theorem 1.2. *Let M be a connected orientable Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$, where the distribution $\mathfrak{D}$ denotes the orthogonal complement of $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$.*

On the other hand, to give some characterization of homogeneous hypersurfaces of Type (A) and (B) in complex projective spaces $\mathbb{C}P^n$ Kimura and Maeda [10] introduced the notion of a *η-parallel shape operator*, which was defined by

$$g((\nabla_X A)Y, Z) = 0 \tag{1.1}$$

for any vector fields X, Y, Z orthogonal to the Reeb vector field ξ where g and ∇ denote the induced Riemannian metric and the Levi-Civita connection, respectively. This kind of notion was extended to Hopf hypersurfaces in complex hyperbolic space $\mathbb{C}H^n$ by Suh [13]. As non-Hopf hypersurfaces in $M_n(c)$ with η-parallel shape operator we can give a class of ruled real hypersurfaces in $M_n(c)$ (see [10] for $c > 0$ and [2] for $c < 0$, respectively). Recently, Kon and Loo [11] gave a complete classification of real hypersurfaces in a non-flat complex space form with η-parallel shape operator as follows:

Theorem 1.3. *Let M be a real hypersurface in non-flat complex space form $M_n(c)$, $n \geq 3$. Then M has a η-parallel shape operator if and only if it is locally congruent to a ruled real hypersurface, or a real hypersurface of Type* (A) *or Type* (B) *from Takagi's list or Montiel's list.*

From such a point of view, naturally we ask the following problem:

> *Can we classify real hypersurfaces in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with η-parallel shape operator?*

Corresponding to this problem, in this paper we give the following theorem:

Theorem 1.4 (Main Theorem)**.** *There does not exist any connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with η-parallel shape operator in Levi-Civita connection, that is, $g((\nabla_X A)Y, Z) = 0$ for any tangent vector fields $X, Y, Z \in \mathfrak{h}$, where the distribution $\mathfrak{h}$ denotes the orthogonal complement of $[\xi] = \mathrm{Span}\{\xi\}$.*

Remark 1.1. Actually, in [6] the authors gave a characterization of $\mathfrak{D}$-invariant real hypersurfaces of Type (A), that is, a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ or a ruled real hypersurface foliated by complex hypersurfaces which includes a maximal totally geodesic submanifold $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in terms of η-parallel shape operator.

Moreover, for a real hypersurface in a complex two-plane Grassmannian the authors in [7] and [8] have introduced the notion of *generalized Tanaka-Webster* (in short, *g-Tanaka-Webster*) *connection* $\hat{\nabla}^{(k)}$, where k is a non-zero real number. Then by using this connection, we consider a η-parallel shape operator in g-Tanaka-Webster connection, that is,

$$g((\hat{\nabla}^{(k)}_X A)Y,Z)=0 \tag{1.2}$$

for any vector fields X,Y,Z orthogonal to the Reeb vector field ξ. We say that the shape operator is *generalized Tanaka-Webster η-parallel*, if the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ satisfies the condition (1.2). Using this notion, we have the following corollary:

Corollary 1.1. *There does not exist any connected orientable Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m\geq 3$ with generalized Tanaka-Webster η-parallel shape operator.*

In Section 2, we recall Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. In Section 3, the generalized Tanaka-Webster connection and some fundamental formulas including the Codazzi equation for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ will be also recalled. In Sections 4 and 5, we will give a complete proof of our main theorem according to the geodesic Reeb flow satisfying $\xi\in\mathfrak{D}^{\perp}$ or $\xi\in\mathfrak{D}$. In Section 6 we will show that the generalized Tanaka-Webster η-parallel shape operator coincides with usual η-parallel shape operator in $G_2(\mathbb{C}^{m+2})$. Finally, in Section 7, we will give the geometrical meaning of the notion of η-parallel by using the relation between η-parallel and cyclic η-parallel.

2. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section, we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [3–5]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G=SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K=S(U(2)\times U(m))\subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o=eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m}\times\mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}),g)$ is eight.

