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The Generalized Connectivity of Complete Equipartition 3-Partite Graphs

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Abstract. Let G be a nontrivial connected graph of order n, and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair of distinct integers i, j with $1 \leq i, j \leq \ell$. Chartrand *et al.* generalized the concept of connectivity as follows: The *k-connectivity* of G, denoted by $\kappa_k(G)$, is defined by $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Thus $\kappa_2(G) = \kappa(G)$, where $\kappa(G)$ is the connectivity of G; whereas, $\kappa_n(G)$ is the maximum number of edge-disjoint spanning trees contained in G.

This paper mainly focuses on the k-connectivity of complete equipartition 3-partite graphs K_b^3, where $b \geq 2$ is an integer. First, we obtain the number of edge-disjoint spanning trees of a general complete 3-partite graph $K_{x,y,z}$, which is $\lfloor (xy+yz+zx)/(x+y+z-1) \rfloor$. Then, based on this result, we get the k-connectivity of K_b^3 for all $3 \leq k \leq 3b$. Namely,

$$\kappa_k(K_b^3) = \begin{cases} \left\lfloor \frac{\lceil \frac{k^2}{3} \rceil + k^2 - 2kb}{2(k-1)} \right\rfloor + 3b - k & \text{if } k \geq \frac{3b}{2}; \\ \left\lfloor \frac{3b}{2} \right\rfloor & \text{if } k < \frac{3b}{2} \text{ and } k = 0 \pmod 3; \\ \left\lfloor \frac{3bk+3b-k+1}{2k+1} \right\rfloor & \text{if } \frac{3b}{4} < k < \frac{3b}{2} \text{ and } k = 1 \pmod 3; \\ \left\lfloor \frac{3bk+6b-2k+1}{2k+2} \right\rfloor & \text{if } b \leq k < \frac{3b}{2} \text{ and } k = 2 \pmod 3; \\ \left\lfloor \frac{3b+1}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

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1. Introduction

We follow the book [1] for all graph theoretical notation and terminology not defined here.

There is an equivalent definition of the connectivity $\kappa(G)$ of a graph G provided by a well-known theorem of Whitney [14]. For each 2-subset $S = \{u, v\}$ of the vertex set $V(G)$, let $\kappa(S)$ denote the maximum number of internally disjoint uv-paths in G. Then $\kappa(G) = \min\{\kappa(S)\}$, where the minimum is taken over all 2-subsets S of $V(G)$. In [2], the authors generalized this definition and proposed the concept of generalized connectivity.

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Let G be a nontrivial connected graph of order n, and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair of distinct integers i, j with $1 \leq i, j \leq \ell$ (note that the trees are vertex-disjoint in $G \backslash S$). A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G with this property is called a set of *internally disjoint trees connecting S*. The *generalized k-connectivity* of G, abbreviated as *k-connectivity* of G, denoted by $\kappa_k(G)$, is then defined as $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$.

From a theoretical perspective, both extremes of κ_k are fundamental concepts in graph theory. $\kappa_2(G) = \kappa(G)$ is the connectivity of G, and $\kappa_n(G)$ is the maximum number of edge-disjoint spanning trees contained in G. The concept of edge-disjoint spanning trees is another subject we studied. To motivate the edge-disjoint spanning trees problem, assume that our graph represents a communication network, and that for every choice of two vertices we want to be able to find k edge-disjoint paths between them. Menger's theorem [10] tells us that such paths exist as soon as our graph is k-edge-connected, which is clearly also necessary. But the theorem does not tell us how to find those paths; in particular, having found them for one pair of endvertices we are not necessarily better placed to find them for another pair. However, if our graph has k edge-disjoint spanning trees, there will always be k such paths, one in each tree. Once we have stored those trees in our computer, we shall always be able to find the k paths quickly, for any given pair of endvertices. For edge-disjoint spanning trees of a finite graph G, Nash-Williams [11] and Tutte [13] proved the following theorem independently:

Theorem 1.1. [11, 13] *A multigraph contains k edge-disjoint spanning trees if and only if for every partition P of its vertex set it has at least $k(|P|-1)$ cross-edges.*

As a consequence of this theorem, Corollary 1.1 gives a sufficient condition for the existence of k edge-disjoint spanning trees.

Corollary 1.1. [4] *Every $2k$-edge-connected multigraph G has k edge-disjoint spanning trees.*

According to this corollary, Kriesell conjectured a more general statement: Defining a set $S \subseteq V(G)$ to be j-edge-connected in G if S lies in a single component of any graph obtained by deleting fewer than j edges from G, he conjectured that if S is $2k$-edge-connected in G, then G has k edge-disjoint trees containing S. For more details about the spanning tree packing problem, see [12].

From a practical perspective, generalized connectivity can measure the reliability and security of a network. Here is an example. Imagine that a given graph represents a communication network. Suppose that k vertices of the graph are users and other vertices are switchers. The users hope that they can communicate on as many frequencies as possible, so that they can communicate with each other in secrecy even if some of the frequencies are subject to interference or eavesdropping. Users can communicate via a tree connecting all users on each frequency. To avoid interference, each edge can carry only one frequency. And in order to ensure secrecy, each switcher can switch only one frequency. So in essence we need to find the maximum number of internally disjoint trees connecting all users. In a communication network, any k nodes may become users and other nodes become switchers. Thus the reliability and security of a network can be measured by its generalized connectivity.

Since Chartrand *et al.* introduced the concept of generalized connectivity in 1984 [2], there have been only few results about it until they calculated $\kappa_k(K_n)$ for every pair of integers k, n with $2 \le k \le n$ in 2010 in [3]. Since then, more and more mathematicians begin to study the generalized connectivity and get some progress. In [5] the authors gave the sharp bounds of the generalized 3-connectivity $\kappa_3(G)$. They also studied the bounds of $\kappa_3(G)$ for planar graphs. In [8] we calculated $\kappa_k(K_{a,b})$ for any two integers a, b with $1 \le a \le b$ and $2 \le k \le a+b$. In [5] and [6] the authors studied the computational complexity of the generalized connectivity of graphs. In [9] the authors studied the generalized 3-connectivity of Cartesian product graphs. In [7] the authors studied the minimal size among graphs with the generalized 3-connectivity $\kappa_3 = 2$.

A complete equipartition 3-partite graph is a complete 3-partite graph in which every part contains exactly b vertices for some integer b. We denote this graph by K_b^3. Actually, all vertices in the same part of K_b^3 are equivalent. So instead of considering all k-subsets S of $V(K_b^3)$, we can restrict our attention to the k-subsets $S_{x,y,z} = \{u_1, u_2, \ldots, u_x, v_1, v_2, \ldots, v_y, w_1, w_2, \ldots, w_z\}$ for $0 \le x, y, z \le k$ with $x+y+z = k$. Moreover, since the three parts U, V and W have the same order, $S_{x,y,z}$ and $S_{\alpha,\beta,\gamma}$ are equivalent, where α, β, γ is any permutation of x, y, z. So we can assume that $b \ge x \ge y \ge z \ge 0$. If $z = 0$, obviously $\min\{\kappa(S_{x,y,0})\} = b + \kappa_k(K_{b,b})$. So we will restrict our attention to the case that $b \ge x \ge y \ge z > 0$. For convenience, we denote $U_x = \{u_1, u_2, \ldots, u_x\}$, $V_y = \{v_1, v_2, \ldots, v_y\}$ and $W_z = \{w_1, w_2, \ldots, w_z\}$.

In the next two sections, we will give the number of edge-disjoint spanning trees in a complete 3-partite graph $K_{x,y,z}$ and get the k-connectivity of K_b^3 for all $3 \le k \le 3b$, respectively.

2. The number of edge-disjoint spanning trees of a complete 3-partite graph

Before we give the number of edge-disjoint spanning trees of a general complete 3-partite graph $K_{x,y,z}$, we will introduce a method to find $\lfloor ab/(a+b-1) \rfloor$ edge-disjoint spanning trees of a complete bipartite graph $K_{a,b}$ quickly and conveniently. Without loss of generality, we can assume that $a \le b$.

The List Method

Step 1. Calculate $t = \lfloor ab/(a+b-1) \rfloor$.

Step 2. Assign appropriate values to d_j for $1 \le j \le a$. The method of assigning appropriate values to d_j was introduced in [8]. Generally speaking, consider the numbers $1, t+1, 2t+1, \ldots, (a-1)t+1$, where addition is performed modulo a. If $1, t+1, 2t+1, \ldots, (a-1)t+1$ are pairwise distinct, we can assign the values to d_j as follows: Let $a+b-1 = ka+c$, where k, c are integers, and $0 \le c \le a-1$. Then $a+b-1 = (k+1)c + k(a-c)$. If $c = 0$, let $d_j = k$ for all $1 \le j \le a$. If $c > 0$, let $d_{(i-1)t+1} = k+1$ for all $1 \le i \le c$, and let the other $d_j = k$. If some of the numbers $1, t+1, 2t+1, \ldots, (a-1)t+1$ are equal, we can order $1, 2, \ldots, a$ by $1, t+1, 2t+1, \ldots, (j-1)t+1, 2, t+2, 2t+2, \ldots, (j-1)t+2, \ldots, s, t+s, 2t+s, \ldots, (j-1)t+s$. Then we can assign the values of d_j as follows: Let $a+b-1 = ka+c$, where k, c are integers, and $0 \le c \le a-1$. Then $a+b-1 = (k+1)c + k(a-c)$. In the case that $c = 0$, let $d_j = k$ for all $1 \le j \le a$. In the case that $c > 0$ for the first c numbers of our ordering, if d_j uses one of them as subscript, then $d_j = k+1$; else $d_j = k$.

Step 3. Form a list with row headings of $u_1, \ldots, u_a$ and column headings of $v_1, \ldots, v_b$. Denote the entry in row u_i and column v_j by $a_{i,j}$.

