

## Para-Contact Para-Complex Semi-Riemannian Submersions

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**Abstract.** We introduce the concept of para-contact para-complex semi-Riemannian submersions from an almost para-contact metric manifold onto an almost para-Hermitian manifold. We provide an example and show that the vertical and horizontal distributions of such submersions are invariant with respect to the almost para-contact structure of the total manifold. Moreover, we investigate various properties of the O’Neill’s tensors of such submersions and find the integrability of the horizontal distribution. We also obtain curvature relations between the base manifold and the total manifold. The paper is also focused on the transference of structures defined on the total manifold.

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### 1. Introduction

The theory of Riemannian submersion was introduced by O’Neill and Gray in [16] and [9], respectively. Later, Riemannian submersions were considered between almost complex manifolds by Watson in [19] under the name of almost Hermitian submersion. He showed that if the total manifold is a Kähler manifold, the base manifold is also a Kähler manifold. Since then, Riemannian submersions have been used as an effective tool to describe the structure of a Riemannian manifold equipped with a differentiable structure. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. For instances, Riemannian submersions between almost contact manifolds were studied by Chinea in [4] under the name of almost contact submersions. Riemannian submersions have been also considered for quaternionic Kähler manifolds [12] and para-quaternionic Kähler manifolds [2, 11].

In [14] Kaneyuki and Williams defined the almost para-contact structure on a pseudo-Riemannian manifold  $M$  of dimension  $(2m+1)$  and constructed the almost para-complex structure on  $M^{2m+1} \times \mathbb{R}$ . On the other hand, para-complex manifolds, almost para-Hermitian manifolds and para-Kähler manifolds were defined by Libermann [15] in 1952. In fact,

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such manifolds were arose in [18]. Indeed, Rashevskij introduced the properties of para-Kähler manifolds when he considered a metric of signature  $(n, n)$  defined from a potential function the so-called scalar field on a  $2n$ -dimensional locally product manifold called by him stratified space.

Semi-Riemannian submersions were introduced by O'Neill in his book [17]. It is known that such submersions have their applications in Klauza-Klein theories, Yang-Mills equations, strings, supergravity. For applications of semi-Riemannian submersions, see [7]. Since para-Hermitian manifolds and para-contact manifolds are semi-Riemannian manifolds, one should consider semi-Riemannian submersions between such manifolds. In this paper, we define para-contact para-complex semi-Riemannian submersions between almost para-contact metric manifold and almost para-Hermitian manifold, and study the geometry of such submersions. We observe that para-contact para-complex semi-Riemannian submersion has also rich geometric properties.

The paper is organized as follows. In Section 2 we collect basic definitions, some formulas and results for later use. In Section 3 we introduce the notion of para-contact para-complex semi-Riemannian submersions and give an example of para-contact para-complex semi-Riemannian submersion. Moreover, we investigate properties of O'Neill's tensors and show that such tensors have nice algebraic properties for para-contact para-complex semi-Riemannian submersions. Then we find the integrability of the horizontal distribution. In Section 4, we obtain relations between bisectonal curvatures and sectional curvatures of the base manifold, the total manifold and the fibres of a para-contact para-complex semi-Riemannian submersion.

## 2. Preliminaries

In this section we are going to recall main definitions and properties of almost para-contact metric manifolds, almost para-Hermitian manifolds and semi-Riemannian submersions.

### 2.1. Almost para-contact metric manifolds

In this subsection we recall basic definitions and properties of almost para-contact manifolds.

Let  $M$  be a  $(2m+1)$ -dimensional differentiable manifold. Let  $\varphi$  be a  $(1, 1)$ -tensor field,  $\xi$  a vector field and  $\eta$  a 1-form on  $M$ . Then  $(\varphi, \xi, \eta)$  is called an almost para-contact structure on  $M$  if

- (i)  $\eta(\xi) = 1, \varphi^2 = Id - \eta \otimes \xi,$
- (ii) the tensor field  $\varphi$  induces an almost para-complex structure on the distribution  $\mathcal{D} = \ker \eta$ , that is, the eigendistributions  $\mathcal{D}^+, \mathcal{D}^-$  corresponding to the eigenvalues  $1, -1$  of  $\varphi$ , respectively, have equal dimension  $m$ .  $M$  is said to be almost para-contact manifold if it is endowed with an almost para-contact structure [14, 3, 21].

Let  $M$  be an almost para-contact manifold.  $M$  is called an almost para-contact metric manifold if it is additionally endowed with a pseudo-Riemannian metric  $g$  of signature  $(m+1, m)$  such that

$$(2.1) \quad g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad X, Y \in \chi(M).$$

For such a manifold, we additionally have  $\eta(X) = g(X, \xi), \varphi \xi = 0, \eta \circ \varphi = 0$ . Moreover, we can define a skew-symmetric 2-form  $\Phi$  by  $\Phi(X, Y) = g(X, \varphi Y)$ , which is called the

fundamental form corresponding to the structure. Note that  $\eta \wedge \Phi$  is, up to a constant factor, the Riemannian volume element of  $M$ . On an almost para-contact manifold, one defines the  $(1,2)$ -tensor field  $N^{(1)}$  by

$$(2.2) \quad N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) - 2d\eta(X, Y)\xi,$$

where  $[\varphi, \varphi]$  is the Nijenhuis torsion of  $\varphi$  given by

$$(2.3) \quad [\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

If  $N^{(1)}$  vanishes identically, then the almost para-contact manifold (structure) is said to be normal [21]. The normality condition says that the almost para-complex structure  $J$  defined on  $M \times R$  by

$$(2.4) \quad J\left(X, f \frac{d}{dt}\right) = \left(\varphi X + f\xi, \eta(X) \frac{d}{dt}\right)$$

is integrable.