When $m=1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m=2$, we note that the isomorphism $\mathrm{Spin}(6)\simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m\geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$ for $\nu = 1,2,3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\widetilde{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\widetilde{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{2.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3} \left\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\right\} \\ & + \sum_{\nu=1}^{3} \left\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\right\},\end{aligned} \tag{2.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

3. Some fundamental formulas

In this section, we introduce the generalized Tanaka-Webster connection and derive some basic formulas including the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [5, 7–9, 12]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N. Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.1}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X,\xi) \tag{3.2}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in section 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned} & \phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu \xi_\nu = 0, \\ & \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\ & \phi_\nu \phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu, \\ & \phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1} \end{aligned} \tag{3.3}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1,2,3$ in Section 2 and (3.1), the relation between these two contact metric structures (ϕ,ξ,η,g) and $(\phi_\nu,\xi_\nu,\eta_\nu,g)$, $\nu = 1,2,3$, can be given by

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi. \tag{3.4}$$

On the other hand, as J is a Kähler structure, that is, $\widetilde{\nabla}J = 0$ and J_ν a quaternionic Kähler structure (see (2.1)), together with Gauss and Weingarten formulas it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{3.5}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{3.6}$$

$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu. \tag{3.7}$$

Using the above expression (2.2) for the curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi is given by

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\} \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ & + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu.\end{aligned} \tag{3.8}$$

Now let us recall the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for real hypersurfaces in Kähler manifolds as follows:

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX,Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{3.9}$$

for a non-zero real number k (see [7] and [8]). In particular, if a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the g-Tanaka-Webster connection coincides with the Tanaka-Webster connection defined as the canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold.

4. Key lemma

From now on, we assume that M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with η-parallel shape operator with respect to Levi-Civita connection ∇. In other words, the shape operator A of M satisfies the following condition

$$g((\nabla_X A)Y,Z) = 0 \tag{4.1}$$

for any tangent vector fields X,Y,Z on $\mathfrak{h}$ where $\mathfrak{h}$ denotes a distribution orthogonal to the Reeb vector field ξ, that is, $\mathfrak{h} = \{X \in T_xM \mid X \perp \xi\}$.

From the equation of Codazzi (3.8) and (4.1), we have

$$\begin{aligned}\sum_{\nu=1}^{3}\{&\eta_\nu(X)g(\phi_\nu Y,Z) - \eta_\nu(Y)g(\phi_\nu X,Z) - 2g(\phi_\nu X,Y)\eta_\nu(Z) \\ & + \eta_\nu(\phi X)g(\phi_\nu\phi Y,Z) - \eta_\nu(\phi Y)g(\phi_\nu\phi X,Z)\} = 0\end{aligned} \tag{4.2}$$

for any tangent vector fields X, Y, Z on $\mathfrak{h}$.

In this section, our main purpose is to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the orthogonal complement $\mathfrak{D}^\perp$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$ for any point $x \in M$. To show this fact, unless otherwise stated in this section, we consider that the Reeb vector field ξ satisfies

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{4.3}$$

for some unit $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$.

Now, under these situations, we will give the following lemma:

Lemma 4.1. *Let M be a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with η-parallel shape operator related to Levi-Civita connection of M. Then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. Assume that a real hypersurface M has the η-parallel shape operator. From our notation (4.3), we see that $\xi_2 \in \mathfrak{h}$. Substituting $Y = Z = \xi_2 \in \mathfrak{h}$ in (4.2), we have

$$2\eta_1(\xi)\eta_1(\phi X) = 0, \quad \forall X \in \mathfrak{h}. \tag{4.4}$$

Since $\phi\xi = 0$, we see that the vector ϕX_0 also belongs to distribution $\mathfrak{h}$. Thus putting $X = \phi X_0$ in (4.4), it follows

$$2\eta_1^2(\xi)\eta(X_0) = 0.$$

This gives a contradiction. Hence we complete the proof of our lemma. ∎

5. The proof of main theorem

Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with η-parallel shape operator for Levi-Civita connection ∇ on M. Then by Lemma 4.1 we consider the following two cases:

Case I: the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$,

Case II: the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.

First of all, let us consider the Case I, that is, $\xi \in \mathfrak{D}^\perp$. Accordingly, we may put $\xi = \xi_1$. Under these assumptions, we assert the following:

Theorem 5.1. *There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with η-parallel shape operator for Levi-Civita connection when the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$.*

Proof. From our assumptions, the equation (4.2) can be written by

$$\begin{aligned} &g\big(\eta_2(X)\phi_2 Y - \eta_2(Y)\phi_2 X - 2g(\phi_2 X, Y)\xi_2 + \eta_2(\phi X)\phi_2\phi Y - \eta_2(\phi Y)\phi_2\phi X \\ &\quad + \eta_3(X)\phi_3 Y - \eta_3(Y)\phi_3 X - 2g(\phi_3 X, Y)\xi_3 + \eta_3(\phi X)\phi_3\phi Y - \eta_3(\phi Y)\phi_3\phi X, Z\big) = 0 \end{aligned} \tag{5.1}$$

for any tangent vectors $X, Y, Z \in \mathfrak{h}$.

