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Real Hypersurfaces of Codazzi Type in Complex Two-Plane Grassmannians

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Abstract. We prove the non-existence of Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ whose shape operator is of Codazzi type with respect to the generalized Tanaka-Webster connection.

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1. Introduction

The generalized Tanaka-Webster connection (from now on, g-Tanaka Webster connnection) for contact metric manifolds was introduced by Tanno [16] as a generalization of the connection defined by Tanaka in [15] and, independently, by Webster in [17]. The Tanaka-Webster connection is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. A real hypersurface M in a Kähler manifold has an (integrable) CR-structure associated with the almost contact structure (ϕ,ξ,η,g) induced on M by the Kähler structure, but, in general, this CR-structure is not guaranteed to be pseudo-Hermitian. Cho defined the g-Tanaka-Webster connection for a real hypersurface of a Kähler manifold (see [4–6]) by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX,Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.1}$$

for any X,Y tangent to M, where ∇ denotes the Levi-Civita connection on M, A is the shape operator on M and k is a non-zero real number. In particular, if the real hypersurface satisfies $A\phi+\phi A = 2k\phi$, then the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see [6]). Using the g-Tanaka-Webster connection, some geometers have studied some characterizations of real hypersurfaces in complex space forms (see [6, 12]).

Let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. It is known to be the unique compact irreducible Riemannian symmetric space

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equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J(see Berndt and Suh [2]). In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible Kähler, quaternionic Kähler manifold wich is not a hyper-Kähler manifold.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ and N a local normal unit vector field on M. Let also A be the shape operator of M associated to N. Then we define the structure vector field of M by $\xi = -JN$. Moreover, if $\{J_1, J_2, J_3\}$ is a local basis of $\mathfrak{J}$, we define $\xi_i = -J_iN$, $i = 1,2,3$. We will call $\mathscr{D}^\perp = Span\{\xi_1, \xi_2, \xi_3\}$.

M is called Hopf if ξ is principal, that is, $A\xi = \alpha\xi$. Berndt and Suh, [2] proved that if $m \geq 3$, a real hypersurface M of $G_2(\mathbb{C}^{m+2})$ for which both $[\xi]$ and $\mathscr{D}^\perp$ are A-invariant must be an open part of either (A) a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or (B) a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$. In this second case $m = 2n$.

Bearing in mind this result an interesting topic is to find geometric properties that characterize either type (A) or type (B) among real hypersurfaces in complex two-plane Grassmannians. Several results have been obtained as the characterization of type (A) real hypersurfaces as the unique ones for which the structure tensor ϕ and the shape operator A commute, see [3], that we will use in this paper, or the characterization of type (B) real hypersurfaces given by Lee and Suh in [14].

Jeong, Lee and Suh, [9], considered Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with g-Tanaka-Webster parallel shape operator. That is, A satisfies $(\hat{\nabla}^{(k)}_X A)Y = 0$, for any vector fields X, Y on M. They proved the non-existence of such real hypersurfaces if $m \geq 3$ and $\alpha \neq 2k$. Some weaker conditions were studied in [8, 10, 11, 13].

In this paper we will study a different weaker condition. Given a tensor T of type (1,1) on M we will say that T is of Codazzi type with respect to the g-Tanaka-Webster connection if it satisfies $(\hat{\nabla}^{(k)}_X T)Y = (\hat{\nabla}^{(k)}_Y T)X$, for any X, Y tangent to M. So we will consider real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ whose shape operator A is of Codazzi type with respect to the g-Tanaka-Webster connection. Thus $(\hat{\nabla}^{(k)}_X A)Y = (\hat{\nabla}^{(k)}_Y A)X$ for any X, Y tangent to M.

From the Codazzi equation, (see Section 2), it is very easy to see that A is not of Codazzi type for the Levi-Civita connection on M. We will prove the

Theorem 1.1. *There do not exist connected orientable Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose shape operator is of Codazzi type with respect to the g-Tanaka-Webster connection if $\alpha \neq 2k$.*

Recently, in [7], the authors have obtained a similar result but with a further condition: that the $\mathscr{D}$-component or the $\mathscr{D}^\perp$-component of the structure vector field ξ is A-invariant.

2. Preliminaries

For the study of the Riemannian geometry of $G_2(\mathbb{C}^{m+2})$ see [1]. All the notations we will use from now on are the ones in [2] and [3]. We will suppose that the metric g of $G_2(\mathbb{C}^{m+2})$ is normalized for the maximal sectional curvature of the manifold to be eight. Then the Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3} \{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z)\}\end{aligned}$$

$$+\sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX-g(J_\nu JX,Z)J_\nu JY\}, \tag{2.1}$$

where J_1,J_2,J_3 is any canonical local basis of $\mathfrak{J}$.