We note that an almost para-contact metric manifold  $(M, g, \varphi, \xi, \eta)$  is called

- (a) *normal*, if  $N_\varphi - 2d\eta \otimes \xi = 0$ , where  $N_\varphi$  is the Nijenhuis tensor of  $\varphi$ ;
- (b) *para-contact*, if  $\Phi = d\eta$ ;
- (c) *K-para-contact*, if  $M$  is para-contact and  $\xi$  Killing;
- (d) *para-cosymplectic*, if  $\nabla\Phi = 0$  which implies  $\nabla\eta = 0$ , where  $\nabla$  is the Levi-Civita connection on  $M$ ;
- (f) *almost para-cosymplectic*, if  $d\eta = 0$  and  $d\Phi = 0$ ;
- (g) *weakly para-cosymplectic*, if  $M$  is almost para-cosymplectic and  $[R(X, Y), \varphi] = R(X, Y)\varphi - \varphi R(X, Y) = 0$ ;
- (h) *para-Sasakian*, if  $\Phi = d\eta$  and  $M$  is normal;
- (j) *quasi-para-Sasakian*, if  $d\Phi = 0$  and  $M$  is normal [6, 20, 21].

It is known that an almost para-contact manifold is a para-Sasakian manifold if and only if  $(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X$ , for  $X, Y \in \Gamma(TM)$ .

**Lemma 2.1.** [21] *Let  $(M, \varphi, \xi, \eta, g)$  be an almost para-contact metric manifold. Then we have*

$$(2.5) \quad 2g((\nabla_X \varphi)Y, Z) = -d\Phi(X, Y, Z) - d\Phi(X, \varphi Y, \varphi Z) - N^{(1)}(Y, Z, \varphi X) + N^{(2)}(Y, Z)\eta(X) - 2d\eta(\varphi Z, X)\eta(Y) + 2d\eta(\varphi Y, X)\eta(Z),$$

where  $\Phi$  is the fundamental 2-form and

$$(2.6) \quad N^{(2)}(X, Y) = (\mathcal{L}_{\varphi X} \eta)Y - (\mathcal{L}_{\varphi Y} \eta)X,$$

where  $\mathcal{L}$  is the Lie derivative.

Moreover if  $M$  is para-contact, then we have

$$(2.7) \quad 2g((\nabla_X \varphi)Y, Z) = -N^{(1)}(Y, Z, \varphi X) - 2d\eta(\varphi Z, X)\eta(Y) + 2d\eta(\varphi Y, X)\eta(Z).$$

For an almost para-contact metric manifold, the following identities are well known:

$$(2.8) \quad (\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi(\nabla_X Y),$$

$$(2.9) \quad (\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z),$$

$$(2.10) \quad (\nabla_X \eta)Y = g(Y, \nabla_X \xi).$$

## 2.2. Almost para-Hermitian manifolds

In this subsection we recall basic definitions and properties of almost para-complex manifolds.

A  $(1, 1)$ -tensor field  $J$  on an  $2n$ -dimensional smooth manifold  $M$  is said to be an almost product structure if  $J^2 = I$ . In this case the pair  $(M, J)$  is called almost product manifold. An almost para-complex manifold is an almost product manifold  $(M, J)$  such that the two eigenbundles  $T^+M$  and  $T^-M$  associated with the two eigenvalues  $\pm 1$  of  $J$  have the same rank. An almost para-Hermitian manifold  $(M, g, J)$  is a smooth manifold endowed with an almost para-complex structure  $J$  and a pseudo-Riemannian metric  $g$  compatible in the sense that

$$(2.11) \quad g(JX, Y) + g(X, JY) = 0, \quad X, Y \in \chi(M).$$

It follows that the metric  $g$  is neutral, i.e., it has signature  $(n, n)$  and the eigenbundles  $TM^\pm$  are totally isotropic with respect to  $g$ . Let  $e_1, \dots, e_n, e_{n+1}, \dots, e_{2n} = Je_1, \dots, Je_n$  be an orthonormal basis and denote  $\varepsilon_i = \text{sign}(g(e_i, e_i)) = \pm 1, \varepsilon_i = 1, i = 1, \dots, n, \varepsilon_j = -1, j = n+1, \dots, 2n$ . The fundamental 2-form of the almost para-Hermitian manifold is defined by

$$(2.12) \quad \Phi(X, Y) = g(X, JY),$$

it is easy to see that  $\Phi$  is skew-symmetric [5, 13].

