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Cohen-Macaulay Simplicial Complexes of Degree k

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Abstract. For a positive integer k a class of simplicial complexes, to be denoted by CM(k), is introduced. This class generalizes Cohen-Macaulay simplicial complexes. In analogy with the Cohen-Macaulay complexes, we give some homological and combinatorial properties of CM(k) complexes. It is shown that the complex Δ is CM(k) if and only if $I_{\Delta^\vee}$, the Stanley-Reisner ideal of the Alexander dual of Δ, has a k-resolution, i.e. $\beta_{i,j}(I_{\Delta^\vee}) = 0$ unless $j = ik + q$, where q is the degree of $I_{\Delta^\vee}$. As a main result, we characterize all bipartite graphs whose independence complexes are CM(k) and show that an unmixed bipartite graph is CM(k) if and only if it is pure k-shellable. Our result improves a result due to Herzog and Hibi and also a result due to Villarreal.

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1. Introduction

Let Δ be a simplicial complex of dimension $d-1$ with the vertex set $[n] := \{1,2,\dots,n\}$. Let K be a field. The squarefree monomial ideal I_Δ in the polynomial ring $S = K[x_1,\dots,x_n]$ is generated by the monomials $\mathbf{x}^F = \prod_{i\in F} x_i$ which F is a non-face in Δ.

The simplicial complex Δ is said *Cohen-Macaulay* when the quotient ring $K[\Delta] := S/I_\Delta$, called *Stanley-Reisner ring* of Δ, is Cohen-Macaulay. In [18] Reisner showed that the simplicial complex Δ is Cohen-Macaulay over K if and only if, for all faces F of Δ, the i^{th} reduced homology group of the link of F in Δ vanishes unless $i = \dim(\text{link}_\Delta F)$. (This result is known as Reisner's criterion for Cohen-Macaulayness.) In this paper we extend the concept of Cohen-Macaulayness in the language of reduced homology as shown by Reisner. We introduce a new class of simplicial complexes, called *Cohen-Macaulay simplicial complexes of degree* k (*CM(k)* for short), which generalizes the notion of Cohen-Macaulayness for simplicial complexes. Actually, k is an integer between 1 and d and for $k = 1$, CM(1)-ness coincides with Cohen-Macaulayness.

This paper is organized as follows. We begin in Section 2 by introducing CM(k) simplicial complexes and discussing some of their basic properties. Next, in Section 3 we introduce a class of monomial ideals with k-resolution. It is shown that Δ is CM(k) if and

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only if the Alexander dual of I_Δ has a k-resolution which $1 \le k \le d$ (see Theorem 3.1). This result extends [6, Theorem 3]. In Section 4, we give a class of CM(k) complexes, called k-shellable, and prove that k-shellability is equivalent to saying that the Alexander dual of I_Δ has k-quotients (see Theorem 4.1). The notions k-shellable and k-quotients were first introduced in [7]. In this paper, the definitions of k-shellability and having k-quotients are a little bit different from the definitions of these notions in [7]. We also use a Theorem of Hochster [13] and characterize all edge ideals of simple graphs which have a 2-resolution (see Corollary 4.2). The last section is devoted to the study of the bipartite graphs whose independence complex is CM(k). As a main result of the paper, we characterize all CM(k) bipartite graphs which are unmixed (see Theorem 5.1). Our result generalizes [8, Theorem 2.9] and also [10, Theorem 3.4].

For all undefined terms, we refer the reader to [3, 11, 14, 20].

2. The CM(k) simplicial complexes

In this section we introduce CM(k) complexes and discuss some of their basic properties. We also give some characterizations of CM(k) complexes, in terms of vanishing of some relative singular homologies of the geometric realization of the complex and its punctured space.

First, we recall some definitions related to simplicial complexes. Given a simplicial complex Δ on $[n]$, the *link* and the *deletion* of F in Δ are defined, respectively, by

$$\mathrm{link}_\Delta(F) = \{G \in \Delta : F \cap G = \emptyset, F \cup G \in \Delta\} \quad \text{and} \quad \Delta \backslash F = \{G \in \Delta : F \nsubseteq G\}.$$

Moreover, the *Alexander dual* of Δ is defined as $\Delta^\vee = \{F \in \Delta : [n] \backslash F \notin \Delta\}$. For the subset W of the vertex set of Δ, the *restriction* of Δ on W is defined as $\Delta_W = \{F \in \Delta : F \subseteq W\}$. We say that a simplicial complex is *pure* if all facets have the same cardinality.