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ,ξ,η,g). Furthermore, let J_1,J_2,J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M. Using the above expression for the curvature tensor $\bar{R}$, the Gauss and Codazzi equations are respectively given by

$$\begin{aligned}R(X,Y)Z&=g(Y,Z)X-g(X,Z)Y+g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y-2g(\phi X,Y)\phi Z\\&+\sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X-g(\phi_\nu X,Z)\phi_\nu Y-2g(\phi_\nu X,Y)\phi_\nu Z\}\\&+\sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X-g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\}\\&-\sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X-\eta(X)\eta_\nu(Z)\phi_\nu\phi Y\}\\&-\sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y,Z)-\eta(Y)g(\phi_\nu\phi X,Z)\}\xi_\nu\\&+g(AY,Z)AX-g(AX,Z)AY\end{aligned}$$

and

$$\begin{aligned}(\nabla_X A)Y-(\nabla_Y A)X&=\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\\&+\sum_{\nu=1}^{3}\left\{\eta_\nu(X)\phi_\nu Y-\eta_\nu(Y)\phi_\nu X-2g(\phi_\nu X,Y)\xi_\nu\right\}\\&+\sum_{\nu=1}^{3}\left\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\right\}\\&+\sum_{\nu=1}^{3}\left\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\right\}\xi_\nu,\end{aligned}$$

where R denotes the curvature tensor of M in $G_2(\mathbb{C}^{m+2})$.

A real hypersurface of type (A) has three (if $r=\pi/2\sqrt{8}$) or four (otherwise) distinct principal curvatures $\alpha=\sqrt{8}cot(\sqrt{8}r)$, $\beta=\sqrt{2}cot(\sqrt{2}r)$, $\lambda=-\sqrt{2}tan(\sqrt{2}r)$, $\mu=0$, for some radius $r\in(0,\pi/\sqrt{8})$, with corresponding multiplicities $m(\alpha)=1$, $m(\beta)=2$, $m(\lambda)=m(\mu)=2m-2$. The corresponding eigenspaces can be seen in [2].

A real hypersurface of type (B) has five distinct principal curvatures $\alpha=-2tan(2r)$, $\beta=2cot(2r)$, $\gamma=0$, $\lambda=cot(r)$, $\mu=-tan(r)$, for some $r\in(0,\pi/4)$, with corresponding multiplicities $m(\alpha)=1$, $m(\beta)=3=m(\gamma)$, $m(\lambda)=4m-4=m(\mu)$. For the corresponding eigenspaces see [2].

3. Proof of the theorem

If we develop $(\hat{\nabla}_X^{(k)}A)Y = (\hat{\nabla}_Y^{(k)}A)X$, for X,Y tangent to M we get

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = {} & g(\phi AY,AX)\xi - g(\phi AX,AY)\xi - \eta(AX)\phi AY + \eta(AY)\phi AX \\ & - k\eta(Y)\phi AX + k\eta(X)\phi AY - g(\phi AY,X)A\xi + g(\phi AX,Y)A\xi \\ & + \eta(X)A\phi AY - \eta(Y)A\phi AX + k\eta(Y)A\phi X - k\eta(X)A\phi Y.\end{aligned} \tag{3.1}$$

Using the Codazzi equation, (3.1) becomes

$$\begin{aligned}&\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\} \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu \\ &= -2g(A\phi AX,Y)\xi - \eta(AX)\phi AY + \eta(AY)\phi AX - k\eta(Y)\phi AX + k\eta(X)\phi AY \\ &\quad + g((\phi A + A\phi)X,Y)A\xi + \eta(X)A\phi AY - \eta(Y)A\phi AX + k\eta(Y)A\phi X - k\eta(X)A\phi Y.\end{aligned} \tag{3.2}$$

As we suppose M is Hopf, we write $A\xi = \alpha\xi$ for a certain function α. We also can write $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$, where $X_0 \in \mathscr{D}$ is unit. Suppose $\eta(X_0)\eta(\xi_1) \neq 0$. Taking $X = \xi$ in (3.2) we obtain

$$\phi Y + \eta_1(\xi)\phi_1 Y - \sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu\xi + 3\sum_{\nu=1}^{3}\eta_\nu(\phi Y)\xi_\nu = -\alpha\phi AY + k\phi AY + A\phi AY - kA\phi Y \tag{3.3}$$

for any Y tangent to M. The scalar product of (3.3) and ξ yields $4\eta_1(\xi)g(\phi\xi_1,Y) = 0$ for any Y tangent to M. This gives $\eta_1(\xi)\phi\xi_1 = \eta_1(\xi)\phi_1\xi = \eta_1(\xi)\eta(X_0)\phi_1 X_0 = 0$. As $\phi_1 X_0 \neq 0$, we have a contradiction.