An almost para-Hermitian manifold is called

- (i) *para-Kähler*, if  $\nabla J = 0$ ,
- (ii) *para-Hermitian*, if  $N = 0 \Leftrightarrow (\nabla_{JX}J)JY + (\nabla_XJ)Y = 0$ , where  $N$  is the Nijenhuis tensor of  $J$ ,
- (iii) *nearly para-Kähler*, if  $(\nabla_XJ)X = 0$ ,
- (iv) *almost para-Kähler*, if  $d\Phi = 0$  [5, 10, 13].

## 2.3. Semi-Riemannian submersions

In this subsection, We recall basic definitions and properties of semi-Riemannian submersions.

Let  $(M, g)$  and  $(B, g')$  be two connected semi-Riemannian manifolds of index  $s (0 \leq s \leq \dim M)$  and  $s' (0 \leq s' \leq \dim B)$  respectively, with  $s > s'$ . Roughly speaking, a semi-Riemannian submersion is a smooth map  $\pi : M \rightarrow B$  which is onto and satisfies the following conditions:

- (i)  $\pi_{*p} : T_pM \rightarrow T_{\pi(p)}B$  is onto for all  $p \in M$ ;
- (ii) The fibres  $\pi^{-1}(p'), p' \in B$ , are semi-Riemannian submanifolds of  $M$ ;
- (iii)  $\pi_*$  preserves scalar products of vectors normal to fibres.

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by  $\mathcal{V}$  the vertical distribution, by  $\mathcal{H}$  the horizontal distribution and by  $v$  and  $h$  the vertical and horizontal projection. A horizontal vector field  $X$  on  $M$  is said to be basic if  $X$  is  $\pi$ -related to a vector field  $X'$  on  $B$ . It is clear that every vector field  $X'$  on  $B$  has a unique horizontal lift  $X$  to  $M$  and  $X$  is basic.

We recall that the sections of  $\mathcal{V}$ , respectively  $\mathcal{H}$ , are called the vertical vector fields, respectively horizontal vector fields. A semi-Riemannian submersion  $\pi : M \rightarrow B$  determines two  $(1, 2)$  tensor field  $T$  and  $A$  on  $M$ , by the formulas:

$$(2.13) \quad T(E, F) = T_E F = h\nabla_{vE}vF + v\nabla_{vE}hF$$

and

$$(2.14) \quad A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF$$

for any  $E, F \in \Gamma(TM)$ , where  $v$  and  $h$  are the vertical and horizontal projections (see [1, 8]). From (2.13) and (2.14), one can obtain

$$(2.15) \quad \nabla_U X = T_U X + h(\nabla_U X);$$

$$(2.16) \quad \nabla_X U = v(\nabla_X U) + A_X U;$$

$$(2.17) \quad \nabla_X Y = A_X Y + h(\nabla_X Y),$$

for any  $X, Y \in \Gamma(\mathcal{H})$ ,  $U \in \Gamma(\mathcal{V})$ . Moreover, if  $X$  is basic then  $h(\nabla_U X) = h(\nabla_X U) = A_X U$ .

We note that for  $U, V \in \Gamma(\mathcal{V})$ ,  $T_U V$  coincides with the second fundamental form of the immersion of the fibre submanifolds and for  $X, Y \in \Gamma(\mathcal{H})$ ,  $A_X Y = \frac{1}{2}v[X, Y]$  reflecting the complete integrability of the horizontal distribution  $\mathcal{H}$ . It is known that  $A$  is alternating on the horizontal distribution:  $A_X Y = -A_Y X$ , for  $X, Y \in \Gamma(\mathcal{H})$  and  $T$  is symmetric on the vertical distribution:  $T_U V = T_V U$ , for  $U, V \in \Gamma(\mathcal{V})$ .

We now recall the following result which will be useful for later.

**Lemma 2.2.** (see [8, 17]) *If  $\pi : M \rightarrow B$  is a semi-Riemannian submersion and  $X, Y$  basic vector fields on  $M$ ,  $\pi$ -related to  $X'$  and  $Y'$  on  $B$ , then we have the following properties*

- (1)  $h[X, Y]$  is a basic vector field and  $\pi_* h[X, Y] = [X', Y'] \circ \pi$ ;
- (2)  $h(\nabla_X Y)$  is a basic vector field  $\pi$ -related to  $(\nabla'_{X'} Y')$ , where  $\nabla$  and  $\nabla'$  are the Levi-Civita connection on  $M$  and  $B$ ;
- (3)  $[E, U] \in \Gamma(\mathcal{V})$ , for any  $U \in \Gamma(\mathcal{V})$  and for any basic vector field  $E$ .

### 3. Para-contact para-complex semi-riemannian submersions

In this section, we define the notion of para-contact para-complex semi-Riemannian submersion, give an example and study the geometry of such submersions. We now define a  $(\varphi, J)$ -para-holomorphic map which is similar to the notion of a  $(\varphi, J)$ -holomorphic map between almost contact metric manifold and almost Hermitian manifold, for  $(\varphi, J)$ -holomorphic map see [8].

**Definition 3.1.** *Let  $(M^{2m+1}, \varphi, \xi, \eta)$  be an almost para-contact manifold and  $(B^{2n}, J)$  an almost para-complex manifold, respectively. Then we say that the map  $\pi : M \rightarrow B$  is  $(\varphi, J)$ -para-holomorphic if  $\pi_* \circ \varphi = J \circ \pi_*$ .*

By using the above definition, we are ready to give the following notion.