Step 4. According to the assignment of d_j for $1 \le j \le a$, mark the edges of the first spanning tree by 1 in the list. Namely, for every row u_i with $1 \le i \le a$, put $a_{i,d_1+d_2+\cdots+d_{i-1}-(i-2)} = \cdots = a_{i,d_1+d_2+\cdots+d_i-(i-1)} = 1$.

Step 5. For every row u_i with $1 \le i \le a$, mark the entry next to the last 1, namely $a_{i,d_1+d_2+\cdots+d_i-(i-2)}$, by 2. For every row u_i with $1 \le i \le a-1$, mark the entry just above the 2 of row u_{i+1}, namely $a_{i,d_1+d_2+\cdots+d_{i+1}-(i-1)}$, by 2. For row u_a, mark the entry in the same column as the last 1 of row u_1, namely a_{a,d_1}, by 2. Finally, for every row u_i with $1 \le i \le a$, mark the entries between the two 2 by 2. Thus the edges marked by 2 consist of a spanning tree T_2.

Step 6. For ℓ with $3 \le \ell \le t$, we can find a spanning tree T_ℓ similarly. For every row u_i with $1 \le i \le a$, mark the entry next to the last $\ell-1$ by ℓ. For every row u_i with $1 \le i \le a-1$, mark the entry just above the ℓ of row u_{i+1} by ℓ. For row u_a, mark the entry in the same column as the last $\ell-1$ of row u_1 by ℓ. Finally, for every row u_i with $1 \le i \le a$, mark the entries between the two ℓ by ℓ. Thus the edges marked by ℓ consist of a spanning tree T_ℓ.

Finally, we find all $\lfloor ab/(a+b-1) \rfloor$ edge-disjoint spanning trees.

If ℓ is less than $t = \lfloor ab/(a+b-1) \rfloor$, similarly we can use this method to find ℓ edge-disjoint spanning trees of $K_{a,b}$ such that for every pair of vertices u_i and u_j with $1 \le i, j \le a$, the difference between the number of unused edges incident with u_i and the number of unused edges incident with u_j is at most 1. For every pair of vertices v_i and v_j with $1 \le i, j \le b$, we also can use this method to find ℓ edge-disjoint spanning trees of $K_{a,b}$ such that the difference between the number of unused edges incident with v_i and the number of unused edges incident with v_j is at most 1. Just replace $t = \lfloor ab/(a+b-1) \rfloor$ by ℓ.

With this method, we can prove the next theorem.

Theorem 2.1. *For a complete 3-partite graph $K_{x,y,z}$, we have*

$$\kappa_{x+y+z}(K_{x,y,z}) = \left\lfloor \frac{xy+yz+zx}{x+y+z-1} \right\rfloor.$$

Proof. Let $U = \{u_1, \ldots, u_x\}$, $V = \{v_1, \ldots, v_y\}$ and $W = \{w_1, \ldots, w_z\}$ be the three parts of $K_{x,y,z}$. Without loss of generality, we may assume that $z \le y \le x$. Since $K_{x,y,z}$ contains $xy+yz+zx$ edges and a spanning tree needs $x+y+z-1$ edges, the number of edge-disjoint spanning trees of $K_{x,y,z}$ is at most $\lfloor (xy+yz+zx)/(x+y+z-1) \rfloor$, namely, $\kappa_{x+y+z}(K_{x,y,z}) \le \lfloor (xy+yz+zx)/(x+y+z-1) \rfloor$. Thus, it suffices to prove that $\kappa_{x+y+z}(K_{x,y,z}) \ge \lfloor (xy+yz+zx)/(x+y+z-1) \rfloor$. To this end, we want to find out all the $\lfloor (xy+yz+zx)/(x+y+z-1) \rfloor$ edge-disjoint spanning trees. In other words, we will prove that after we have found some edge-disjoint spanning trees of $K_{x,y,z}$, the number of unused edges is at most $x+y+z-2$, namely they are not enough to form a spanning tree.

Firstly, consider the complete bipartite graph $K_{y,x}$, which is a subgraph of $K_{x,y,z}$. We can use The List Method to find $\lfloor yx/(y+x-1) \rfloor$ edge-disjoint spanning trees of $K_{y,x}$, and leave at most $y+x-2$ unused edges. If we connect each w_i to some spanning tree of $K_{y,x}$, we can get a spanning tree of $K_{x,y,z}$. So we can get $\lfloor yx/(y+x-1) \rfloor$ edge-disjoint spanning trees of $K_{x,y,z}$ as long as we guarantee that the edges which we used to connect w_i are all distinct. So we can first use The List Method to find $t = \lfloor (z(y+x) - z(\lfloor yx/(y+x-1) \rfloor))/(z+y+x-1) \rfloor$ spanning trees of $K_{z,y+x}$. Now, since for every pair of vertices w_i and w_j with $1 \le i, j \le z$, the difference between the number of unused edges incident with w_i and the number of unused edges incident with w_j is at most 1, every w_i is incident with at least $\lfloor yx/(y+x-1) \rfloor$ unused edges. So we can indeed find $\lfloor yx/(y+x-1) \rfloor$ edge-disjoint spanning trees of $K_{x,y,z}$. Now the number of unused edges is at most $y+x-2+z+y+x-2<2(z+y+x-1)$. If it is less than $z+y+x-1$, we are done. If it is at least $z+y+x-1$, we need to find one more spanning tree using the rest unused edges.

Let R be the set of the rest unused edges. If $G=(V,R)$ is connected, we are done. If there are at least two components in $G=(V,R)$, there must be a component containing a cycle. Since $|R|\geq z+y+x-1$ and the number of unused edges in $K_{y,x}$ is at most $y+x-2$, the number of unused edges in $K_{z,y+x}$ is at least $z+1$. According to The List Method, each w_i has almost the same number of unused edges. So each w_i has degree at least 1 in G. Again, according to The List Method, the unused edges in $K_{y,x}$ can not form a cycle, neither can the unused edges in $K_{z,y+x}$. So the component containing a cycle must contain some unused edges both in $K_{y,x}$ and in $K_{z,y+x}$. And the cycle must be one of the two cases shown in Figure 1. In case 1, w_iu_j and u_jv_k are edges and w_iv_k is a path. In case 2, w_iv_k and u_jv_k are edges and w_iu_j is a path. Now consider another component. Without loss of generality, we can assume that it contains a vertex u_q. If the cycle is the first case, we can exchange the signs of column u_j and column u_q in the list of $K_{z,y+x}$, but keep the list of $K_{y,x}$ unchanged. Namely, for every vertex w_i with $1\leq i\leq z$, which is adjacent to exactly one of u_j and u_q originally, say u_j, now w_i is adjacent to u_q, the other one of the two vertices. But the other adjacency relations are kept unchanged. Similarly, if the cycle is the second case, we can exchange the signs of column v_k and column u_q in the list of $K_{z,y+x}$, but keep the list of $K_{y,x}$ unchanged. Now with the new list, we have the edges w_iu_q, u_jv_k. Since u_q and $u_j(v_k)$ can not appear in the original path $w_iv_k(w_iu_j)$, the path still exists and keeps unchanged. Then the two components become one, but the other components remain the same as before. So the total number of components is reduced by 1. Repeating the procedure, we can finally make G connected and find out a spanning tree of G. Now the number of rest unused edges is less than $z+y+x-1$. So we have already found $\lfloor (xy+yz+zx)/(x+y+z-1)\rfloor$ edge-disjoint spanning trees of $K_{x,y,z}$, and hence, $\kappa_{x+y+z}(K_{x,y,z})\geq \lfloor (xy+yz+zx)/(x+y+z-1)\rfloor$. So we have proved that $\kappa_{x+y+z}(K_{x,y,z})=\lfloor (xy+yz+zx)/(x+y+z-1)\rfloor$. ∎

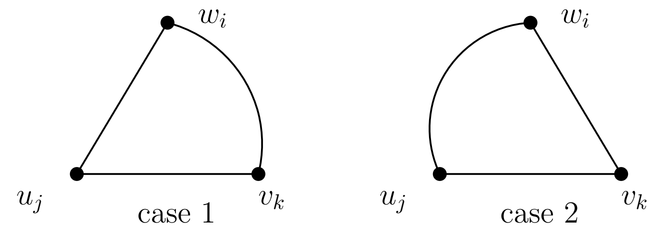

Figure 1. The component containing a cycle

3. The k-connectivity of a complete equipartition 3-partite graph

For simplicity, denote K_b^3 by G. Now, let $\mathfrak{A}_0$ be the set of trees connecting $S_{x,y,z}$ whose vertex set is $S_{x,y,z}$, let $\mathfrak{A}_1$ be the set of trees connecting $S_{x,y,z}$ whose vertex set is $S_{x,y,z}\cup\{u\}$, where $u\notin S_{x,y,z}$, and let $\mathfrak{A}_2$ be the set of trees connecting $S_{x,y,z}$ whose vertex set is $S_{x,y,z}\cup\{u,v\}$, where $u,v\notin S_{x,y,z}$ and they belong to distinct parts.