Definition 2.1. *Let Δ be a simplicial complex of dimension $d-1$ and $1 \le k \le d$ an integer. We say that Δ is* Cohen-Macaulay of degree k *(*CM(k) *for short) if, for all faces F of Δ, $\tilde{H}_{i-1}(\mathrm{link}_\Delta F; K) = 0$ unless $ik = \dim(\mathrm{link}_\Delta F) + 1$.*

Note that if $k = 1$, then by Reisner's criterion, the CM(1)-ness of the simplicial complex Δ is equivalent to Cohen-Macaulayness.

Proposition 2.1. *Let Δ be a simplicial complex of dimension $d-1$ and an integer k with $1 \le k \le d$. Suppose that Δ is CM(k). Then:*

(a) *For every face F in Δ, $\mathrm{link}_\Delta F$ is CM(k).*
(b) *$k = d$ if and only if Δ is disconnected.*
(c) *If $k < d$ then Δ is connected and pure. Furthermore, Δ is connected in codimension k, i.e. for every two facets $F, G \in \Delta$ there exists a sequence of facets $F = F_0, F_1, \dots, F_r = G$ such that $|F_i \cap F_{i+1}| = d - k$.*

Proof. The ideas of the proofs of (a) and (c) are the same as used in proofs of Reisner's criterion or [11, Lemma 9.1.12]. (b) follows from the definition and the fact that the number of connected components of Δ coincides with $\dim_K(\tilde{H}_0(\Delta; K)) + 1$. ∎

The following result is due to Munkres which says that Cohen-Macaulayness is a topological property, i.e. if Δ is Cohen-Macaulay and its geometric realization is homeomorphic with geometric realization of the simplicial complex Δ', then Δ' is also Cohen-Macaulay.

For the concept of geometric realization of a simplicial complex we refer the reader to the books [15, 20].

Theorem 2.1. [16, Corollary 3.4] *Let Δ be a simplicial complex of dimension $d-1$. Suppose that X is the geometric realization of Δ. Then the following are equivalent:*
(a) *Δ is Cohen-Macaulay over K;*
(b) *For all $p \in X$ and all i with $i < d-1$, $\tilde{H}_i(X;K) \cong H_i(X, X-p;K) = 0$.*

The following theorem shows that CM(k)-ness is a topological property and also gives some information about how specific algebraic notion, topological notion and combinatorial notion are related to each other.

Theorem 2.2. *Let Δ be a simplicial complex of dimension $d-1$ and an integer k with $1 \leq k \leq d$. Suppose that X is the geometric realization of Δ. Then the following conditions are equivalent:*
(a) *Δ is CM(k);*
(b) *For all $p \in X$, all faces $F \in \Delta$ containing p and all i with $ik \neq d - |F| + (|F|-1)k$, $\tilde{H}_i(X;K) \cong H_i(X, X-p;K) = 0$.*

Proof. By [16, Lemma 3.3], for any face $F \in \Delta$ and any interior point p of F we have

$$H_i(X, X-p;K) \cong \tilde{H}_{i-|F|}(\mathrm{link}_\Delta F;K).$$

Now the assertion follows from this fact and the definition. ∎

Recall that, the *support* of $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ denoted by $\mathrm{supp}(\mathbf{a})$ is the set of i such that $a_i \neq 0$. Also the i^{th} local cohomology module of $K[\Delta]$ is denoted by $H^i_{\mathfrak{m}}(K[\Delta])$.

Theorem 2.3. *(Hochster [11]) Let $\mathbb{Z}^n_- = \{\mathbf{a} \in \mathbb{Z}^n : a_i \leq 0$ for $i = 1, \ldots, n\}$. Then*

$$\dim H^i_{\mathfrak{m}}(K[\Delta])_{\mathbf{a}} = \begin{cases} \dim \tilde{H}_{i-|F|-1}(\mathrm{link}_\Delta F;K), & \text{if } \mathbf{a} \in \mathbb{Z}^n_-, \text{ where } F = \mathrm{supp}(\mathbf{a}) \\ 0, & \text{if } \mathbf{a} \notin \mathbb{Z}^n_-. \end{cases}$$

Corollary 2.1. *Let Δ be a CM(k) simplicial complex of dimension $d-1$. Then $H^i_{\mathfrak{m}}(K[\Delta]) = 0$ unless $i = \lceil d - j(k-1)/k \rceil$, where $j = 0, 1, \ldots, d$. In particular,* $\mathrm{depth}(K[\Delta]) = \lceil d/k \rceil$ *and the projective dimension of $K[\Delta]$ is $n - \lceil d/k \rceil$. ($\lceil x \rceil$ means the least integer greater than or equal to x)*