**Definition 3.2.** *Let  $(M, \varphi, \xi, \eta, g)$  be an almost para-contact metric manifold and  $(B, J, g')$  be an almost para-Hermitian manifold. A semi-Riemannian submersion  $\pi : M \rightarrow B$  is called para-contact para-complex semi-Riemannian submersion if it is  $(\varphi, J)$ -para-holomorphic, as well.*

We now give an example for para-contact para-complex submersion. But we first recall that the para-contact structure on  $R^{2m+1}$  is given by

$$\varphi = \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \eta = dz, \quad \xi = \frac{\partial}{\partial z}.$$

Also the semi-Riemannian metric compatible with  $\varphi$  is

$$g = \sum_{i=1}^m (-dx_i \otimes dx_i + dy_i \otimes dy_i) + \eta \otimes \eta.$$

On the other hand, the canonical para-complex structure on  $R^{2n}$  is given by

$$J(x_1, \dots, x_{2n}) = (x_{2n}, x_{2n-1}, \dots, x_2, x_1),$$

where scalar product or semi-Riemannian metric is standard inner product defined on  $R^{2n}$ .

**Example 3.1.** Consider the following submersion defined by

$$\begin{aligned} \pi : R_2^5 &\rightarrow R_1^2 \\ (x_1, x_2, y_1, y_2, z) &\rightarrow \left( \frac{x_1 + x_2}{\sqrt{2}}, \frac{y_1 + y_2}{\sqrt{2}} \right). \end{aligned}$$

Then, the kernel of  $\pi_*$  is

$$\mathcal{V} = Ker\pi_* = Span \left\{ V_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, V_2 = -\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \xi = \frac{\partial}{\partial z} \right\}$$

and the horizontal distribution is spanned by

$$\mathcal{H} = (Ker\pi_*)^\perp = Span \left\{ X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Y = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \right\}.$$

Hence, we have

$$g(X, X) = g'(\pi_*X, \pi_*X) = -2, \quad g(Y, Y) = g'(\pi_*Y, \pi_*Y) = 2.$$

Thus,  $\pi$  is a semi-Riemannian submersion. Moreover, we can easily obtain that  $\pi$  satisfies

$$\pi_*\varphi X = J\pi_*X, \quad \pi_*\varphi Y = J\pi_*Y.$$

Thus,  $\pi$  is a para-contact para-complex semi-Riemannian submersion.

By using Definition 3.1, we have the following result.

**Proposition 3.1.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from an almost para-contact metric manifold  $M$  onto an almost para-Hermitian manifold  $B$ , and let  $X$  be a basic vector field on  $M$ ,  $\pi$ -related to  $X'$  on  $B$ . Then,  $\varphi X$  is also a basic vector field  $\pi$ -related to  $JX'$ .*

The following result can be proved in a standard way.

**Proposition 3.2.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from an almost para-contact metric manifold  $M$  onto an almost para-Hermitian manifold  $B$ . If  $X, Y$  are basic vector fields on  $M$ ,  $\pi$ -related to  $X', Y'$  on  $B$ , Then, we have*

- (i)  $h(\nabla_X \varphi)Y$  is the basic vector field  $\pi$ -related to  $(\nabla_{X'}^J)Y'$ ;
- (ii)  $h[X, Y]$  is the basic vector field  $\pi$ -related to  $[X', Y']$ .

Next proposition shows that a para-contact para-complex semi-Riemannian submersion puts some restrictions on the distributions  $\mathcal{V}$  and  $\mathcal{H}$ .

**Proposition 3.3.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from an almost para-contact metric manifold  $M$  onto an almost para-Hermitian manifold  $B$ . Then, the horizontal and vertical distributions are  $\varphi$ -invariant.*

*Proof.* Consider a vertical vector field  $U$ ; it is known that  $\pi_*(\varphi U) = J(\pi_*U)$ . Since  $U$  is vertical and  $\pi$  is a semi-Riemannian submersion, we have  $\pi_*U = 0$  from which  $\pi_*(\varphi U) = 0$  follows and implies that  $\varphi U$  is vertical, being in the kernel of  $\pi_*$ . As concerns the horizontal distribution, let  $X$  be a horizontal vector field. We have  $g(\varphi X, U) = -g(X, \varphi U) = 0$  because  $\varphi U$  is vertical and  $X$  is horizontal. From  $g(\varphi X, U) = 0$  we deduce that  $\varphi X$  is orthogonal to  $U$  and then  $\varphi X$  is horizontal. ■

**Proposition 3.4.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from an almost para-contact metric manifold  $M$  onto an almost para-Hermitian manifold  $B$ . Then, we have*

- (i)  $\pi^*\Phi' = \Phi$  holds on the horizontal distribution, only;
- (ii)  $\xi$  is a vertical vector field;
- (iii)  $\eta(X) = 0$ , for all horizontal vector fields  $X$ ;
- (iv) The fibres are invariant almost para-contact metric manifolds.