Lemma 3.1. *Let A be a maximum set of internally disjoint trees connecting $S_{x,y,z}$. Then we can always find a set A' of internally disjoint trees connecting $S_{x,y,z}$, such that $\mid A \mid = \mid A' \mid$ and $A' \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$.*

Proof. Let $A = \{T_1, T_2, \dots, T_p\}$. If for some tree T_j in A, $T_j \notin \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$, then let $V(T_j) = S_{x,y,z} \cup U' \cup V' \cup W'$, where $(U' \cup V' \cup W') \cap S_{x,y,z} = \emptyset$, $U' \subseteq U$, $V' \subseteq V$ and $W' \subseteq W$. At most two of U', V' and W' can be empty. If at least two of them are nonempty, say U', V', let $u' \in U'$ and $v' \in V'$. The tree T'_j with vertex set $V(T'_j) = S_{x,y,z} \cup \{u', v'\}$ and edge set $E(T'_j) = \{u'w_1, \dots, u'w_z, u'v_1, \dots, u'v_y, v'u_1, \dots, v'u_x, u'v'\}$ is a tree in $\mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ (See Figure 2). Since $V(T_j) \cap V(T_i) = S_{x,y,z}$ and $E(T_j) \cap E(T_i) = \emptyset$ for every tree $T_i \in A$, where $i \neq j$, T_i does not contain u', v' nor the edges incident with u', v'. Therefore, $V(T'_j) \cap V(T_i) = S_{x,y,z}$ and $E(T'_j) \cap E(T_i) = \emptyset$ for $1 \le i \le p, i \neq j$. If exactly one of U', V' and W' is nonempty, say U', let $U' = \{u'_1, u'_2, \dots, u'_q\}$. Then we delete $u'_2, \dots, u'_q$. It may produce some connected components. For each component which does not contain u'_1, there must be an edge connecting one of $u'_2, \dots, u'_q$ with it originally. Thus each component which does not contain u'_1 must contain a vertex in $V \cup W$. Find such a vertex and connect it with u'_1 by an edge. Obviously, the new graph we obtain is a tree $T'_j \in \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ that connects $S_{x,y,z}$ (See Figure 3). For every tree $T_i \in A$, where $i \neq j$, T_i does not contain u'_1 nor the edges incident with u'_1. Therefore, $V(T'_j) \cap V(T_i) = S_{x,y,z}$ and $E(T'_j) \cap E(T_i) = \emptyset$ for $1 \le i \le p, i \neq j$. Replacing each $T_j \notin \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ by T'_j, we finally get the set $A' \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ which has the same cardinality as A. ∎

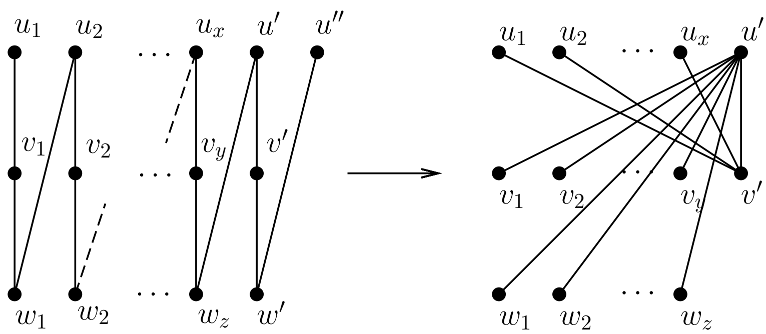

Figure 2. If U' and V' are not empty

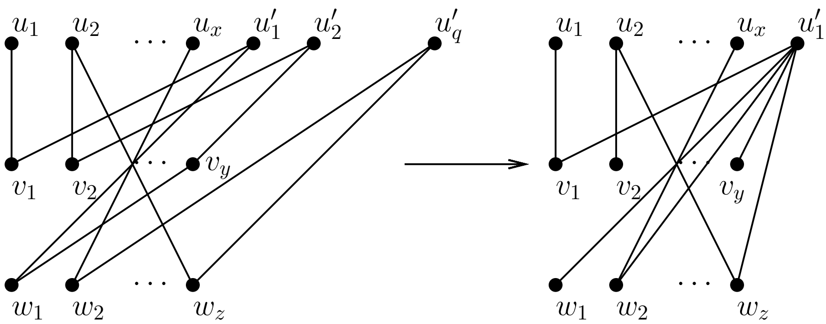

Figure 3. If only U' is nonempty

So, we can assume that the maximum set A of internally disjoint trees connecting $S_{x,y,z}$ is contained in $\mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$. Next, we will define the standard structure of trees in $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively.

Every tree in $\mathfrak{A}_0$ is of standard structure. A tree T in $\mathfrak{A}_1$ with vertex set $V(T) = S_{x,y,z} \cup \{u\}$, where $u \in U \setminus S_{x,y,z}$, is of standard structure if u is adjacent to every vertex in $S_{x,y,z} \cap (V \cup W)$. Since $|E(T)| = |V(T)| - 1 = k$ and $d_T(u) = |S_{x,y,z} \cap (V \cup W)| = k - x$, there are x edges incident with $S_{x,y,z} \cap U$. We know that $|S_{x,y,z} \cap U| = x$ and each vertex must have degree at least 1 in T. So every vertex in $S_{x,y,z} \cap U$ has degree 1. A tree T in $\mathfrak{A}_1$ with vertex set $V(T) = S_{x,y,z} \cup \{v\}$, where $v \in V \setminus S_{x,y,z}$, is of standard structure if v is adjacent to every vertex in $S_{x,y,z} \cap (U \cup W)$. Similarly, every vertex in $S_{x,y,z} \cap V$ has degree 1. A tree T in $\mathfrak{A}_1$ with vertex set $V(T) = S_{x,y,z} \cup \{w\}$, where $w \in W \setminus S_{x,y,z}$, is of standard structure if w is adjacent to every vertex in $S_{x,y,z} \cap (U \cup V)$. Similarly, every vertex in $S_{x,y,z} \cap W$ has degree 1. A tree T in $\mathfrak{A}_2$ with vertex set $V(T) = S_{x,y,z} \cup \{u, v\}$, where $u \notin S_{x,y,z}$, $v \notin S_{x,y,z}$ and u, v belong to distinct parts, is of standard structure if u is adjacent to v and every vertex in $S_{x,y,z}$ is adjacent to exactly one of u, v. We then denote the set of trees in $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with the standard structure by $\mathscr{A}_0$, $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively. Clearly, $\mathscr{A}_0 = \mathfrak{A}_0$.

Lemma 3.2. *Let A be a maximum set of internally disjoint trees connecting $S_{x,y,z}$, $A \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$. Then we can always find a set A'' of internally disjoint trees connecting $S_{x,y,z}$, such that $|A| = |A''|$ and $A'' \subset \mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2$.*

Proof. Let $A = \{T_1, T_2, \ldots, T_p\}$. Suppose that there is a tree T_j in A such that $T_j \in \mathfrak{A}_1$, but $T_j \notin \mathscr{A}_1$. Let $V(T_j) = S_{x,y,z} \cup \{u\}$, where $u \in U \setminus S_{x,y,z}$. Note that the case $u \in V \setminus S_{x,y,z}$ and the case $u \in W \setminus S_{x,y,z}$ are similar. Since $T_j \notin \mathscr{A}_1$, there are some vertices in $S_{x,y,z} \cap (V \cup W)$, say $v'_1, \ldots, v'_s, w'_1, \ldots, w'_t$, not adjacent to u. Then we can connect v'_1 to u by a new edge. It will produce a unique cycle. Delete the other edge incident with v'_1 on the cycle. The graph remains a tree. Do the same operation on $v'_2, \ldots, v'_s, w'_1, \ldots, w'_t$ in turn. Finally we get a tree T'_j whose vertex set is $S_{x,y,z} \cup \{u\}$ and u is adjacent to every vertex in $S_{x,y,z} \cap (V \cup W)$, that is, T'_j is of standard structure. For each tree $T_r \in A \setminus \{T_j\}$, clearly T_r does not contain u nor the edges incident with u. So $V(T'_j) \cap V(T_r) = S_{x,y,z}$ and $E(T'_j) \cap E(T_r) = \emptyset$. Suppose that there is a tree T_j in A such that $T_j \in \mathfrak{A}_2$, but $T_j \notin \mathscr{A}_2$. Let $V(T_j) = S_{x,y,z} \cup \{u, v\}$, where $u, v \notin S_{x,y,z}$ and they belong to distinct parts. Without loss of generality, suppose that $u \in U \setminus S_{x,y,z}$ and $v \in V \setminus S_{x,y,z}$. If u and v are not adjacent, connect u and v by the edge uv. It will produce a unique cycle. Delete the other edge incident with u on the cycle. Then for every vertex $w \in S_{x,y,z}$, if w is adjacent to neither u nor v, connect w with an edge to one of them which is in the different part from w. It will produce a unique cycle. Delete the other edge incident with w on the cycle. The graph remains a tree, denoted by T_j'. By our operation, T_j' is a tree in $\mathscr{A}_2$. For each tree $T_r \in A \setminus \{T_j\}$, $V(T_j') \cap V(T_r) = S_{x,y,z}$ and $E(T_j') \cap E(T_r) = \emptyset$. Replacing each $T_j \notin \mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2$ by T_j', we finally get the set $A'' \subset \mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2$ which has the same cardinality as A. ∎

So, we can assume that the maximum set A of internally disjoint trees connecting $S_{x,y,z}$ is contained in $\mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2$. Namely, all trees in A are of standard structure.

For simplicity, we denote the union of the vertex sets of all trees in set A by $V(A)$ and the union of the edge sets of all trees in set A by $E(A)$. Let $A_0 := A \cap \mathscr{A}_0$, $A_1 := A \cap \mathscr{A}_1$ and $A_2 := A \cap \mathscr{A}_2$. Then $A = A_0 \cup A_1 \cup A_2$.