Proof. For $\mathbf{a} \in \mathbb{Z}^n_-$ and $F = \mathrm{supp}(\mathbf{a})$ suppose $|F| = d-j$. Now using the definition and Theorem 2.3, the assertion obtains. The second statement follows from [2, Theorem 6.2.7] which says that $\mathrm{depth}(K[\Delta])$ is the least integer i such that $H^i_{\mathfrak{m}}(K[\Delta]) \neq 0$. For the projective dimension of $K[\Delta]$ we use the Auslander-Buchsbaum formula (see for example [3, Theorem 1.3.3]). ∎

3. The ideals with k-resolution

In this section we investigate behavior of the Alexander dual of the Stanley-Reisner ideal associated to a CM(k) complex. First we present the following more general definition.

Definition 3.1. *An $\mathbb{N}^n$-graded module M generated in degree $\mathbf{b} \in \mathbb{N}^n$ with $|\mathbf{b}| = q$ for some fixed $q \in \mathbb{N}$, has a k-*resolution *if for all $i \geq 0$ the minimal i^{th} syzygies of M lie in degrees $\mathbf{b} \in \mathbb{N}^n$ with $|\mathbf{b}| = q + ik$. Equivalently, $\beta_{i,j}(M) = 0$ for each $j \neq q + ik$ whenever $0 \leq i \leq$* $\mathrm{proj.dim}(M)$.

Notice that the concepts of having linear resolution and 1-resolution coincides. Recall from [14] that, a monomial matrix is an array of scalar entries λ_{qp} whose columns are labeled by source degrees $\mathbf{a}_p$, whose rows are labeled by target degrees $\mathbf{a}_q$, and whose entry $\lambda_{qp} \in K$ is zero unless $\mathbf{a}_q \preceq \mathbf{a}_p$. "$\mathbf{a} \preceq \mathbf{b}$" means that $a_i \leq b_i$ for all $i = 1, \dots, n$ which $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$.

Remark 3.1. Let $k > 0$ be an integer. The $\mathbb{N}^n$-graded free resolution $\mathscr{F}_\bullet$ is a k-resolution if and only if there is a choice of monomial matrices for the differentials of $\mathscr{F}_\bullet$ such that in each matrix, $|\mathbf{a}_p - \mathbf{a}_q| = k$ whenever the scalar λ_{qp} is nonzero.

Example 3.1. Let $I = (abc, bde, bfg)$ be a monomial ideal in the polynomial ring $K[a,b,c,d,e,f,g]$. Then we have:

$$0 \longrightarrow S(-7) \xrightarrow{\begin{array}{c|c} & abcdefg \\ \hline bdefg & -1 \\ abcfg & 1 \\ abcde & 1 \end{array}} S(-5)^3 \xrightarrow{\begin{array}{c|ccc} & bdefg & abcfg & abcde \\ \hline abc & 0 & 1 & 1 \\ bde & 1 & 0 & -1 \\ bfg & -1 & -1 & 0 \end{array}} S(-3)^3 \longrightarrow I \longrightarrow 0$$

Observe that for every nonzero scalar λ_{qp}, $|\mathbf{a}_p - \mathbf{a}_q| = 2$ and also $\beta_{i,j}(I) = 0$ whenever $j \neq 2i+3$ for $i = 0,1,2$. This means that I has a 2-resolution.

Example 3.2. Let X be an $r \times s$ matrix with $r < s$ whose entries are forms of degree k. Let $I_r(X)$ be the ideal generated by the r-minors of X (the determinants of $r \times r$ submatrices). Suppose $\mathrm{ht}(I_r(X)) = s - r + 1$. It was shown by Eagon and Northcott [5] that minimal free resolution of $S/I_r(X)$ is of the form

$$0 \longrightarrow S(-ks)^{\beta_{s-r+1}} \longrightarrow \cdots \longrightarrow S(-k(r+1))^{\beta_2} \longrightarrow S(-kr)^{\beta_1} \longrightarrow I_r(X) \longrightarrow 0.$$

Therefore $I_r(X)$ has a k-resolution.

In the following we present a homological property of $\mathbb{Z}$-graded modules over an standard K-algebra. The first statement was proved by Römer [19] and the second by Olteanu [17] for the special case $k = 1$.

Lemma 3.1. *Let R be a standard graded K-algebra and*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of $\mathbb{Z}$-graded R-modules. Let $k > 0$.