*Proof.* We prove only statement (i), the other assertions can be obtained in a similar way. If  $X$  and  $Y$  are basic vector fields on  $M$ ,  $\pi$ -related to  $X', Y'$  on  $B$ , then using the definition of a para-contact para-complex semi-Riemannian submersion, we have

$$\begin{aligned} \pi^*\Phi'(X, Y) &= \Phi'(\pi_*X, \pi_*Y) = g'(\pi_*X, J\pi_*Y) = g'(\pi_*X, \pi_*\varphi Y) \\ &= \pi^*g'(X, \varphi Y) = g(X, \varphi Y) = \Phi(X, Y) \end{aligned}$$

which gives the proof of assertion (i). ■

In the sequel, we show that base space is a para-Hermitian manifold if the total space is a normal.

**Proposition 3.5.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion. If the almost para-contact structure of  $M$  is normal, then the base space is a para-Hermitian manifold.*

*Proof.* Let  $X$  and  $Y$  be basic vector fields on  $M$ ,  $\pi$ -related to  $X'$  and  $Y'$  on  $B$ . From (2.2), we have

$$\pi_*(N^{(1)}(X, Y)) = \pi_*([\varphi, \varphi](X, Y) - 2d\eta(X, Y)\xi) = \pi_*([\varphi, \varphi](X, Y)).$$

On the other hand,  $\pi_*\varphi = J\pi_*$  implies that

$$\begin{aligned} \pi_*([\varphi, \varphi](X, Y)) &= \pi_*\varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] \\ &= [\pi_*X, \pi_*Y] - \eta[X, Y]\pi_*\xi + [\pi_*\varphi X, \pi_*\varphi Y] - J\pi_*[\varphi X, Y] - J\pi_*[X, \varphi Y] \\ &= [X', Y'] + [JX', JY'] - J[JX', Y'] - J[X', JY']. \end{aligned}$$

Then, we have

$$\pi_*[\varphi, \varphi](X, Y) = N'(X', Y') = 0$$

which shows that  $B$  is a para-Hermitian manifold. ■

We now investigate what kind of para-complex structure is defined on the base manifold, when the total manifold has some special para-contact structures.

**Corollary 3.1.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion. If the total space  $M$  is an almost para-contact metric manifold with  $(\nabla_X\varphi)Y = 0$ , for  $X, Y \in \Gamma(\mathcal{H})$ , then the base space  $B$  is a para-Kähler manifold.*

*Proof.* Let  $X, Y$  and  $Z$  be basic vector fields on  $M$ ,  $\pi$ -related to  $X', Y'$  and  $Z'$  on  $B$ . Since  $(\nabla_X \varphi)Y = 0$  for  $X, Y \in \Gamma(\mathcal{H})$ , we get

$$0 = g(Z, \nabla_X \varphi Y - \varphi \nabla_X Y)$$

for  $Z \in \Gamma(\mathcal{H})$ . Using (2.17) we obtain

$$0 = g(Z, h\nabla_X \varphi Y) - g(Z, h\varphi \nabla_X Y)$$

Then, by using  $\pi_* \varphi = J\pi_*$ , we get

$$0 = g'(Z', \nabla'_{X'} JY) - g'(Z', J\nabla'_{X'} Y').$$

Hence  $0 = g'(Z', (\nabla'_{X'} J)Y')$  which shows that  $B$  is para-Kähler manifold.  $\blacksquare$

Since for a para-cosymplectic manifold (respectively, almost para-cosymplectic manifold)  $\nabla \phi = 0$ , (resp.  $d\phi = 0$ ) we have the following result.

**Corollary 3.2.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion. If the total space  $M$  is an almost para-cosymplectic manifold, then the base space  $B$  is an almost para-Kähler manifold.*

*Proof.* Let  $X, Y$  and  $Z$  be basic vector fields on  $M$ ,  $\pi$ -related to  $X', Y'$  and  $Z'$  on  $B$ . Since  $M$  is an almost para-cosymplectic manifold, we have  $d\Phi(X, Y, Z) = 0$ , so that we have

$$X(\Phi(Y, Z)) - Y(\Phi(X, Z)) + Z(\Phi(X, Y)) - \Phi([X, Y], Z) + \Phi([X, Z], Y) - \Phi([Y, Z], X) = 0.$$

On the other hand, by direct calculations, we obtain

$$\begin{aligned} 0 &= g(\nabla_X Y, \varphi Z) + g(Y, \nabla_X \varphi Z) - g(\nabla_Y X, \varphi Z) - g(X, \nabla_Y \varphi Z) + g(\nabla_Z X, \varphi Y) \\ &\quad + g(X, \nabla_Z \varphi Y) - g([X, Y], \varphi Z) + g([X, Z], \varphi Y) - g([Y, Z], \varphi X). \end{aligned}$$