Lemma 3.3. *Let $A \subset \mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2$ be a maximum set of internally disjoint trees connecting $S_{x,y,z}$. Then at most one of $U \setminus V(A)$, $V \setminus V(A)$ and $W \setminus V(A)$ is nonempty.*

Proof. If at least two of $U \setminus V(A)$, $V \setminus V(A)$ and $W \setminus V(A)$ are nonempty, without loss of generality, let $u \in U \setminus V(A)$ and $v \in V \setminus V(A)$. Then the tree $T' \in \mathscr{A}_2$ with vertex set $V(T') = S_{x,y,z} \cup \{u,v\}$ and edge set $E(T') = \{u_1v, \ldots, u_xv, v_1u, \ldots, v_yu, w_1u, \ldots, w_zu, uv\}$ is a tree that connects $S_{x,y,z}$. Moreover, $V(T') \cap V(A) = S_{x,y,z}$ and since all edges of T' are incident with u or v, T' and T are edge-disjoint for any tree $T \in A$. So, $A \cup \{T'\}$ is also a set of internally disjoint trees connecting $S_{x,y,z}$, contradicting to the maximality of A. ∎

Lemma 3.4. *Let $A \subset \mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2$ be a maximum set of internally disjoint trees connecting $S_{x,y,z}$, and $A = A_0 \cup A_1 \cup A_2$. If $V(A) \neq V(G)$ and $A_0 \neq \emptyset$, then we can find a maximum set $A' = A_0' \cup A_1' \cup A_2'$ of internally disjoint trees connecting $S_{x,y,z}$, such that $|A_0'| = |A_0| - 1$, $|A_1'| = |A_1| + 1$, and $A_2' = A_2$.*

Proof. Let $u \in V(G) \setminus V(A)$ and $T \in A_0$. Without loss of generality, suppose $u \in U$. Then we can add the edge uv_1 to T and get a tree $T' \in \mathfrak{A}_1$. Using the method in Lemma 3.2, we can transform T' into a tree T'' of standard structure. Then $T'' \in \mathscr{A}_1$. Since for $T_j \in A \setminus \{T\}$, T_j does not contain u nor the edges incident with u and $E(T_j) \cap E(T) = \emptyset$, T'' and T_j are edge-disjoint, and $V(T'') \cap V(T_j) = S_{x,y,z}$. Let $A_0' := A_0 \setminus \{T\}$, $A_1' := A_1 \cup \{T''\}$ and $A_2' = A_2$. It is easy to see that $A' = A_0' \cup A_1' \cup A_2'$ is a set of internally disjoint trees connecting $S_{x,y,z}$. Since $|A_0'| = |A_0| - 1$, $|A_1'| = |A_1| + 1$, and $A_2' = A_2$, A' is a maximum set of internally disjoint trees connecting $S_{x,y,z}$ we want to find. ∎

So, we can assume that for the maximum set A of internally disjoint trees connecting $S_{x,y,z}$, either $V(A) = V(G)$ or $A_0 = \emptyset$. Moreover, if A' is a set of internally disjoint trees connecting $S_{x,y,z}$ which we find currently, $V(A') \neq V(G)$ and the edges in $E(G[S_{x,y,z}]) \setminus E(A')$ can form a tree T in $\mathscr{A}_0$, then by Lemma 3.4 we will add to A' a tree $T'' \in \mathscr{A}_1$ rather than a tree $T \in \mathscr{A}_0$ to form a larger set of internally disjoint trees connecting $S_{x,y,z}$. In a similar way, we can prove the following lemma.

Lemma 3.5. *Let $A \subset \mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2$ be a maximum set of internally disjoint trees connecting $S_{x,y,z}$, and $A = A_0 \cup A_1 \cup A_2$. If $V(A) \neq V(G)$, $(V(G) \setminus V(A)) \subseteq X$ and $V(A_1) \setminus (S_{x,y,z} \cup X) \neq \emptyset$, where $X = U, V, or\ W$, then we can find a maximum set $A' = A_0' \cup A_1' \cup A_2'$ of internally disjoint trees connecting $S_{x,y,z}$, such that $A_0' = A_0$, $|A_1'| = |A_1| - 1$, and $|A_2'| = |A_2| + 1$.*

Lemma 3.6. *We can always find a maximum set A of internally disjoint trees connecting $S_{x,y,z}$, such that $V(A) = V(G)$.*

Proof. Let A be the maximum set of internally disjoint trees connecting $S_{x,y,z}$ we find. If $V(A) \neq V(G)$, by Lemmas 3.4 and 3.5, we can assume that $A_0 = \emptyset$, $(V(G) \setminus V(A)) \subseteq X$ and $(V(A_1) \setminus S_{x,y,z}) \subseteq X$, where $X = U, V,$ or W. Since $A_0 = \emptyset$, $A_1 \neq \emptyset$ by the maximality of A.

Case 1. $X = U$.

Since $(V(G) \setminus V(A)) \subseteq U$ and $(V(A_1) \setminus S_{x,y,z}) \subseteq U$, any vertex $v \in (V \cup W) \setminus S_{x,y,z}$ is in some tree $T \in A_2$.

Claim: If there is a tree $T \in A_2$ with vertex set $S_{x,y,z} \cup \{v, w\}$, where $v \in V \setminus S_{x,y,z}$ and $w \in W \setminus S_{x,y,z}$, then $|V(G) \setminus V(A)| = 1$.

Proof. By contradiction, suppose that there are two vertices in $V(G) \setminus V(A)$, say u', u''. Let T_1 and T_2 be two trees in $\mathscr{A}_2$ with vertex sets $S_{x,y,z} \cup \{v, u'\}$ and $S_{x,y,z} \cup \{u'', w\}$, respectively. Since for $T' \in A \setminus \{T\}$, T' does not contain u', u'', v, w nor the edges incident with them, T_i and T' are edge-disjoint, and $V(T_i) \cap V(T') = S_{x,y,z}$ for $i = 1, 2$. Clearly, $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S_{x,y,z}$. Let $A' = A \setminus \{T\} \cup \{T_1, T_2\}$. It is easy to see that A' is a set of internally disjoint trees connecting $S_{x,y,z}$. But $|A'| = |A| + 1$, contradicting to the maximality of A. ∎

Since $|W \setminus S_{x,y,z}| = b - z \geq b - x = |U \setminus S_{x,y,z}| > |U \setminus V(A_1)|$, there must be a tree $T \in A_2$ with vertex set $S_{x,y,z} \cup \{v, w\}$, where $v \in V \setminus S_{x,y,z}$ and $w \in W \setminus S_{x,y,z}$. So $|V(G) \setminus V(A)| = 1$. Denote the vertex in $V(G) \setminus V(A)$ by u'. Take a tree $T_1 \in A_1$ with vertex set $S_{x,y,z} \cup \{u''\}$, where $u'' \in U \setminus S_{x,y,z}$. Let T_2 and T_3 be two trees in $\mathscr{A}_2$ with vertex sets $S_{x,y,z} \cup \{v, u'\}$ and $S_{x,y,z} \cup \{u'', w\}$, respectively. Since for $T' \in A \setminus \{T, T_1\}$, T' does not contain u', u'', v, w nor the edges incident with them, T_i and T' are edge-disjoint, and $V(T_i) \cap V(T') = S_{x,y,z}$ for $i = 2, 3$. Clearly, $E(T_3) \cap E(T_2) = \emptyset$ and $V(T_3) \cap V(T_2) = S_{x,y,z}$. Let $A' = A \setminus \{T, T_1\} \cup \{T_3, T_2\}$. It is easy to see that A' is a set of internally disjoint trees connecting $S_{x,y,z}$. Since $|A'| = |A|$, A' is a maximum set of internally disjoint trees connecting $S_{x,y,z}$, such that $V(A') = V(G)$.

Case 2. $X = V$.

The proof is similar to that of Case 1.

Case 3. $X = W$.

In this case, since $|W \setminus S_{x,y,z}| = b - z \geq b - y \geq b - x = |U \setminus S_{x,y,z}|$, it seems that it is possible that there is no tree $T \in A_2$ with vertex set $S_{x,y,z} \cup \{v, u\}$, where $v \in V \setminus S_{x,y,z}$ and $u \in U \setminus S_{x,y,z}$, and hence any vertex $v \in (V \cup U) \setminus S_{x,y,z}$ is in some tree $T \in A_2$ with vertex set $S_{x,y,z} \cup \{v, w\}$, where $w \in W \setminus V(A_1)$. But actually this is impossible. This is because it implies that $b - x + b - y < b - z$, namely, $b + z < x + y$. Since $(V(A_1) \setminus S_{x,y,z}) \subseteq W$, $|A_1| = |V(A_1) \setminus S_{x,y,z}| < b - z < b + z < x + y$ and $E(A) \cap E(G[U_x \cup V_y]) = \emptyset$. Now $|A_1| < x + y$ implies that for every vertex $w_i \in S_{x,y,z}$, $i = 1, \ldots, z$, there is at least one unused edge in $E(G[S_{x,y,z}])$ incident with w_i. These edges together with the edges in $E(G[U_x \cup V_y]$ will form a tree in $\mathscr{A}_0$, which is internally disjoint with all trees in A, contradicting to the maximality of A. So there must be a tree $T \in A_2$ with vertex set $S_{x,y,z} \cup \{v, u\}$, where $v \in V \setminus S_{x,y,z}$ and $u \in U \setminus S_{x,y,z}$. Similar to the proof of Case 1, we can transform A to A', which is a maximum set of internally disjoint trees connecting $S_{x,y,z}$, such that $V(A') = V(G)$. ∎

So, we can assume that if A is a maximum set of internally disjoint trees connecting $S_{x,y,z}$, then $V(A) = V(G)$.