(1) *If M' and M'' both are generated in the same degree and have k-resolution, then M has a k-resolution.*
(2) *If M and M' are generated in degrees q and $q+k$, respectively, and have k-resolution, then M'' has a k-resolution.*

Proof. Applying $\mathrm{Tor}(K,.)$ functor on the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

one obtains

$$\cdots \longrightarrow \mathrm{Tor}_i^S(K,M')_{ik+j} \longrightarrow \mathrm{Tor}_i^S(K,M)_{ik+j} \longrightarrow \mathrm{Tor}_i^S(K,M'')_{ik+j}$$
$$\longrightarrow \mathrm{Tor}_{i-1}^S(K,M')_{ik+j} \longrightarrow \mathrm{Tor}_{i-1}^S(K,M)_{ik+j} \longrightarrow \cdots.$$

Let $j \neq q$. Then by the definition, $\mathrm{Tor}_i^S(K,M)_{ik+j} = 0$ and $\mathrm{Tor}_{i-1}^S(K,M')_{ik+j} = 0$. Consequently, for all $j \neq q$, $\mathrm{Tor}_i^S(K,M)_{ik+j} = 0$. Therefore M'' has a k-resolution. ∎

Theorem 3.1. *Let Δ be a pure simplicial complex of dimension $d-1$ and an integer k with $1 \leq k \leq d$. Then the following conditions are equivalent:*
(a) *Δ is CM(k);*
(b) *$I_{\Delta^\vee}$ has a k-resolution.*

Proof. Let $F \in \Delta$ with $|F| = n-j$. Let $q = \deg(I_{\Delta^\vee})$. It is known that $I_{\Delta^\vee} = I(\Delta^c)$ which $\Delta^c = \{[n]\backslash F : F \text{ a facet of } \Delta\}$ and $I(\Delta^c)$ is a monomial ideal which is generated by those squarefree monomials $\mathbf{x}^F$ with F a facet of Δ^c. Then $q = n-d$ and so $\dim(\mathrm{link}_\Delta F) = d-(n-j)-1 = j-q-1$. On the other hand, from [11, Corollary 8.1.4], we have

$$\mathrm{Tor}_i^S(K, I_\Delta)_{\mathbf{a}} \cong \tilde{H}_{i-1}(\mathrm{link}_{\Delta^\vee} F; K) \quad \text{for all } i$$

which $\mathbf{a} \in \mathbb{Z}^n$ is squarefree and $F = [n]\backslash \mathrm{supp}(\mathbf{a})$. Using the above isomorphism, the condition (a) holds if and only if for all i, $\beta_{i,j}(I_{\Delta^\vee}) = 0$ unless $ik = j-q$. ∎

Remark 3.2. For $k=1$, Theorem 3.1 is the same as Eagon-Reiner's theorem [6, Theorem 3].

Recall that, the *Castelnuovo-Mumford regularity* of graded S-module M is defined as

$$\mathrm{reg}(M) = \max\{j-i : \beta_{i,j}(M) \neq 0\}.$$

Corollary 3.1. *Let Δ be a pure simplicial complex of dimension $d-1$ and an integer k with $1 \leq k \leq d$. If Δ is CM(k) then* $\mathrm{reg}(I_{\Delta^\vee}) = q + (k-1)\mathrm{proj.dim}(I_{\Delta^\vee})$*, where* $q = \deg(I_{\Delta^\vee})$*. In particular,* $\mathrm{proj.dim}(K[\Delta]) = q + (k-1)\mathrm{proj.dim}(I_{\Delta^\vee})$.

Proof. The first equality follows from Theorem 3.1, while the second equality follows from [21, Corollary 1.6] and the first one. ∎

4. A class of CM(k) complexes

In [7] the authors introduced the concepts of d-shellability and d-quotients. In this section, we use the same notions of d-shellability and d-quotients to introduce a class of simplicial complexes which are CM(k) and a class of monomial ideals which have k-resolution. Actually, our definitions are special cases of the definitions of [7]. We will see that the concepts of being 1-shellable and having 1-quotients, respectively, coincide with being shellable and having linear quotients. We refer the reader to [1] for the definition of shellability and to [12] for the definition of ideals with linear quotients.