Let $\{e_1, e_2, \cdots, e_{4m-2}, \xi\}$ be an orthonormal basis of $T_xM = \mathfrak{h} \oplus [\xi]$. Let us denote by W_{XY} the tangent vector field on M given by

$$\begin{aligned} W_{XY} = {}&\eta_2(X)\phi_2 Y - \eta_2(Y)\phi_2 X - 2g(\phi_2 X, Y)\xi_2 + \eta_2(\phi X)\phi_2\phi Y - \eta_2(\phi Y)\phi_2\phi X \\ &+ \eta_3(X)\phi_3 Y - \eta_3(Y)\phi_3 X - 2g(\phi_3 X, Y)\xi_3 + \eta_3(\phi X)\phi_3\phi Y - \eta_3(\phi Y)\phi_3\phi X \end{aligned}$$

for any tangent vectors $X, Y \in \mathfrak{h}$.

From the equation (5.1), the tangent vector $W_{XY} \in T_xM$ becomes

$$\begin{aligned} W_{XY} &= \sum_{j=1}^{4m-2} g(W_{XY}, e_j)e_j + \eta(W_{XY})\xi \\ &= g\big(\eta_2(X)\phi_2 Y - \eta_2(Y)\phi_2 X - 2g(\phi_2 X, Y)\xi_2 + \eta_2(\phi X)\phi_2\phi Y - \eta_2(\phi Y)\phi_2\phi X \\ &\quad + \eta_3(X)\phi_3 Y - \eta_3(Y)\phi_3 X - 2g(\phi_3 X, Y)\xi_3 + \eta_3(\phi X)\phi_3\phi Y - \eta_3(\phi Y)\phi_3\phi X, \xi\big)\xi \\ &= 4\big\{\eta_2(X)\eta_3(Y) - \eta_3(X)\eta_2(Y)\big\}\xi, \end{aligned}$$

that is,

$$\begin{aligned} &\eta_2(X)\phi_2 Y - \eta_2(Y)\phi_2 X - 2g(\phi_2 X, Y)\xi_2 + \eta_2(\phi X)\phi_2\phi Y - \eta_2(\phi Y)\phi_2\phi X \\ &+ \eta_3(X)\phi_3 Y - \eta_3(Y)\phi_3 X - 2g(\phi_3 X, Y)\xi_3 + \eta_3(\phi X)\phi_3\phi Y - \eta_3(\phi Y)\phi_3\phi X \\ &= 4\big\{\eta_2(X)\eta_3(Y) - \eta_3(X)\eta_2(Y)\big\}\xi \end{aligned} \tag{5.2}$$

for any tangent vectors $X, Y \in \mathfrak{h}$.

Taking the inner product with ξ_2 in (5.2), we have $g(\phi_2 X, Y) = 0$ for $X, Y \in \mathfrak{h}$. Thus we obtain that

$$\phi_2 X = \sum_{j=1}^{4m-2} g(\phi_2 X, e_i)e_i + g(\phi_2 X, \xi)\xi = \eta_3(X)\xi \tag{5.3}$$

for any tangent vector field $X \in \mathfrak{h}$ where $\{e_1 = \xi_2, e_2 = \xi_3, e_3, e_4, \cdots, e_{4m-2}\}$ is an orthonormal basis of $\mathfrak{h}$. Using (3.3), it follows that

$$X = \eta_2(X)\xi_2 + \eta_3(X)\xi_3, \quad \forall X \in \mathfrak{h}. \tag{5.4}$$

From this, we obtain that any vector e_j is zero for $j = 3, 4, \cdots, 4m-2$. By the constitution of tangent vector space T_xM, we see that $\dim M = 3$. But, since we consider the dimension of M is $4m-1$ where $m \geq 3$, we can assert our theorem. ∎

Remark 5.1. Now, we define a new notion related to the parallelism of shape operator on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, namely, $\mathfrak{D}$-parallel shape operator. That is, if the shape operator A of M satisfies the condition (4.1) for $X, Y, Z \in \mathfrak{D}$ then we say that A is *$\mathfrak{D}$-parallel*. Furthermore, when a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ has such a shape operator, M is said to be a *$\mathfrak{D}$-parallel hypersurface*. Using the derivative formula of the shape operator for real hypersurfaces of Type (A) in Theorem 1.1, it can be easily verified that the shape operator of this type is $\mathfrak{D}$-parallel, but not η-parallel (see [5, 6]).