Then, by using  $\pi_* \varphi = J\pi_*$ , we get

$$\begin{aligned} 0 &= g'(\nabla'_{X'} Y', JZ') + g'(Y', \nabla'_{X'} JZ') - g'(\nabla'_{Y'} X', JZ') - g'(X', \nabla'_{Y'} JZ') + g'(\nabla'_{Z'} X', JY') \\ &\quad + g'(X', \nabla'_{Z'} JY') - g'([X', Y'], JZ') + g'([X', Z'], JY') - g'([Y', Z'], JX') \\ 0 &= X'(\Phi'(Y', Z')) - Y'(\Phi'(X', Z')) + Z'(\Phi'(X', Y')) \\ &\quad - \Phi'([X', Y'], Z') + \Phi'([X', Z'], Y') - \Phi'([Y', Z'], X') \\ 0 &= d\Phi'(X', Y', Z'). \end{aligned}$$

Thus, if the total space  $M$  is an almost para-cosymplectic manifold, then the base space  $B$  is an almost para-Kähler manifold.  $\blacksquare$

**Proposition 3.6.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion. If the total space  $M$  is a para-Sasakian manifold, then the base space  $B$  is a para-Kähler manifold.*

*Proof.* Let  $X$  and  $Y$  be basic vector fields on  $M$ ,  $\pi$ -related to  $X'$  and  $Y'$  on  $B$ . Since  $M$  is a para-Sasakian manifold, we have

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X = -g(X, Y)\xi.$$

Since  $\pi$  is a semi-Riemannian submersion, we get

$$\pi_*((\nabla_X \varphi)Y) = -g(X, Y)\pi_* \xi = 0.$$

Then, by using  $\pi_* \varphi = J\pi_*$ , we obtain  $\pi_*((\nabla_X \varphi)Y) = (\nabla'_{X'} J)Y' = 0$ , which proves the assertion.  $\blacksquare$



**Proposition 3.7.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion. If the total space  $M$  is a quasi para-Sasakian manifold, then the base space  $B$  is a nearly para-Kähler manifold.*

*Proof.* Let  $X$  and  $Z$  be basic vector fields on  $M$ ,  $\pi$ -related to  $X'$  and  $Z'$  on  $B$ . By (2.5), we have

$$2g((\nabla_X \varphi)X, Z) = -d\Phi(X, X, Z) - d\Phi(X, \varphi X, \varphi Z) - N^{(1)}(X, Z, \varphi X) \\ + N^{(2)}(X, Z)\eta(X) - 2d\eta(\varphi Z, X)\eta(X) + 2d\eta(\varphi X, X)\eta(Z).$$

Since  $M$  is a quasi-para-Sasakian manifold, we obtain

$$2g((\nabla_X \varphi)X, Z) = -2d\eta(\varphi Z, X)\eta(X) + 2d\eta(\varphi X, X)\eta(Z).$$

Since  $\eta$  vanishes on the horizontal distribution, we have

$$g((\nabla_X \varphi)X, Z) = 0.$$

Thus, we deduce that

$$\pi_*((\nabla_X \varphi)X) = 0 = (\nabla_{X'}^J)X',$$

which shows that the base space is a nearly para-Kähler manifold.  $\blacksquare$

We now check the properties of the tensor fields  $T$  and  $A$  for a para-contact para-complex semi-Riemannian submersion, we will see that such tensors have extra properties for such submersions.

**Lemma 3.1.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from a para-cosymplectic manifold  $M$  onto an almost para-Hermitian manifold  $B$ , and let  $X$  and  $Y$  be horizontal vector fields. Then, we have*

- (i)  $A_X \varphi Y = \varphi A_X Y$ ,
- (ii)  $A_{\varphi X} Y = \varphi A_X Y$ .

*Proof.* (i) Let  $X$  and  $Y$  be horizontal vector fields, and  $U$  vertical. Since  $M$  is a para cosymplectic manifold, we have

$$(\nabla_X \Phi)(U, Y) = g((\nabla_X \varphi)Y, U) = g(\nabla_X \varphi Y - \varphi \nabla_X Y, U) = 0$$

Thus, since the vertical and horizontal distributions are invariant, from (2.17) we obtain

$$g(A_X \varphi Y - \varphi A_X Y, U) = 0.$$

Then, we have

$$A_X \varphi Y = \varphi A_X Y.$$

In a similar way, we obtain (ii).  $\blacksquare$

For the tensor field  $T$  we have the following.

**Lemma 3.2.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from a para-cosymplectic manifold  $M$  onto an almost para-Hermitian manifold  $B$ , and let  $U$  and  $V$  be vertical vector fields. Then, we have*

- (i)  $T_U \varphi V = \varphi T_U V$ ,
- (ii)  $T_{\varphi U} V = \varphi T_U V$ .

We now investigate the integrability of the horizontal distribution  $\mathcal{H}$ .