Lemma 3.7. *Let $A \subset \mathscr{A}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2$ be a maximum set of internally disjoint trees connecting $S_{x,y,z}$, and $A = A_0 \cup A_1 \cup A_2$. If $A_0 \neq \emptyset$ and $A_2 \neq \emptyset$, then we can find a maximum set $A' = A'_0 \cup A'_1 \cup A'_2$ of internally disjoint trees connecting $S_{x,y,z}$, such that $|A'_0| = |A_0| - 1$, $|A'_1| = |A_1| + 2$, and $|A'_2| = |A_2| - 1$.*

Proof. Let T be a tree in A_2 with vertex set $S_{x,y,z} \cup \{u,v\}$, where $u \in U \setminus S_{x,y,z}$ and $v \in V \setminus S_{x,y,z}$. Let T_1 be a tree in A_0. We want to transform T and T_1 to two trees $T_2, T_3 \in \mathscr{A}_1$ with vertex sets $S_{x,y,z} \cup \{u\}$ and $S_{x,y,z} \cup \{v\}$ respectively. In T_2, every vertex in U_x must be incident with an edge in $E(T_1)$. In T_3, every vertex in V_y must be incident with an edge in $E(T_1)$. And all these edges must be distinct. To do this, let the vertices in W_z be in Layer 0. Let the vertices having distance i to W_z in T_1 be in Layer i. Every vertex in $U_x \cup V_y$ is in some Layer i, $1 \leq i \leq k$, since T_1 is connected. For each vertex $u' \in U_x \cup V_y$, assume that u' is in Layer i. Take one edge connecting u' with some vertex in Layer $i-1$. There must be such an edge by our construction. According to our choices of edges, these edges are all distinct. So T_2 and T_3 are edge-disjoint and $V(T_3) \cap V(T_2) = S_{x,y,z}$. Since for $T' \in A \setminus \{T, T_1\}$, T' does not contain u, v nor the edges incident with them and T' does not contain the edges of T_1, T_i and T' are edge-disjoint, and $V(T_i) \cap V(T') = S_{x,y,z}$ for $i = 2,3$. Let $A' = A \setminus \{T, T_1\} \cup \{T_3, T_2\}$. It is easy to see that A' is a set of internally disjoint trees connecting $S_{x,y,z}$. Since $|A'_0| = |A_0| - 1$, $|A'_1| = |A_1| + 2$, and $|A'_2| = |A_2| - 1$, $|A'| = |A|$. Thus A' is a maximum set of internally disjoint trees connecting $S_{x,y,z}$ we want to find. ∎

So, we can assume that if A is a maximum set of internally disjoint trees connecting $S_{x,y,z}$, then either A_0 or A_2 is empty.

From the above lemmas, we can form our principle in finding the maximum set of internally disjoint trees connecting $S_{x,y,z}$. First we find as many trees in $\mathscr{A}_1$ as possible. If $V(A_1) \neq V(G)$, we then find as many trees in $\mathscr{A}_2$ as possible; else if $V(A_1) = V(G)$, we then find as many trees in $\mathscr{A}_0$ as possible. So the final maximum set A of internally disjoint trees connecting $S_{x,y,z}$ is of the form $A_0 \cup A_1$ or $A_1 \cup A_2$. If $A = A_0 \cup A_1$, then every vertex $v \in V(G) \setminus S_{x,y,z}$ is contained in some tree $T \in A_1$ with vertex set $S_{x,y,z} \cup \{v\}$. Since $|V(G) \setminus S_{x,y,z}| = 3b - k$, there are $(3b-k)$ trees in A_1. So $|A| \geq 3b - k$. If $A = A_1 \cup A_2$, every tree in A must contain one vertex in $V(G) \setminus S_{x,y,z}$ and some trees may contain two such vertices. So $|A| \leq |V(G) \setminus S_{x,y,z}| = 3b - k$. Since $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$, for fixed k, if there exists $x = x_1$, $y = y_1$ and $z = z_1$ such that $A = A_1 \cup A_2$, then we need not consider other $x = x_2$, $y = y_2$ and $z = z_2$ such that $A = A_0 \cup A_1$. But for some k, $A = A_0 \cup A_1$ holds for any x, y, z such that $x + y + z = k$. Next we will give a necessary and sufficient condition to $A = A_0 \cup A_1$.

Lemma 3.8. *Let A be the final maximum set of internally disjoint trees connecting $S_{x,y,z}$ we find. Then $A_2 = \emptyset$ if and only if $x(b-x) + y(b-y) + z(b-z) \leq xy + yz + zx$.*

Proof. If $A_2 = \emptyset$, then every vertex $v \in V(G) \setminus S_{x,y,z}$ is contained in some tree $T \in A_1$ with vertex set $S_{x,y,z} \cup \{v\}$. There are $(b-x)$ trees in A_1 with vertex set $S_{x,y,z} \cup \{u\}$, where $u \in U \setminus S_{x,y,z}$. We denote the trees by $\mathscr{T}^U = \{T^U_1, \ldots, T^U_{b-x}\}$. In T^U_i, every vertex in U_x is incident with an edge in $E(G[S_{x,y,z}])$ and these edges must be distinct. So T^U_i contains x edges in $E(G[S_{x,y,z}])$. $T^U_1, \ldots, T^U_{b-x}$ contain altogether $x(b-x)$ edges in $E(G[S_{x,y,z}])$. Similarly, There are $(b-y)$ trees in A_1 with vertex set $S_{x,y,z} \cup \{v\}$, where $v \in V \setminus S_{x,y,z}$. We denote the trees by $\mathscr{T}^V = \{T^V_1, \ldots, T^V_{b-y}\}$. These trees contain altogether $y(b-y)$ edges in $E(G[S_{x,y,z}])$. There are $(b-z)$ trees in A_1 with vertex set $S_{x,y,z} \cup \{w\}$, where $w \in W \setminus S_{x,y,z}$. We denote the trees by

$\mathscr{T}^W = \{T_1^W, \ldots, T_{b-z}^W\}$. These trees contain altogether $z(b-z)$ edges in $E(G[S_{x,y,z}])$. Since these trees are edge-disjoint, $x(b-x)+y(b-y)+z(b-z) \leq |E(G[S_{x,y,z}])| = xy+yz+zx$.

If $x(b-x)+y(b-y)+z(b-z) \leq xy+yz+zx$, we want to prove that $A_2 = \emptyset$. Let $d_S(v)$ denote the number of unused edges incident with vertex v in $E(G[S_{x,y,z}])$ currently. Let $d_{X,Y}(v)$ denote the number of unused edges incident with vertex v in $E(G[X \cup Y])$ currently, where $X,Y \in \{U_x, V_y, W_z\}$. If we replace the vertex v by a set Q in above notation, then $d_S(Q) = \sum_{v \in Q} d_S(v)$, and other notation is similar. Then we can find $(b-x)$ trees $T_1^U, \ldots, T_{b-x}^U$ with vertex set $S_{x,y,z} \cup \{u\}$, where $u \in U \setminus S_{x,y,z}$ if and only if for every vertex u_i with $1 \leq i \leq x$, $d_S(u_i) \geq b-x$, and we can find $b-y$ trees $T_1^V, \ldots, T_{b-y}^V$ with vertex set $S_{x,y,z} \cup \{v\}$, where $v \in V \setminus S_{x,y,z}$ if and only if for every vertex v_i with $1 \leq i \leq y$, $d_S(v_i) \geq b-y$, and we also can find $b-z$ trees $T_1^W, \ldots, T_{b-z}^W$ with vertex set $S_{x,y,z} \cup \{w\}$, where $w \in W \setminus S_{x,y,z}$ if and only if for every vertex w_i with $1 \leq i \leq z$, $d_S(w_i) \geq b-z$. Since $x(b-x)+y(b-y)+z(b-z) \leq xy+yz+zx$, at least one of $x(b-x) \leq xy$, $y(b-y) \leq yz$ and $z(b-z) \leq zx$ must hold. We distinct three cases.

Case 1. $y(b-y) \leq yz$.

Since $y(b-y) \leq yz$, $b-y \leq z$. So $b \leq y+z \leq x+z \leq x+y$, and hence $b-x \leq y$ and $b-z \leq x$ hold. Since $d_{U_x,W_z}(w_i) = x \geq b-z$, we can find $b-z$ trees $T_1^W, \ldots, T_{b-z}^W$ with vertex sets $S_{x,y,z} \cup \{w_{z+1}\}, S_{x,y,z} \cup \{w_{z+2}\}, \ldots, S_{x,y,z} \cup \{w_b\}$, respectively. Let the neighbors of w_1 in $T_1^W, \ldots, T_{b-z}^W$ be $u_1, \ldots, u_{b-z}$, respectively. Let the neighbors of w_2 in $T_1^W, \ldots, T_{b-z}^W$ be $u_{b-z+1}, \ldots, u_{2(b-z)}$ respectively. Let the neighbors of w_i in $T_1^W, \ldots, T_{b-z}^W$ be $u_{(i-1)(b-z)+1}, \ldots$, $u_{i(b-z)}$, respectively, and so on and so forth. Here and in what follows, the subscript j of $u_j \in U_x$ is expressed modulo x as one of $1,2,\ldots,x$. Now, since $d_{V_y,W_z}(v_i) = z \geq b-y$, we can find $b-y$ trees $T_1^V, \ldots, T_{b-y}^V$ with vertex sets $S_{x,y,z} \cup \{v_{y+1}\}, S_{x,y,z} \cup \{v_{y+2}\}, \ldots, S_{x,y,z} \cup \{v_b\}$, respectively. Let the neighbors of v_1 in $T_1^V, \ldots, T_{b-y}^V$ be $w_1, \ldots, w_{b-y}$, respectively. Let the neighbors of v_2 in $T_1^V, \ldots, T_{b-y}^V$ be $w_{b-y+1}, \ldots, w_{2(b-y)}$, respectively. Let the neighbors of v_i in $T_1^V, \ldots, T_{b-y}^V$ be $w_{(i-1)(b-y)+1}, \ldots, w_{i(b-y)}$, respectively, and so on and so forth. Here and in what follows, the subscript j of $w_j \in W_z$ is expressed modulo z as one of $1,2,\ldots,z$. Now, since $d_{U_x,V_y}(u_i) = y \geq b-x$, we can find $b-x$ trees $T_1^U, \ldots, T_{b-x}^U$ with vertex sets $S_{x,y,z} \cup \{u_{x+1}\}, S_{x,y,z} \cup \{u_{x+2}\}, \ldots, S_{x,y,z} \cup \{u_b\}$, respectively. Let the neighbors of u_1 in $T_1^U, \ldots, T_{b-x}^U$ be $v_1, \ldots, v_{b-x}$, respectively. Let the neighbors of u_2 in $T_1^U, \ldots, T_{b-x}^U$ be $v_{b-x+1}, \ldots, v_{2(b-x)}$, respectively. Let the neighbors of u_i in $T_1^U, \ldots, T_{b-x}^U$ be $v_{(i-1)(b-x)+1}, \ldots, v_{i(b-x)}$, respectively, and so on and so forth. Here and in what follows, the subscript j of $v_j \in V_y$ is expressed modulo y as one of $1,2,\ldots,y$. Now we have found $3b-k$ trees in A_1, and thus every vertex in $V(G) \setminus S_{x,y,z}$ is contained in a tree in A_1. Thus, $V(A_1) = V(G)$, and so $A_2 = \emptyset$.