Next we consider the case $\xi \in \mathfrak{D}$. Then, by Theorem 1.3, we see that M is locally congruent to a real hypersurface of Type (B) under our assumptions. Thus from now on, let us check whether the shape operator A for real hypersurfaces of Type (B) satisfies the condition (4.1) for any tangent vector fields $X, Y, Z \in \mathfrak{h}$. In order to solve this problem, let us recall the following proposition given by Berndt and Suh in [4]:

Proposition 5.1. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n-4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi = \mathrm{Span}\{\xi\}, \quad T_\beta = \mathfrak{J}J\xi = \mathrm{Span}\{\xi_\nu \mid \nu = 1,2,3\},$$
$$T_\gamma = \mathfrak{J}\xi = \mathrm{Span}\{\phi_\nu \xi \mid \nu = 1,2,3\}, \quad T_\lambda, \quad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^\perp$ is the orthogonal complement of $\mathbb{H}\mathbb{C}\xi$ where

$$\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

Let us denote by M_B hypersurface in $G_2(\mathbb{C}^{m+2})$ mentioned in Proposition A. Suppose M_B has η-parallel shape operator. From Proposition A we see that four eigenspaces T_β, T_γ, T_λ and T_μ belong to the distribution $\mathfrak{h}$. It follows that $Y \in \mathfrak{h}$ and $AY = \beta Y$ for any tangent vector field $Y \in T_\beta$. From this, differentiating with respect to any direction $X \in T_xM_B$, we get

$$(\nabla_X A)Y = \beta(\nabla_X Y) - A(\nabla_X Y).$$

Taking the inner product with $Z \in T_\lambda \subset \mathfrak{h}$, it follows

$$\begin{aligned} g\big((\nabla_X A)Y, Z\big) &= \beta g(\nabla_X Y, Z) - g\big(A(\nabla_X Y), Z\big) \\ &= \beta g(\nabla_X Y, Z) - \lambda g(\nabla_X Y, Z) = (\beta - \lambda) g(\nabla_X Y, Z) \end{aligned} \tag{5.5}$$

for any tangent vector fields $X \in T_xM_B$, $Y \in T_\beta$ and $Z \in T_\lambda$. Now since $Y \in T_\beta$, we may put $Y = \xi_\mu$ $(\mu = 1,2,3)$. Then from (3.6), we have

$$g(\nabla_X Y, Z) = g(\nabla_X \xi_\mu, Z) = g(\phi_\mu AX, Z), \quad \mu = 1,2,3$$

for $X \in T_xM_B$, $Y \in T_\beta$ and $Z \in T_\lambda$.

On the other hand, we know that $\phi_\mu Z \in T_\lambda$ for $Z \in T_\lambda$. Thus, we get

$$g(\nabla_X Y, Z) = -g(A\phi_\mu Z, X) = -\lambda g(\phi_\mu Z, X), \quad \mu = 1,2,3 \tag{5.6}$$

for $X \in T_xM_B$, $Y \in T_\beta$ and $Z \in T_\lambda$.

From (5.5) and (5.6), our assumption that M_B has η-parallel shape operator implies that

$$\lambda(\lambda - \beta) g(\phi_\mu Z, X) = 0, \quad \mu = 1,2,3 \tag{5.7}$$

for any $X \in \mathfrak{h}$, $Y \in T_\beta$ and $Z \in T_\lambda$.

Replacing X by $\phi_\mu Z \in T_\lambda \subset \mathfrak{h}$ in (5.7), we have

$$\lambda(\lambda - \beta) = 0.$$

But, from Proposition 5.1, we see that $\lambda^2 - \lambda\beta = 1$ for some $r \in (0, \pi/4)$. This gives a contradiction. So this case can not occur.

Therefore we give the following:

Lemma 5.1. *The shape operator A of a real hypersurface of Type (B) in $G_2(\mathbb{C}^{m+2})$ does not satisfy the η-parallel condition* (4.1).