Thus $\eta(X_0)\eta(\xi_1) = 0$. That means that either $\xi \in \mathscr{D}$ or $\xi \in \mathscr{D}^\perp$. If $\xi \in \mathscr{D}$, by [14], M is locally congruent to a real hypersurface of type (B).

Therefore we will study the case $\xi \in \mathscr{D}^\perp$. We can write $\xi = \xi_1$. Taking $X \in \mathscr{D}$, $Y = \xi$ in (3.2) we have

$$-\phi X - \phi_1 X = (\alpha - k)\phi AX + kA\phi X - A\phi AX \tag{3.4}$$

for any $X \in \mathscr{D}$. Continue taking $X \in \mathscr{D}$ in (3.2). Its scalar product with ξ yields

$$-2g(\phi X,Y) - 2g(\phi_1 X,Y) = -2g(A\phi AX,Y) + \alpha g((\phi A + A\phi)X,Y) \tag{3.5}$$

for any Y tangent to M. Thus $-2\phi X - 2\phi_1 X = -2A\phi AX + \alpha(\phi A + A\phi)X$ for any $X \in \mathscr{D}$. This and (3.4) imply

$$(\alpha - 2k)A\phi X = (\alpha - 2k)\phi AX \tag{3.6}$$

for any $X \in \mathscr{D}$. As we suppose $\alpha \neq 2k$, we obtain

$$A\phi X = \phi AX \tag{3.7}$$

for any $X \in \mathscr{D}$. Take $X = \xi_2$ in (3.2). This gives

$$
\begin{aligned}
&-\sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu\xi_2 - 2\sum_{\nu=1}^{3}g(\phi_\nu\xi_2,Y)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\phi\xi_2)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi\xi_2)\}\\
&\quad -\sum_{\nu=1}^{3}\eta(Y)\eta_\nu(\phi\xi_2)\xi_\nu + \eta(Y)\xi_3 + 2\eta_3(Y)\xi + \phi_2 Y\\
&= -2g(A\phi A\xi_2,Y)\xi + \alpha\eta(Y)\phi A\xi_2 - k\eta(Y)\phi A\xi_2\\
&\quad + \alpha g((\phi A + A\phi)\xi_2,Y)\xi - \eta(Y)A\phi A\xi_2 + k\eta(Y)A\phi\xi_2
\end{aligned} \tag{3.8}
$$

for any Y tangent to M. Taking $Y = \xi$ and its scalar product with ξ_2 we get $(\alpha - 2k)g(A\xi_2, \xi_3) = 0$. As $\alpha \neq 2k$ we obtain

$$g(A\xi_2,\xi_3) = 0. \tag{3.9}$$

And taking its scalar product with ξ_3 we have

$$2 = (k-\alpha)g(A\xi_2,\xi_2) - kg(A\xi_3,\xi_3) - g(A\phi A\xi_2,\xi_3). \tag{3.10}$$

A similar reasoning taking $X = \xi_3$, $Y = \xi$ and the scalar product with ξ_2 gives

$$-2 = -(k-\alpha)g(A\xi_3,\xi_3) + kg(A\xi_2,\xi_2) + g(A\phi A\xi_2,\xi_3). \tag{3.11}$$

From (3.10) and (3.11) we obtain $0 = (2k-\alpha)g(A\xi_2,\xi_2) - (2k-\alpha)g(A\xi_3,\xi_3)$. As $\alpha \neq 2k$ we arrive to

$$g(A\xi_2,\xi_2) = g(A\xi_3,\xi_3). \tag{3.12}$$

From (3.9) and (3.12) we can write $A\xi_2 = \beta\xi_2 + \gamma X_2$, $A\xi_3 = \beta\xi_3 + \mu X_3$, for some functions β, γ, μ on M, X_2 and X_3 being unit vector fields in $\mathscr{D}$.