Case 2. $y(b-y) > yz$, $z(b-z) \leq zx$, and

Case 3. $y(b-y) > yz$, $z(b-z) > zx$ and $x(b-x) \leq xy$ can be dealt with similarly. The details are omitted. ∎

Now we know that if $x(b-x)+y(b-y)+z(b-z) \leq xy+yz+zx$, then $A_2 = \emptyset$. It is clear that $x(b-x)+y(b-y)+z(b-z) \leq xy+yz+zx \Leftrightarrow (x+y+z)b \leq x^2+y^2+z^2+xy+yz+zx \Leftrightarrow 2kb \leq (x+y)^2+(y+z)^2+(z+x)^2 \Leftrightarrow 2kb \leq (k-z)^2+(k-x)^2+(k-y)^2 \Leftrightarrow 2kb-k^2 \leq x^2+y^2+z^2$. Since $x+y+z=k$, $x^2+y^2+z^2 \geq k^2/3$. If $k^2/3 \geq 2kb-k^2$, $A = A_0 \cup A_1$ holds for any x,y,z such that $x+y+z=k$. Since $k^2/3 \geq 2kb-k^2 \Leftrightarrow k \geq 3b/2$, then when $k \geq 3b/2$, $A = A_0 \cup A_1$.

If $A = A_0 \cup A_1$, $|A_1| = 3b-k$. Next we will consider $|A_0|$.

Lemma 3.9. *When $k \geq \frac{3b}{2}$, we can find $\lfloor((xy+yz+zx) - [x(b-x)+y(b-y)+z(b-z)])/(k-1)\rfloor$ trees in A_0 and $3b-k$ trees in A_1.*

Proof. For convenience, denote $\lfloor((xy+yz+zx) - [x(b-x)+y(b-y)+z(b-z)])/(k-1)\rfloor = a$. Similar to the proof of Lemma 3.8, we will distinct three cases to prove this lemma. Since $a \leq \lfloor(xy+yz+zx)/(k-1)\rfloor$, we can find a trees in A_0 using the method in the proof of Theorem 2.1. If $z+y \leq z+x < b \leq y+x$, then $a < \lfloor(xy-x(b-x))/(k-1)\rfloor$. Namely, we can use The List Method to find a edge-disjoint spanning trees of $K_{y,x}$, such that $d_{U_x,V_y}(u_i) \geq b-x$ for $1 \leq i \leq x$. So we can find $b-x$ trees $T_1^U, \dots, T_{b-x}^U$ with vertex sets $S_{x,y,z} \cup \{u_{x+1}\}, S_{x,y,z} \cup \{u_{x+2}\}, \dots, S_{x,y,z} \cup \{u_b\}$, respectively, without using the edges in $E(G[U_x \cup W_z])$. According to The List Method, for every pair of vertices v_i and v_j with $1 \leq i, j \leq y$, the difference between the number of unused edges incident with v_i and the number of unused edges incident with v_j is at most 1. For simplicity, we refer V_y to satisfy property P. Then for every spanning tree of $K_{y,x}$, we can connect each w_i to some v_j to form a spanning tree of $K_{x,y,z}$ and keep V_y satisfying property P. Now the number of unused edges incident with each w_i is $x+y-a$. Since $x+y-a > b-z$, we can find $b-z$ trees $T_1^W, \dots, T_{b-z}^W$ with vertex sets $S_{x,y,z} \cup \{w_{z+1}\}, S_{x,y,z} \cup \{w_{z+2}\}, \dots, S_{x,y,z} \cup \{w_b\}$, respectively. Since $x < b-z$, all edges in $E(G[U_x \cup W_z])$ are used. So all unused edges are incident with V_y. Since the number of all unused edges is at least $y(b-y)$ and V_y satisfies property P, $d_S(v_i) \geq b-y$ for $1 \leq i \leq y$. So we can find $b-y$ trees $T_1^V, \dots, T_{b-y}^V$ with vertex sets $S_{x,y,z} \cup \{v_{y+1}\}, S_{x,y,z} \cup \{v_{y+2}\}, \dots, S_{x,y,z} \cup \{v_b\}$, respectively. So we can find altogether $b-x+b-y+b-z = 3b-k$ trees in A_1. The proofs for the other two cases are similar. ∎

Now we know that, when $k \geq 3b/2$,

$$\kappa(S_{x,y,z}) = \left\lfloor \frac{(xy+yz+zx) - [x(b-x)+y(b-y)+z(b-z)]}{k-1} \right\rfloor + 3b-k.$$

Next, we will calculate $\kappa_k(K_b^3)$ for $k \geq 3b/2$.

Lemma 3.10. *When $k \geq 3b/2$,*

$$\kappa_k(K_b^3) = \left\lfloor \frac{\lceil \frac{k^2}{3} \rceil + k^2 - 2kb}{2(k-1)} \right\rfloor + 3b-k.$$

Proof. Since $\kappa(S_{x,y,z}) = \lfloor((xy+yz+zx) - [x(b-x)+y(b-y)+z(b-z))/(k-1)\rfloor + 3b-k$, $k-1$ and $3b-k$ are constants, all we have to do is to calculate $\min\{(xy+yz+zx) - [x(b-x)+y(b-y)+z(b-z)]\}$. Since

$$\begin{aligned}
&(xy+yz+zx) - [x(b-x)+y(b-y)+z(b-z)] \\
=&\frac{(x+y)^2+(y+z)^2+(z+x)^2-2kb}{2} \\
=&\frac{(k-z)^2+(k-x)^2+(k-y)^2-2kb}{2} \\
=&\frac{x^2+y^2+z^2-2kb+k^2}{2},
\end{aligned}$$

then

$$\min\{(xy+yz+zx) - [x(b-x)+y(b-y)+z(b-z)]\}$$

$$=\min\left\{\frac{x^2+y^2+z^2-2kb+k^2}{2}\right\}$$
$$=\frac{\min\{x^2+y^2+z^2\}}{2}-kb+\frac{k^2}{2}.$$

Now the problem is reduced to the one of calculating $\min\{x^2+y^2+z^2\}$, where $x+y+z=k$ and x,y,z are integers. We know that, when $x+y+z=k$ and x,y,z are real numbers, the minimum value is $k^2/3$. If $k=0 \pmod 3$, obviously when $x=y=z=k/3$, it attains the minimum value. However, if $k=1 \pmod 3$ or $k=2 \pmod 3$, namely $k^2=1 \pmod 3$, it can not attain the minimum value. But since x,y,z are integers, $\min\{x^2+y^2+z^2\}$ should also be an integer, and $(k^2+2)/3=\lceil k^2/3\rceil$ should be the minimum value, when x,y,z are integers. If $k=1 \pmod 3$, when $x=k+2/3, y=z=(k-1)/3$, it indeed attains $(k^2+2)/3$. If $k=2 \pmod 3$, when $x=y=(k+1)/3, z=(k-2)/3$, it indeed attains $(k^2+2)/3$, too. So for $z\neq 0$,

$$\begin{aligned}&\min\{\kappa(S_{x,y,z})\}\\ =&\min\left\{\left\lfloor\frac{(xy+yz+zx)-[x(b-x)+y(b-y)+z(b-z)]}{k-1}\right\rfloor+3b-k\right\}\\ =&\left\lfloor\frac{\lceil\frac{k^2}{3}\rceil+k^2-2kb}{2(k-1)}\right\rfloor+3b-k.\end{aligned}$$

On the other hand, if $z=0$,

$$\min\{\kappa(S_{x,y,0})\}=b+\kappa_k(K_{b,b})=\begin{cases}2b-\frac{k}{2}+\lfloor\frac{k^2}{4(k-1)}\rfloor & \text{if } k \text{ is even };\\ 2b-\frac{k-1}{2}+\lfloor\frac{(k-1)^2}{4(k-1)}\rfloor & \text{if } k \text{ is odd }.\end{cases}$$

Note that $b\geq x\geq y\geq z=0$, $k=x+y\leq 2b$ and $k\geq 3$. If k is even, we have

$$\begin{aligned}&\left\lceil\frac{\lceil\frac{k^2}{3}\rceil+k^2-2kb+2(k-1)(3b-k)}{2(k-1)}\right\rceil-\left\lceil\frac{4(k-1)(2b-\frac{k}{2})+k^2}{4(k-1)}\right\rceil\\ \leq&\left\lceil\frac{\frac{k^2+2}{3}+k^2-2kb+2(k-1)(3b-k)}{2(k-1)}\right\rceil-\left\lceil\frac{4(k-1)(2b-\frac{k}{2})+k^2}{4(k-1)}\right\rceil\\ =&\frac{2k-4b-\frac{k^2}{3}+\frac{4}{3}}{4(k-1)}\\ \leq&0,\end{aligned}$$

and so

$$\left\lfloor\frac{\lceil\frac{k^2}{3}\rceil+k^2-2kb+2(k-1)(3b-k)}{2(k-1)}\right\rfloor\leq\left\lfloor\frac{4(k-1)(2b-\frac{k}{2})+k^2}{4(k-1)}\right\rfloor,$$

namely,

$$\left\lfloor\frac{\lceil\frac{k^2}{3}\rceil+k^2-2kb}{2(k-1)}\right\rfloor+3b-k\leq 2b-\frac{k}{2}+\left\lfloor\frac{k^2}{4(k-1)}\right\rfloor.$$

The proof for the case that k is odd is similar.