As mentioned before, by virtue of Theorem 1.3, when $\xi \in \mathfrak{D}$ we see that M is locally congruent to a real hypersurface of Type (B) under our assumptions. But a real hypersurface of Type (B) does not have η-parallel shape operator (see Lemma 5.1). From these facts, we obtain the following theorem:

Theorem 5.2. *There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with η-parallel shape operator in Levi-Civita connection, when the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.*

Summing up Lemma 4.1, and Theorems 5.1 and 5.2, we give a complete proof of Theorem 1.4.

6. The proof of Corollary 1.1

In this section, we consider the condition of η-parallel shape operator with respect to generalized Tanaka-Webster connection defined on real hypersurfaces of complex two-plane Grassmannians, that is,

$$g\big((\hat{\nabla}_X^{(k)} A)Y, Z\big) = 0 \tag{6.1}$$

for any tangent vector fields $X, Y, Z \in \mathfrak{h}$ where $\mathfrak{h} = \{X \in T_xM \mid X \perp \xi\}$.

Proposition 6.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the shape operator A of M satisfies the η-parallel shape operator for generalized Tanaka-Webster connection, then A becomes η-parallel shape operator for Levi-Civita connection.*

Proof. First, in order to prove this proposition, we introduce the following fundamental equation for covariant derivative of shape operator with respect to generalized Tanaka-Webster connection:

$$\begin{aligned}(\widehat{\nabla}_X^{(k)} A)Y = (\nabla_X A)Y &+ g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY \\ &- g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y\end{aligned} \tag{6.2}$$

for any tangent vector fields X, Y on M (see [7,8]). This equation is derived from the definition of generalized Tanaka-Webster connection for a real hypersurface on Kähler manifolds, $\widehat{\nabla}_X^{(k)} Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$ (see (3.9)).

Restricting $X, Y \in \mathfrak{h}$ in (6.2), it can be written by

$$(\widehat{\nabla}_X^{(k)} A)Y = (\nabla_X A)Y + g(\phi AX, AY)\xi - \alpha g(\phi AX, Y)\xi$$

for $X, Y \in \mathfrak{h}$. Taking inner product with $Z \in \mathfrak{h}$, we have

$$g((\widehat{\nabla}_X^{(k)} A)Y, Z) = g((\nabla_X A)Y, Z)$$

for X, Y and $Z \in \mathfrak{h}$. Thus we can assert our Proposition 6.1. ∎

From this proposition and the proof of Theorem 1.4 in Sections 4 and 5, we give a complete proof of Corollary 1.1.

7. The geometrical meaning of η-parallel shape operator

Let $\bar{M}$ be a Riemannian manifold with the Riemannian metric G and Riemannian connection $\bar{\nabla}$. Let M be a real hypersurface of $\bar{M}$ with induced metric g and induced Riemannian connection ∇. Since M is a real hypersurface of $\bar{M}$, there exist only one normal vector field N on M in $\bar{M}$. Thus we have the following two formulae:

$$\begin{aligned}&\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N && \text{(Gauss formula)} \\ &\bar{\nabla}_X N = -AX && \text{(Weingarten formula)}\end{aligned} \tag{7.1}$$

for arbitrary tangent vector fields X, Y on M.

Now, we introduce some notion for parallelism of shape operator: A real hypersurface M is called *cyclic parallel* (or *cyclic η-parallel*, resp.) if it satisfies

$$\mathfrak{S}_{X,Y,Z}g((\nabla_X A)Y,Z)=g((\nabla_X A)Y,Z)+g((\nabla_Y A)Z,X)+g((\nabla_Z A)X,Y)=0$$

for any tangent vector fields X,Y,Z on M (or $X,Y,Z\in\mathfrak{h}$, resp.).

Under these situations, for arbitrary geodesic γ on M, we assert:

Lemma 7.1. *The shape operator A of M is cyclic parallel if and only if*

$$(\mathrm{C}_1)\begin{cases}\textit{the first curvature function of } \gamma \textit{ as a curve in the ambient space } \bar{M} \textit{ is}\\ \textit{a constant function.}\end{cases}$$