Definition 4.1. *Let k be an integer with $1 \leq k \leq d$. The simplicial complex Δ on $[n]$ is called* k-shellable *if its facets can be ordered $F_1, \ldots, F_r$, called k-shelling order, such that for all $j = 2, \ldots, r$, the subcomplex $\langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$ satisfies the following properties:*
(i) *It is generated by a nonempty set of proper faces of $\langle F_j \rangle$ of dimension $|F_j| - k - 1$;*
(ii) *For every two disjoint facets $\sigma, \tau \in \langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$, $(F_j \backslash \sigma) \cap (F_j \backslash \tau) = \emptyset$.*

Definition 4.2. *The monomial ideal $I \subset S$ has* k-quotients *whenever there is the ordering $u_1, \ldots, u_r$ from the minimal generators of I such that for all $j = 2, \ldots, r$, the minimal generators of $(u_1, \ldots, u_{j-1}) : u_j$ are of degree k and form a S-sequence.*

It is easily seen that a sequence $\mathbf{u} = u_1, \ldots, u_r$ of monomials in S is a S-sequence precisely when $\gcd(u_i, u_j) = 1$ for each $i \neq j$. In particular, in this case if $u_1, \ldots, u_r$ have the same degree q, then the ideal $(u_1, \ldots, u_r)$ has a q-resolution. Of course, this fact holds for more general elements other than monomials. In fact if $f_1, \ldots, f_r$ are homogeneous elements of the same degree q in S which form a regular sequence, then the homogeneous ideal $(f_1,\dots,f_r)$ has a q-resolution. Moreover, for every positive integer m, the ideal $(f_1^m,\dots,f_r^m)$ has qm-resolution. This fact can help to produce interesting examples of k-resolutions.

Theorem 4.1. *The simplicial complex Δ of dimension $d-1$ for $1\le k\le d$ is k-shellable if and only if $I_{\Delta^\vee}$ has k-quotients.*

Proof. It is a straightforward consequence of [7, Theorem 6.8]. ∎

Lemma 4.1. *Every monomial ideal with k-quotients for some $k>0$ which is generated in one degree, has a k-resolution.*

Proof. We prove the claim by induction on the number of generators of the given ideal. Suppose that I is generated in degree q and has k-quotients with respect to the ordering $u_1,\dots,u_r$. Set $I'=(u_1,\dots,u_{r-1})$ and $I''=(u_r)$. By induction hypothesis, I' and I'' have k-quotients and so have k-resolution. Also, it is easily verified that $I'\cap I''$ is generated in degree $q+k$ and has k-quotients.

Now from the exact sequence

$$0\longrightarrow I'\longrightarrow I'\oplus I''\longrightarrow I''\longrightarrow 0$$

and Lemma 3.1(1) we get that $I'\oplus I''$ has k-resolution. Again, by the exact sequence

$$0\longrightarrow I'\cap I''\longrightarrow I'\oplus I''\longrightarrow I'+I''\longrightarrow 0$$

and Lemma 3.1(2) we conclude that $I'+I''$ has k-resolution, as desired. ∎

Corollary 4.1. *Every pure k-shellable complex is CM(k), where $k>0$.*

Proof. This follows from Theorems 3.1 and 4.1 and also Lemma 4.1. ∎

In the remaining part of this section, we are going to study the simple graphs (graphs for short) whose edge ideal has a 2-resolution. Recall that, a graph is called *chordal* if for each cycle of length four or more there is an edge joining two non-adjacent vertices in the cycle. Fröberg [9] showed that a graph G is chordal if and only if $I(G^c)$ has a linear resolution, where G^c is the complementary graph of G. Actually, Fröberg's result characterizes all edge ideals of graphs which have a 1-resolution. Here we are interested in finding some properties of G when $I(G^c)$ has a 2-resolution. We start with a theorem of Hochster [13]. Before stating the theorem, we recall some notions related to simplicial complexes and graphs.

For a graph G, let $\Delta(G)$ be the *clique complex* of G which is a simplicial complex on the vertex set of G whose faces are the complete subgraphs (cliques) of G. It is easily seen that $I(G)=I_{\Delta(G^c)}$. The *independence complex* of G is denoted by Δ_G and F is a face of Δ_G if and only if there is no edge of G joining any two vertices of F. It is easy to see that $\Delta_G=\Delta(G^c)$. If Δ_G is pure, we say that G is *unmixed*.