Thus $\min\{\kappa(S_{x,y,z})\} \le \min\{\kappa(S_{x,y,0})\}$, and for $k \ge 3b/2$,

$$\kappa_k(K_b^3) = \left\lfloor \frac{\lceil \frac{k^2}{3} \rceil + k^2 - 2kb}{2(k-1)} \right\rfloor + 3b - k. \qquad \blacksquare$$

Next, we will calculate $\kappa_k(K_b^3)$ for $k < b$. Notice that now $x^2+y^2+z^2 \le k^2 < 2kb-k^2$, namely, $x(b-x)+y(b-y)+z(b-z) > xy+yz+zx$ for any x,y,z such that $x+y+z=k$. So $A = A_1 \cup A_2$, then not every vertex $v \in V(G) \setminus S_{x,y,z}$ is contained in some tree $T \in A_1$. Thus the problem appears: which vertices should we choose to form trees in A_1? We know that every T_i^U needs x edges in $E(G[S_{x,y,z}])$, every T_i^V needs y edges in $E(G[S_{x,y,z}])$ and every T_i^W needs z edges in $E(G[S_{x,y,z}])$. Since $x \ge y \ge z$, a natural idea is that we first pick up as many trees in $\mathscr{T}^W$ as possible, then pick up as many trees in $\mathscr{T}^V$ as possible, and finally pick up as many trees in $\mathscr{T}^U$ as possible. Since $k < b$, namely, $x+y < b-z$, we can pick up $x+y$ trees in $\mathscr{T}^W$, and run out of all edges incident with W_z. Since $x < b-y$, we can pick up x trees in $\mathscr{T}^V$, and now we run out of all edges in $E(G[S_{x,y,z}])$. So the rest of vertices can only form trees in A_2. Now there remain $b-x$ vertices in $U \setminus S_{x,y,z}$, $b-y-x$ vertices in $V \setminus S_{x,y,z}$ and $b-z-y-x$ vertices in $W \setminus S_{x,y,z}$. If $b-x \le b-y-x+b-z-y-x$, namely, $y \le b-k$, we can pair up all these vertices except for at most one vertex. In this case, we do not waste any vertices. So we have found the maximum number of internally disjoint trees connecting $S_{x,y,z}$.

Lemma 3.11. *If $k < b$ and $y \le b-k$, then $\kappa(S_{x,y,z}) = \lfloor (3b+k-y-2z)/2 \rfloor$.*

Proof. As the above statement, there are $x+y+x$ trees in A_1 and $\lfloor (b-x+b-x-y+b-x-y-z)/2 \rfloor$ trees in A_2. So

$$\kappa(S_{x,y,z}) = \left\lfloor \frac{3b-3x+4x-2y+2y-z}{2} \right\rfloor = \left\lfloor \frac{3b+k-y-2z}{2} \right\rfloor. \qquad \blacksquare$$

However, if $b-x > b-y-x+b-z-y-x$, the thing is not so simple, because we may have "wasted" vertices. How to avoid wasting? We should pick up the trees more carefully. After we have picked up $x+y$ trees in $\mathscr{T}^W$, there remain $b-k$ vertices in $W \setminus S_{x,y,z}$. Since we have run out of all edges incident with W_z, these vertices can not form a tree in A_1. Not to waste them, we pair them up with $b-k$ vertices in $U \setminus S_{x,y,z}$. Now there remain $b-x-(b-k) = k-x$ vertices in $U \setminus S_{x,y,z}$ and $b-y$ vertices in $V \setminus S_{x,y,z}$. Now we pick up $b-y-(k-x) = b-k+x-y$ trees in $\mathscr{T}^V$, and there remain $k-x$ vertices in both $U \setminus S_{x,y,z}$ and $V \setminus S_{x,y,z}$. Since we can find altogether at most x trees in $\mathscr{T}^V$ and we have already picked up $b-k+x-y$ of them, we can pick up at most $x-(b-k+x-y) = k-b+y$ more trees in $\mathscr{T}^V$. Since $b-y > x$, $k-b+y < k-x$. If we pick up trees as in the case that $y \le b-k$, then with these $k-x$ pairs of vertices in $U \setminus S_{x,y,z}$ and $V \setminus S_{x,y,z}$, we can find $k-b+y$ trees in $\mathscr{T}^V$ and $k-x-(k-b+y) = b-x-y$ trees in $\mathscr{T}^{U,V}$. There are altogether $k-b+y+b-x-y = k-x$ trees and $k-b+y$ vertices in $U \setminus S_{x,y,z}$ remaining unused. But we can simply find $k-x$ trees in $\mathscr{T}^{U,V}$ by pairing up these $k-x$ pairs of vertices without using any edges in $E(G[S_{x,y,z}])$. It means that we do not use the edges efficiently. The most efficient way to use the edges is that we pick up as many pairs of trees in $\mathscr{T}^U$ and $\mathscr{T}^V$ as possible and then pair up the remaining vertices. The following lemma gives $\kappa(S_{x,y,z})$ for $k < b$ and $y \ge b-k$.

Lemma 3.12. *If $k < b$ and $y \ge b-k$, then $\kappa(S_{x,y,z}) = 2b-y-z+\lfloor ((k-b)y+y^2)/(k-z) \rfloor$.*

Proof. As the above statement, we find $x+y$ trees in $\mathscr{T}^W$ and $b-k$ trees in $\mathscr{T}^{U,W}$. Then we find $b-k+x-y$ trees in $\mathscr{T}^V$ and there are $xy-(b-k+x-y)y=(k-b+y)y$ unused edges left. So we can find $\lfloor((k-b+y)y)/(x+y)\rfloor$ trees in $\mathscr{T}^U$ and $\lfloor((k-b+y)y)/(x+y)\rfloor$ trees in $\mathscr{T}^V$. Finally, there remain $k-x-\lfloor((k-b+y)y)/(x+y)\rfloor$ pairs of vertices unused, and so we can find $k-x-\lfloor((k-b+y)y)/(x+y)\rfloor$ trees in $\mathscr{T}^{U,V}$. Thus

$$\begin{aligned}
&\kappa(S_{x,y,z})\\
=&x+y+b-k+b-k+x-y+2\left\lfloor\frac{(k-b+y)y}{x+y}\right\rfloor+k-x-\left\lfloor\frac{(k-b+y)y}{x+y}\right\rfloor\\
=&2b-k+x+\left\lfloor\frac{(k-b+y)y}{x+y}\right\rfloor\\
=&2b-y-z+\left\lfloor\frac{(k-b)y+y^2}{k-z}\right\rfloor.
\end{aligned}$$

∎

Next, we will calculate $\min\{\kappa(S_{x,y,z})\}$ for $k<b$.

Lemma 3.13. *For $k<b$ and $y\le b-k$, we have*

$$\min\{\kappa(S_{x,y,z})\}=\begin{cases}2k & \text{if } k\ge\frac{3b}{4};\\ \lfloor\frac{3b}{2}\rfloor & \text{if } k<\frac{3b}{4} \text{ and } k=0 \pmod 3;\\ \lfloor\frac{3b+1}{2}\rfloor & \text{if } k<\frac{3b}{4} \text{ and } k\neq 0 \pmod 3).\end{cases}$$

Proof. To get $\min\{\kappa(S_{x,y,z})\}$, first let us consider the function $f_1(z,y)=(3b+k-y-2z)/2$. We want to find out the optimal solution of

$$\begin{gathered}
\min\{f_1(z,y)\}\\
\text{subject to}\\
2y+z\le k,\\
z-y\le 0,\\
y\le b-k,\\
y,z \text{ are positive integers.}
\end{gathered}$$

To this end, first let us ignore the integer restriction and consider

$$\begin{gathered}
\min\{g_1(z,y)=f_1(z,y)\}\\
\text{subject to}\\
2y+z\le k\\
z-y\le 0\\
y\le b-k.
\end{gathered}$$

Since $\partial g_1/\partial y=-1/2<0$ and $\partial g_1/\partial z=-1<0$, $g_1(z,y)$ is a decreasing function in y, and it is also a decreasing function in z. Next we will illustrate it in two cases.
Case 1. $b-k\le k/3$.

The feasible region of $g_1(z,y)$ is shown in Figure 4. Obviously, $g_1(z,y)$ attains the minimum value at $(b-k,b-k)$. Since $b-k$ is a positive integer, $(b-k,b-k)$ is also the optimal solution of $f_1(z,y)$ in this case. So $\min\{f_1(z,y)\}=f_1(b-k,b-k)=2k$.

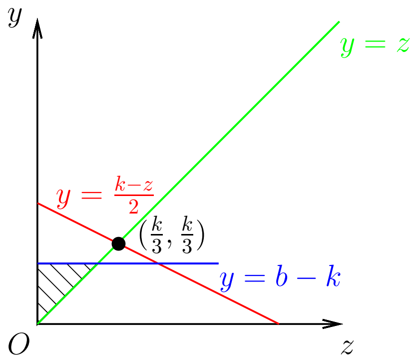

Figure 4. The feasible zone of g_1 for Case 1

Case 2. $b-k>k/3$.