Proof. Assume that the first curvature function for an arbitrary geodesic γ being considered as a curve in $\bar{M}$ is constant. It means by definition, $\bar{\nabla}_{\dot\gamma}\dot\gamma$ has constant length in $\bar{M}$, that is, $G(\bar{\nabla}_{\dot\gamma}\dot\gamma,\bar{\nabla}_{\dot\gamma}\dot\gamma)$ is constant on the interval I. From the Gauss formula in (7.1), we have $G(\bar{\nabla}_{\dot\gamma}\dot\gamma,\bar{\nabla}_{\dot\gamma}\dot\gamma)=g(A\dot\gamma,\dot\gamma)^2$. Hence our assumption is equivalent to the constancy of $g(A\dot\gamma,\dot\gamma)$ on I. By differentiation and using $\nabla_{\dot\gamma}\dot\gamma=0$, we obtain $g((\nabla_{\dot\gamma}A)\dot\gamma,\dot\gamma)=0$ on I. Therefore our assumption is equivalent to

$$g((\nabla_X A)X,X)=0 \tag{7.2}$$

for any tangent vector X of M. Using the linearity of the Riemannian connection, it follows that

$$g\big((\nabla_{X+Y+Z}A)(X+Y+Z),X+Y+Z\big)=2\mathfrak{S}_{X,Y,Z}g((\nabla_X A)Y,Z)=0, \tag{7.3}$$

where we have used

$$\begin{aligned}g\big((\nabla_{X+Y}A)(X+Y),X+Y\big)&=g((\nabla_X A)X,Y)+g((\nabla_X A)Y,X)+g((\nabla_X A)X,Y)\\&\quad+g((\nabla_Y A)X,X)+g((\nabla_Y A)X,Y)+g((\nabla_X A)Y,Y)\end{aligned}$$

for tangent vector fields X,Y,Z on M. Therefore, we can assert M is cyclic parallel under our assumption. The converse is trivial if we put $X=Y=Z$ for arbitrary tangent vector fields $X,Y,Z\in T_pM$. ∎

By virtue of Lemma 7.1, it can be written by

Lemma 7.2. *The shape operator A of M is cyclic η-parallel if and only if*

$$(\mathrm{C}_2)\begin{cases}\textit{every geodesic } \gamma \textit{ as a curve in } \bar{M} \textit{ has constant first curvature}\\ \textit{where } \gamma(0)=p\in M \textit{ and } \dot\gamma(0)=X\in T_pM \textit{ for } X\perp\xi.\end{cases}$$

From now on, in order to give the geometrical meaning of η-parallel shape operator of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, let us restrict tangent vectors X,Y and Z to the distribution $\mathfrak{h}$ where $\mathfrak{h}=\{X\in T_pM\,|\,X\perp\xi,\ p\in M\}$ and put M as a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with η-parallel shape operator. From the definition of η-parallel, we know that M naturally becomes cyclic η-parallel.

Summing up this fact and Lemma 7.2, we obtain the geometrical meaning of η-parallel as follows:

Lemma 7.3. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with η-parallel shape operator, $m\geq 3$. Then every geodesic γ of M in $G_2(\mathbb{C}^{m+2})$ satisfies the condition* (C_2).

By virtue of the equation of Codazzi (3.8), we know that any cyclic η-parallel hypersurface in $G_2(\mathbb{C}^{m+2})$ can not be η-parallel. From such a view point, the converse of Lemma 7.3 does not hold.

Remark 7.1. As a ambient space, let us consider a complex projective space $\mathbb{C}P^n$. Then from the equation of Codazzi,

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi,$$

it follows that

$$g((\nabla_X A)Y, Z) = g((\nabla_Y A)X, Z), \quad \forall X, Y, Z \in \mathfrak{h}.$$

Furthermore, it implies that

$$\mathfrak{S}_{X,Y,Z}\, g((\nabla_X A)Y, Z) = 3g((\nabla_X A)Y, Z)$$

for any tangent vectors $X, Y, Z \in \mathfrak{h}$. Thus we assert the following facts:

(1) When M is a η-parallel hypersurface in $\mathbb{C}P^n$, it coincide with cyclic η-parallel ones.
(2) A real hypersurface M in $\mathbb{C}P^n$ is η-parallel if and only if every geodesic γ has constant first curvature where the curve $\gamma : I \to M$ in $\mathbb{C}P^n$ has the initial conditions $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = X \in T_pM$ for $X \perp \xi$.
(3) A real hyperesurface M in $\mathbb{C}P^n$ is locally congruent to a real hypersurface of type A or B, or a ruled real hypersurface if and only if for every geodesic γ on M the first curvature function of γ in $\mathbb{C}P^n$ is a constant function (see [11]).

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