Theorem 4.2. [13] *Let Δ be a simplicial complex and $\mathbf{a}\in\mathbb{Z}^n$. Then we have:*

(a) $\mathrm{Tor}_i(K,I_\Delta)_{\mathbf{a}}=0$ *if $\mathbf{a}$ is not squarefree;*
(b) *if $\mathbf{a}$ is squarefree and $W=\mathrm{supp}(\mathbf{a})$, then*

$$\mathrm{Tor}_i(K,I_\Delta)_{\mathbf{a}}\cong\tilde{H}_{|W|-i-2}(\Delta_W;K)\quad \text{for all } i.$$

Lemma 4.2. *For $k>0$ and the graph G, $I(G)$ has a k-resolution if and only if $\tilde{H}_j(\Delta(G^c)_W;K)=0$ unless $j=i(k-1)$ and $|W|=ik+2$ which $i\ge 0$ and $W\subseteq[n]$.*

Proof. Let $W \subseteq [n]$ and $|W| = j$. By Theorem 4.2, we conclude that

$$\beta_{i,j}(I_{\Delta(G^c)}) = 0 \text{ for } j \neq ik+2 \Leftrightarrow \sum_{W\subseteq[n],|W|=j} \dim \tilde{H}_{|W|-i-2}(\Delta(G^c)_W;K) = 0 \text{ for } j \neq ik+2$$
$$\Leftrightarrow \tilde{H}_{j-i-2}(\Delta(G^c)_W;K) = 0 \text{ for } j \neq ik+2 \text{ and } |W| = j$$
$$\Leftrightarrow \tilde{H}_j(\Delta(G^c)_W;K) = 0 \text{ for } j \neq i(k-1) \text{ and } |W| \neq ik+2. \quad \blacksquare$$

For every squarefree monomial ideal I of degree 2 there is a graph whose edge ideal is equal to I. In the following corollary, we characterize all graphs which the edge ideal of their complement has a 2-resolution.

Corollary 4.2. *Let I be a squarefree monomial ideal generated in degree 2. If I has a 2-resolution then I has 2-quotients. In particular, it follows that if G is a graph, then $G \cong K_{2,2,\ldots,2}$ if and only if $I(G^c)$ has a 2-resolution. ($K_{2,2,\ldots,2}$ is a complete r-partite graph whose each part contains 2 elements.)*

Proof. Let $I = I(G)$ for some graph G. By Lemma 4.2, for all $W \subseteq [n]$ and all $i \geq 0$ with $|W| \neq 2i+2$ and $j \neq i$ we have $\tilde{H}_j(\Delta(G^c)_W;K) = 0$. We need to show that the minimal generators of $G(I)$ are relatively prime. By relabeling the vertices of G, suppose, on the contrary, that $x_1x_2, x_1x_3 \in G(I)$ and $W = \{1,2,3\}$. Then $\tilde{H}_0(\Delta(G^c)_W;K) = 0$. But $\Delta(G^c)_W = \{1,23\}$ or $\Delta(G^c)_W = \{1,2,3\}$. Therefore $\Delta(G^c)_W$ is disconnected, which is a contradiction.
The second statement is easily verified. $\blacksquare$

5. CM(k) bipartite graphs

In this section we study the properties of CM(k)-ness and k-shellability of bipartite graphs. We say that the graph G is CM(k) or k-shellable if, the independence complex of G has this property.

Let G be a bipartite graph with the vertex partition $V \cup V'$ and the edge set $E(G)$. The following results were proved by the authors in [8, 10]:

- [8, Theorem 2.9] G is Cohen-Macaulay if and only if it is pure shellable.
- [10, Theorem 3.4] G is Cohen-Macaulay if and only if $|V| = |V'|$ and the vertices $V = \{x_1,\ldots,x_n\}$ and $V' = \{y_1,\ldots,y_n\}$ can be labelled such that:
  (i) $x_iy_i \in E(G)$ for $i = 1,\ldots,n$;
  (ii) if $x_iy_j \in E(G)$ then $i \leq j$;
  (iii) if $x_iy_j, x_jy_k \in E(G)$ then $x_iy_k \in E(G)$.

In the following we extend the above results to the case that G is CM(k) for some $k > 0$. For a vertex x of a graph G, we denote by $N_G(x)$ the set of vertices y of G such that $xy \in E(G)$. Also, for every subset A of the vertex set of G, $V(G)$, we set $N_G(A) = \bigcup_{x\in A} N_G(x)$ and $N_G[A] = \bigcup_{x\in A}(N_G(x) \cup \{x\})$.