The feasible region of $g_1(z,y)$ is shown in Figure 5. Obviously, $g_1(z,y)$ attains the minimum value at some point on the segment $y=(k-z)/2$, $3k-2b\leq z\leq k/3$. When $y=(k-z)/2$, $g_1(z,y)=(3b+k-y-2z)/2=(6b+k-3z)/4$, which is decreasing in z. So $g_1(z,y)$ attains the minimum value at $(k/3,k/3)$. If $k=0 \pmod 3$, $(k/3,k/3)$ is also the optimal solution of $f_1(z,y)$ in this case and $\min\{f_1(z,y)\}=f_1(k/3,k/3)=3b/2$. If $k=1 \pmod 3$, $f_1(z,y)$ can attain the minimum value only at $((k-1)/3,(k-1)/3)$ or $((k-4)/3,(k+2)/3)$. Since $f_1((k-4)/3,(k+2)/3)-f_1((k-1)/3,(k-1)/3)=1/2>0$, $\min\{f_1(z,y)\}=f_1((k-1)/3,(k-1)/3)=(3b+1)/2$. If $k=2 \pmod 3$, $f_1(z,y)$ can attain the minimum value only at $((k-2)/3,(k+1)/3)$. So $\min\{f_1(z,y)\}=f_1((k-2)/3,(k+1)/3)=(3b+1)/2$.

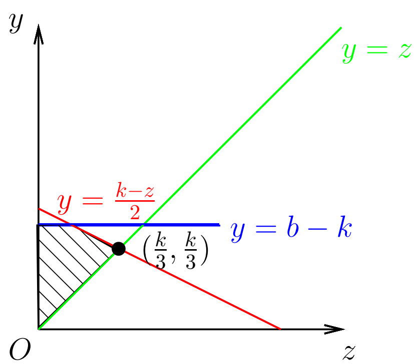

Figure 5. The feasible zone of g_1 for Case 2

Thus, for $k<b$ and $y\leq b-k$, we have

$$\min\{\kappa(S_{x,y,z})\}=\lfloor\min\{f_1(z,y)\}\rfloor=\begin{cases}2k & \text{if } k\geq\frac{3b}{4};\\ \lfloor\frac{3b}{2}\rfloor & \text{if } k<\frac{3b}{4} \text{ and } k=0 \pmod 3;\\ \lfloor\frac{3b+1}{2}\rfloor & \text{if } k<\frac{3b}{4} \text{ and } k\neq 0 \pmod 3.\end{cases} \qquad \blacksquare$$

Similarly, we can get the next result.

Lemma 3.14. *For $k < b$ and $y \geq b-k$, we have*

$$\min\{\kappa(S_{x,y,z})\} = \begin{cases} 3b-2k & \text{if } k \leq \frac{3b}{4}; \\ \lfloor \frac{3b}{2} \rfloor & \text{if } k > \frac{3b}{4} \text{ and } k = 0 \pmod 3; \\ \lfloor \frac{3bk+3b-k+1}{2k+1} \rfloor & \text{if } k > \frac{3b}{4} \text{ and } k = 1 \pmod 3; \\ \lfloor \frac{3b+1}{2} \rfloor & \text{if } k > \frac{3b}{4} \text{ and } k = 2 \pmod 3. \end{cases}$$

Combine Lemma 3.13 and Lemma 3.14, we can get the following result.

Lemma 3.15. *When $k < b$,*

$$\kappa_k(K_b^3) = \begin{cases} \lfloor \frac{3b}{2} \rfloor & \text{if } k = 0 \pmod 3; \\ \lfloor \frac{3bk+3b-k+1}{2k+1} \rfloor & \text{if } k > \frac{3b}{4} \text{ and } k = 1 \pmod 3; \\ \lfloor \frac{3b+1}{2} \rfloor & \text{otherwise.} \end{cases}$$

Finally, we will calculate $\kappa_k(K_b^3)$ for $b \leq k < 3b/2$. First, we will calculate $\kappa(S_{x,y,z})$. Notice that it suffices to calculate $\kappa(S_{x,y,z})$ such that $x(b-x)+y(b-y)+z(b-z) > xy+yz+zx$. Namely, at least one of $x(b-x) > xy$, $y(b-y) > yz$ and $z(b-z) > zx$ must hold. Since $b-x > y$ and $b-z > x$ both imply that $b-z > y$, $b-z > y$ must hold and this will be used later.

Lemma 3.16. *If $b \leq k < 3b/2$, then $\kappa(S_{x,y,z}) = 2b-z-y+\lfloor((k-b)z+y^2)/(k-z)\rfloor$.*

Proof. The way we find trees is similar to that in the case $k < b$. Since $b \leq k = x+y+z$, $b-z \leq x+y$ and we can find $b-z$ trees in $\mathscr{T}^W$ and run out of all vertices in $W \setminus S_{x,y,z}$. Then we find $x-y$ trees in $\mathscr{T}^V$ and there are $b-x$ unused vertices left in both $U \setminus S_{x,y,z}$ and $V \setminus S_{x,y,z}$. Now we have used altogether $(b-z)z+(x-y)y$ edges in $E(G[S_{x,y,z}])$ and left $xy+yz+zx-(b-z)z-(x-y)y = yz+zx+z^2+y^2-bz$ edges unused. So we can find $\lfloor(yz+zx+z^2+y^2-bz)/(x+y)\rfloor$ trees in $\mathscr{T}^U$ and $\lfloor(yz+zx+z^2+y^2-bz)/(x+y)\rfloor$ trees in $\mathscr{T}^V$. Finally, there remain $b-x-\lfloor(yz+zx+z^2+y^2-bz)/(x+y)\rfloor$ pairs of vertices unused, and so we can find $b-x-\lfloor(yz+zx+z^2+y^2-bz)/(x+y)\rfloor$ trees in $\mathscr{T}^{U,V}$. Thus,

$$\begin{aligned} &\kappa(S_{x,y,z}) \\ =&b-z+x-y+2\left\lfloor \frac{yz+zx+z^2+y^2-bz}{x+y} \right\rfloor + b-x-\left\lfloor \frac{yz+zx+z^2+y^2-bz}{x+y} \right\rfloor \\ =&2b-z-y+\left\lfloor \frac{yz+zx+z^2+y^2-bz}{x+y} \right\rfloor \\ =&2b-z-y+\left\lfloor \frac{(k-b)z+y^2}{k-z} \right\rfloor. \end{aligned}$$

∎

Similar to Lemmas 3.10, 3.13, 3.14 and 3.15, we can get the next results.

Lemma 3.17. *For $b \leq k < 3b/2$, we have*

$$\min\{\kappa(S_{x,y,z})\} = \begin{cases} \lfloor \frac{3b}{2} \rfloor & \text{if } k = 0 \pmod 3; \\ \lfloor \frac{3bk+3b-k+1}{2k+1} \rfloor & \text{if } k = 1 \pmod 3; \\ \lfloor \frac{3bk+6b-2k+1}{2k+2} \rfloor & \text{if } k = 2 \pmod 3. \end{cases}$$

Lemma 3.18. *When $b \le k < 3b/2$, we have*

$$\kappa_k(K_b^3) = \begin{cases} \lfloor \frac{3b}{2} \rfloor & \text{if } k = 0 \pmod 3; \\ \lfloor \frac{3bk+3b-k+1}{2k+1} \rfloor & \text{if } k = 1 \pmod 3; \\ \lfloor \frac{3bk+6b-2k+1}{2k+2} \rfloor & \text{if } k = 2 \pmod 3. \end{cases}$$

Now we can give our main result.

Theorem 3.1. *Given any positive integer $b \ge 2$, let K_b^3 denote a complete 3-partite graph in which every part contains exactly b vertices. Then we have*

$$\kappa_k(K_b^3) = \begin{cases} \left\lfloor \frac{\lceil \frac{k^2}{3} \rceil + k^2 - 2kb}{2(k-1)} \right\rfloor + 3b - k & \text{if } k \ge \frac{3b}{2}; \\ \lfloor \frac{3b}{2} \rfloor & \text{if } k < \frac{3b}{2} \text{ and } k = 0 \pmod 3; \\ \lfloor \frac{3bk+3b-k+1}{2k+1} \rfloor & \text{if } \frac{3b}{4} < k < \frac{3b}{2} \text{ and } k = 1 \pmod 3; \\ \lfloor \frac{3bk+6b-2k+1}{2k+2} \rfloor & \text{if } b \le k < \frac{3b}{2} \text{ and } k = 2 \pmod 3; \\ \lfloor \frac{3b+1}{2} \rfloor & \text{otherwise.} \end{cases}$$

Proof. The result follows directly from Lemmas 3.10, 3.15 and 3.18. ∎

Remark 3.1. Note that

$$\left\lfloor \frac{3b}{2} \right\rfloor \le \left\lfloor \frac{3bk+3b-k+1}{2k+1} \right\rfloor \le \left\lfloor \frac{3b+1}{2} \right\rfloor,$$

$$\left\lfloor \frac{3b}{2} \right\rfloor \le \left\lfloor \frac{3bk+6b-2k+1}{2k+2} \right\rfloor \le \left\lfloor \frac{3b+1}{2} \right\rfloor,$$

and

$$\left\lfloor \frac{3b+1}{2} \right\rfloor - \left\lfloor \frac{3b}{2} \right\rfloor \le 1.$$

Also, note that when $k = 3b/2$, $\kappa_k(K_b^3) = \lfloor 3b/2 \rfloor$. So, when $k \le 3b/2$, the k-connectivity of K_b^3 is almost the same. But $\kappa_k(K_b^3)$ is neither increasing nor decreasing on k.

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