**Theorem 3.1.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from an almost para-cosymplectic manifold  $M$  onto an almost para-Hermitian manifold  $B$ . Then, the horizontal distribution is integrable.*

*Proof.* Let  $X$  and  $Y$  be basic vector fields. It suffices to prove that  $\nu([X, Y]) = 0$ , for basic vector fields on  $M$ . Since  $M$  is an almost para-cosymplectic manifold, it implies  $d\Phi(X, Y, V) = 0$ , for any vertical vector  $V$ . Then, one obtains

$$X(\Phi(Y, V)) - Y(\Phi(X, V)) + V(\Phi(X, Y)) - \Phi([X, Y], V) + \Phi([X, V], Y) - \Phi([Y, V], X) = 0.$$

Since  $[X, V], [Y, V]$  are vertical and the two distributions are  $\phi$ -invariant, the last two and the first two terms vanish. Thus, one gets

$$g([X, Y], \phi V) = V(g(X, \phi Y)).$$

On the other hand, if  $X$  is basic then  $h(\nabla_V X) = h(\nabla_X V) = A_X V$ , thus we have

$$V(g(X, \phi Y)) = g(\nabla_V X, \phi Y) + g(\nabla_V \phi Y, X) = g(A_X V, \phi Y) + g(A_{\phi Y} V, X).$$

Since,  $A$  is skew-symmetric and alternating operator, we get  $V(g(X, \phi Y)) = 0$ . This proves the assertion.  $\blacksquare$

Since for a quasi-para-Sasakian manifold  $d\Phi = 0$ , applying Theorem 3.1, we have the following result.

**Corollary 3.3.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from a quasi-para-Sasakian manifold  $M$  onto an almost para-Hermitian manifold  $B$ . Then, the horizontal distribution is integrable.*

**Corollary 3.4.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from a para-cosymplectic manifold  $M$  onto an almost para-Hermitian manifold  $B$ . Then, the horizontal distribution is completely integrable.*

**Theorem 3.2.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from an almost para-cosymplectic manifold  $M$  onto an almost para-Hermitian manifold  $B$  with  $\dim \mathcal{V}_p \geq 2, \forall p \in M$ . If  $X$  horizontal vector field is an infinitesimal automorphism of  $\phi$ -tensor field, then  $T_V X = 0$ , for any  $V \in \Gamma(\mathcal{V})$ , if and only if  $\eta(\nabla_X V) = \eta([X, V])$ .*

*Proof.* Let  $W$  and  $V$  be vertical vector fields on  $M$ ,  $X$  horizontal. Since  $M$  is an almost para-cosymplectic manifold, it implies  $d\Phi = 0$ . Then, we obtain

$$\begin{aligned} d\Phi(W, \phi V, X) &= W(\Phi(\phi V, X)) - \phi V(\Phi(W, X)) + X(\Phi(W, \phi V)) \\ &\quad - \Phi([W, \phi V], X) + \Phi([W, X], \phi V) - \Phi([\phi V, X], W) \\ &= 0. \end{aligned}$$

Since  $[W, \phi V]$  is vertical and the two distributions are  $\phi$ -invariant, the first two terms vanish. Thus, we get

$$X(\Phi(W, \phi V)) + \Phi([W, X], \phi V) - \Phi([\phi V, X], W) = 0.$$

By direct computations, one obtains:

$$\begin{aligned} 0 &= X(g(W, V) - \eta(V)g(W, \xi)) + g([W, X], V - \eta(V)\xi) - g([\phi V, X], \phi W) \\ 0 &= g(W, \nabla_V X + [X, V]) + g(\nabla_W X, V) - g(\phi[X, \phi V], W) - \eta(V)(g(\nabla_X \xi, W) \\ &\quad + g(\nabla_W X, \xi)) - X(\eta(V))\eta(W). \end{aligned}$$

Using (2.15) we derive

$$(3.1) \quad \begin{aligned} 0 &= g(T_V X, W) + g(T_W X, V) - 2g(T_\xi X, W)\eta(V) + \eta(W)(\eta([X, V]) \\ &\quad - X(\eta(V)) - \eta(V)\eta([X, \xi])). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \eta([X, V]) - X(\eta(V)) - \eta(V)\eta([X, \xi]) &= -\eta(\nabla_V X) - g(\nabla_X \xi, V) + \eta(V)\eta(\nabla_\xi X) \\ &= g(T_V \xi, X) - g(\nabla_X \xi, V) - \eta(V)g(T_\xi \xi, X). \end{aligned}$$

Substituting in (3.1), we obtain

$$(3.2) \quad \begin{aligned} 0 &= 2g(T_V X, W) - 2g(T_\xi X, W)\eta(V) + \eta(W)(g(T_V \xi, X) \\ &\quad + \eta(V)g(T_\xi X, \xi) - g(\nabla_X \xi, V)). \end{aligned}$$

Now, assume that  $T_V X = 0$ , for any  $X \in \Gamma(\mathcal{V})$ . Then (3.2) implies  $g(\nabla_X \xi, V) = 0$ , for any  $V$  and we have

$$\eta([X, V]) = g(\nabla_X V, \xi) - g(\nabla_V X, \xi) = X(\eta(V)) - g(\nabla_X \xi, V) - g(T_V X, \xi) = \eta(\nabla_X V).$$

On the other hand, for any  $X \in \Gamma(\mathcal{H})$  and  $V \in \Gamma(\mathcal{V})$ , the hypothesis  $\eta([X, V]) = \eta(\nabla_X V)$  implies  $g(T_V X, \xi) = g(\nabla_V X, \xi) = g(\nabla_X V + [V, X], \xi) = 0$ . So, (3.2) reduces to

$$0 = 2g(T_V X, W) - \eta(W)g(\nabla_X \xi, V),$$

for any  $V, W \in \Gamma(\mathcal{V})$ . Thus, for any vertical vector field  $W$  orthogonal to  $\xi$ , we get  $g(T_V X, W) = 0$ . Since  $g(T_V X, \xi) = 0$ , one has  $T_V X = 0$ ,  $V \in \Gamma(\mathcal{V})$  and the proof is completed. ■

From Theorem 3.2, we have the following result.