If $\gamma = \mu = 0$, from (3.7) we should have $A\phi = \phi A$. Thus M would be locally congruent to a type (A) real hypersurface (see [3]).

Suppose that, at least, one among γ and μ does not vanish. From the above expressions we have $\gamma X_2 = A\xi_2 - \beta\xi_2$. Thus $\gamma A\phi X_2 = A\phi A\xi_2 + \beta^2\xi_3 + \beta\mu X_3$. On the other hand $\gamma\phi A X_2 = \phi A^2\xi_2 + \beta^2\xi_3 - \gamma\beta\phi X_2$. Therefore $A\phi A\xi_2 + \beta\mu X_3 = \phi A^2\xi_2 - \gamma\beta\phi X_2$. Its scalar product with ξ_3 yields $g(A\phi A\xi_2,\xi_3) = -g(A^2\xi_2,\phi\xi_3) = -g(A^2\xi_2,\phi_3\xi_1) = -g(A^2\xi_2,\xi_2) = -(\beta^2+\gamma^2)$. But $g(A\phi A\xi_2,\xi_3) = g(\phi A\xi_2,A\xi_3) = g(\beta\phi\xi_2 + \gamma\phi X_2, \beta\xi_3 + \mu X_3) = -\beta^2 + \gamma\mu g(\phi X_2,X_3)$. This gives

$$\gamma\mu g(\phi X_2,X_3) = -\gamma^2. \tag{3.13}$$

Observe that if $\mu = 0$, $\gamma = 0$. So we have $\mu \neq 0$.

As $\mu X_3 = A\xi_3 - \beta\xi_3$, we can make a similar reasoning and obtain

$$\mu\gamma g(\phi X_3,X_2) = \mu^2. \tag{3.14}$$

Now, if $\gamma = 0$, $\mu = 0$. So we obtain $\gamma \neq 0$. From (3.13) and (3.14) $g(\phi X_2,X_3) = -\mu/\gamma = -\gamma/\mu$. Thus $\mu^2 = \gamma^2$ and $g(\phi X_2,X_3) = \pm 1$. Let us see that this means $\phi X_2 = \pm X_3$. Consider an orthonormal basis of $\mathscr{D}$ given by $\{X_3,Y_i\}_{i=2,\dots,4m-4}$. We can write $\phi X_2 = g(\phi X_2,X_3)X_3 + \sum_{i=2}^{4m-4} g(\phi X_2,Y_i)Y_i$. As X_2 is unit we have $1 = g(\phi X_2,X_3)^2 + \sum_{i=2}^{4m-4} g(\phi X_2,Y_i)^2$. As $g(\phi X_2, X_3)^2 = 1$, this implies $g(\phi X_2,Y_i) = 0$, $i = 2,\dots,4m-4$ and $\phi X_2 = \pm X_3$. Recall that $A\xi_2 = \beta\xi_2 + \gamma X_2$, $A\xi_3 = \beta\xi_3 + \mu X_3$ with $\gamma^2 = \mu^2 = 1$. Now $A\phi\xi_2 = A\phi_2\xi_1 = -A\xi_3 = -\beta\xi_3 - \mu X_3$ and $\phi A\xi_2 = \beta\phi\xi_2 + \gamma\phi X_2 = \beta\phi_2\xi_1 + \gamma\phi X_2 = -\beta\xi_3 + \gamma\phi X_2$. Similarly $A\phi\xi_3 = A\phi_3\xi_1 = A\xi_2 = \beta\xi_2 + \gamma X_2$ and $\phi A\xi_3 = \beta\phi\xi_3 + \mu\phi X_3 = \beta\phi_3\xi_1 + \mu\phi X_3 = \beta\xi_2 + \mu\phi X_3$. Thus $A\phi\xi_2 =$

$\phi A\xi_2$ if and only if $-\mu X_3 = \gamma\phi X_2$ and $A\phi\xi_3 = \phi A\xi_3$ if and only if $\gamma X_2 = \mu\phi X_3$. As $\phi X_2 = \pm X_3$ and $\gamma = \pm\mu$, we have that $A\phi = \phi A$ on M and M is locally congruent to a type (A) real hypersurface.

If M is of type (A), taking $X = \xi_2$, $Y = \xi_3$ in (3.1) we should obtain $1 = 2$. Thus type (A) real hypersurfaces do not satisfy our condition.

If M is of type (B), taking $X = \xi_2$, $Y \in T_\lambda$, in order to (3.2) to be satisfied $\phi_2 Y = 0$. This is impossible and finishes our proof.

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