Theorem 5.1. *Let G be a bipartite graph without isolated vertices and $V \cup V'$ be a vertex partition for G. Let $V = \{x_1,\ldots,x_m\}$ and $V' = \{y_1,\ldots,y_n\}$.*

(a) *If $k \geq m$ and $k \geq n$ then the following conditions are equivalent:*
  (i) *G is unmixed and CM(k);*
  (ii) *$m = n = k$ and $G \cong K_{k,k}$;*
  (iii) *G is unmixed and k-shellable.*

(b) *If $k < m$ or $k < n$ then the following conditions are equivalent:*
  (i) *G is CM(k);*

(ii) *$m = n$, $k|n$ and the elements of V and V' can be labelled such that:*
(1) *$x_iy_i \in E(G)$ for all $i = 1,\dots,n$;*
(2) *if $x_iy_j, x_jy_l \in E(G)$, then $x_iy_l \in E(G)$;*
(3) *if $x_iy_j \in E(G)$ then either $i \le j$ or there is some $l \in \{0,1,\dots,n/k-1\}$ such that $n-(l+1)k+1 \le j < i \le n-lk$;*
(4) *for all $l \in \{0,1,\dots,n/k-1\}$, the induced subgraph on the vertices x_i and y_j with $n-(l+1)k+1 \le i, j \le n-lk$ is complete bipartite.*

(iii) *G is unmixed and k-shellable.*

Proof. (a) (i)⇒(ii): Since G is unmixed and $1 \le k \le \dim(\Delta_G)+1$, we have $m = n = k$. In order to see that G is complete bipartite, it suffices to show that there is no other facet, except $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$ in Δ_G. This follows immediately from the fact that Δ_G is disconnected, by Proposition 2.1(b). (ii)⇒(iii) is easily verified and (iii)⇒(i) holds by Corollary 4.1.

(b) (i)⇒(ii): By Proposition 2.1(c), Δ_G is pure, and so $m = n$. Now we want to show that $k|n$. Since

$$\mathrm{link}_{\Delta_G}(\{x_{n-[n/k]k+1},\dots,x_n\}) = \Delta_{G\setminus\{x_{n-[n/k]k+1},\dots,x_n,y_{n-[n/k]k+1},\dots,y_n\}},$$

it follows from Proposition 2.1(a) that $H := G\setminus\{x_{n-[n/k]k+1},\dots,x_n,y_{n-[n/k]k+1},\dots,y_n\}$ is CM(k). Also, the unmixedness of G immediately implies that H is unmixed. Suppose, on the contrary, that $n-[n/k]k \ge 1$. Therefore H is a bipartite graph with the vertex partition $\{x_1,\dots,x_{n-[n/k]k}\} \cup \{y_1,\dots,y_{n-[n/k]k}\}$. Since $k \ge n-[n/k]k$, it follows from (a) that $n-[n/k]k = k$, which is a contradiction.

(1) and (2) are proved exactly similar to [10, Theorems 3.3 and 3.4]. It remains to show (3) and (4) are hold.

By Proposition 2.1, Δ_G is connected in codimension k and the link of every face of Δ is CM(k). Now because V and V' are both in Δ_G, after a suitable change of the labeling of variables $x_1,\dots,x_n$ and $y_1,\dots,y_n$, the subset $\{y_1,\dots,y_{n-lk},x_{n-lk+1},\dots,x_n\}$ is a facet of Δ_G for every $l \in \{0,1,\dots,n/k-1\}$. This implies that $x_iy_j \notin E(G)$ if $j \le n-lk < i$ for every $l \in \{0,1,\dots,n/k-1\}$. Consequently, for $x_iy_j \in E(G)$ either $i \le j$ or $i > j$ and there exists some l with $l = 0,1,\dots,n/k-1$ such that $i \le n-lk$ and $j > n-(l+1)k$. This proves (3).

To prove (4), we fix k and proceed by induction on the number of vertices of G. Since $\mathrm{link}_{\Delta_G}(\{y_1,\dots,y_k\}) = \Delta_{G\setminus\{x_1,\dots,x_k,y_1,\dots,y_k\}}$, it follows that $G\setminus\{x_1,\dots,x_k,y_1,\dots,y_k\}$ is CM(k). By induction hypothesis, for all $l \in \{0,1,\dots,n/k-2\}$, the induced subgraph on the vertices x_i and y_j with $n-(l+1)k+1 \le i, j \le n-lk$ is complete bipartite. Furthermore, $\mathrm{link}_{\Delta_G}(\{x_{k+1},\dots,x_n\})$ is $(k-1)$-dimensional and CM(k). Therefore $G\setminus\{x_{k+1},\dots,x_n,y_{k+1},\dots,y_n\}$ is CM(k) with the vertex partition $\{x_1,\dots,x_k\} \cup \{y_1,\dots,y_k\}$. Now it follows from (a) that $G\setminus\{x_{k+1},\dots,x_n,y_{k+1},\dots,y_n\}$ is complete bipartite, as desired.

(ii)⇒(iii): We use induction on the number of vertices of G. Assume that the assertion holds for every bipartite graph with the vertex partition $W \cup W'$ which satisfies the condition (ii) and $|W| = |W'| < n$.

Since $G\setminus\{x_1,\dots,x_k,y_1,\dots,y_k\}$ satisfies the condition (ii), it is k-shellable, by induction hypothesis. Hence $G\setminus\{x_1,\dots,x_k\}$ is k-shellable, too, because $y_1,\dots,y_k$ are isolated.

Now we want to show that $G\setminus N_G[x_1,\dots,x_k]$ is k-shellable. First, note that if $y_t \in N_G(x_1,\dots,x_k)$ for some $t > k$, then x_t is isolated in $G\setminus N_G[x_1,\dots,x_k]$. Because otherwise there would exist an edge $x_t y_j$ in $G\backslash N_G[x_1,\dots,x_k]$. Then, by (2), $y_j \in N_G(x_1,\dots,x_k)$, which is a contradiction. Furthermore, if $n-(l+1)k+1 \leq t \leq n-lk$ for some $l \in \{0,\dots,n/k-2\}$, then for every j with $n-(l+1)k+1 \leq j \leq n-lk$, we will have $y_j \in N_G(x_1,\dots,x_k)$.

The above statements show that

$$H := (G\backslash N_G[x_1,\dots,x_k])\backslash\{\text{isolated vertices of } G\backslash N_G[x_1,\dots,x_k]\}$$

satisfies the conditions in (ii). Hence H is k-shellable, by induction hypothesis. Therefore $G\backslash N_G[x_1,\dots,x_k]$ is k-shellable.

Set $G' = G\backslash\{x_1,\dots,x_k\}$ and $G'' = G\backslash N_G[x_1,\dots,x_k]$. Then $\Delta_{G'} = \Delta_G\backslash\{x_1,\dots,x_k\}$ and $\Delta_{G''} = \mathrm{link}_{\Delta_G}(\{x_1,\dots,x_k\})$. Let $F_1,\dots,F_r$ and $G_1,\dots,G_s$ be k-shelling orders for $\Delta_{G'}$ and $\Delta_{G''}$, respectively. Then it is easily verified that

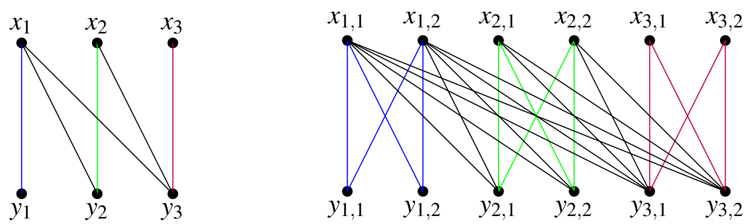
$$F_1,\dots,F_r, G_1\cup\{x_1,\dots,x_k\},\dots,G_s\cup\{x_1,\dots,x_k\}$$

is a k-shelling order for Δ_G. Therefore G is k-shellable.

(iii)⇒(i) holds by Corollary 4.1. ∎

Remark 5.1. Theorem 5.1 provides a way to construct a CM($k+1$) bipartite graph starting with a CM(k) bipartite graph. To see this, let G be a CM(k) bipartite graph with the vertex partition $V \cup V'$ which $|V| = |V'| = n$. For every $l \in \{0,1,\dots,n/k-1\}$, let $K^l_{k,k}$ denote the induced complete bipartite subgraph of G on the vertex partition $\{x_{l,1},\dots,x_{l,k}\} \cup \{y_{l,1},\dots,y_{l,k}\}$. To construct a CM($k+1$) bipartite graph H, we replace every subgraph $K^l_{k,k}$ by a complete bipartite graph $K^l_{k+1,k+1}$ with the vertex partition $\{x_{l,1},\dots,x_{l,k+1}\} \cup \{y_{l,1},\dots,y_{l,k+1}\}$ such that if $x_{r,i}y_{s,j}$ is an edge of G then $x_{r,i'}y_{s,j'}$ is an edge of H for all $1 \leq i', j' \leq k+1$.

The following figure shows of two bipartite graphs G and H which satisfy the condition (b)(ii) of Theorem 5.1. The graph G is Cohen-Macaulay, while H is constructed from G and is CM(2).



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