**Corollary 3.5.** *Let  $\pi : M \rightarrow B$  be a para-contact para-complex semi-Riemannian submersion from a quasi-para-Sasakian manifold  $M$  onto an almost para-Hermitian manifold  $B$  with  $\dim \mathcal{V}_p \geq 2$ ,  $\forall p \in M$ . If  $X$  horizontal vector field is an infinitesimal automorphism of  $\varphi$ -tensor field, then  $T_V X = 0$ , for any  $V \in \Gamma(\mathcal{V})$ , if and only if  $\eta(\nabla_X V) = \eta([X, V])$ .*

#### 4. Curvature relations for para-contact para-complex semi-riemannian submersions

We begin this section relating the  $\varphi$ -para-holomorphic bisectonal and sectional curvatures of the total space, the base and the fibres of a para-contact para-complex semi-Riemannian submersions.

Let us recall the sectional curvature of semi-Riemannian manifolds for non-degenerate planes. Let  $M$  be a semi-Riemannian manifold and  $P$  a non-degenerate tangent plane to  $M$  at  $p$ . The number

$$K(U, V) = \frac{g(R(U, V)U, V)}{g(U, U)g(V, V) - g(U, V)^2}$$

is independent on the choice of basis  $U, V$  for  $P$  and is called the sectional curvature.

Let  $\pi$  be a para-contact para-complex semi-Riemannian submersion between an almost para-contact metric manifold  $M$  and an almost para-Hermitian manifold  $N$ . We denote the Riemannian curvatures of  $M, N$  and any fibre  $\pi^{-1}(x)$  by  $R, R'$  and  $\hat{R}$ , respectively. For  $X, Y, Z, W \in \Gamma(\mathcal{H})$ , we have

$$R^*(X, Y, Z, W) = R'(\pi_* X, \pi_* Y, \pi_* Z, \pi_* W) \circ \pi.$$

Let  $\pi : M \rightarrow N$  be a para-contact para-complex semi-Riemannian submersion from an almost para-contact manifold  $(M, \varphi, \xi, \eta, g)$  onto an almost para-Hermitian manifold  $(N, J, g')$ .

We denote by  $B$  the  $\varphi$ -para-holomorphic bisectonal curvature, defined for any pair of nonzero nonlightlike vectors  $X$  and  $Y$  on  $M$  orthogonal to  $\xi$  by the formula:

$$B(X, Y) = \frac{R(X, \varphi X, Y, \varphi Y)}{\|X\|^2 \|Y\|^2}.$$

We note that if  $X$  is a nonlightlike vector field, the  $\varphi X$  is also a nonlightlike vector field.

The  $\varphi$ -para-holomorphic sectional curvature is  $H(X) = B(X, X)$  for any nonzero nonlightlike vector  $X$  orthogonal to  $\xi$ . We denote by  $B'$  and  $H'$  the  $\varphi$ -para-holomorphic bisectonal and  $\varphi$ -para-holomorphic sectional curvatures of  $B$ . Similarly,  $\hat{B}$  and  $\hat{H}$  denote the bisectonal and the sectional para-holomorphic curvatures of a fibre.

The following is a translation of the results of Gray[9] and O'Neill[16] to the present situation:

**Proposition 4.1.** *Let  $\pi : M \rightarrow N$  a para-contact para-complex semi-Riemannian submersion from an almost para-contact metric manifold  $M$  onto an almost para-Hermitian manifold  $N$ . Let  $U$  and  $V$  be nonzero nonlightlike unit vertical vectors, and  $X$  and  $Y$  nonzero nonlightlike unit horizontal vectors orthogonal to  $\xi$ . Then, we have*

- (a)  $B(U, V) = \hat{B}(U, V) - \varepsilon_U \varepsilon_V [g(T_U V, T_{\varphi U} \varphi V) - g(T_{\varphi U} V, T_U \varphi V)];$
- (b)  $B(X, U) = \varepsilon_U \varepsilon_X [g((\nabla_U A)_X \varphi X, \varphi U) - g((\nabla_{\varphi U} A)_X \varphi X, U) + g(A_X U, A_{\varphi X} \varphi U) - g(A_X \varphi U, A_{\varphi X} U) - g(T_U X, T_{\varphi U} \varphi X) + g(T_{\varphi U} X, T_U \varphi X)];$
- (c)  $B(X, Y) = B'(X', Y') \circ \pi - \varepsilon_X \varepsilon_Y [2g(A_X \varphi X, A_Y \varphi Y) - g(A_{\varphi X} Y, A_X \varphi Y) + g(A_X Y, A_{\varphi X} \varphi Y)],$

where  $\varepsilon_U = g(U, U) \in \{\pm 1\}$ ,  $\varepsilon_V = g(V, V) \in \{\pm 1\}$ ,  $\varepsilon_X = g(X, X) \in \{\pm 1\}$  and  $\varepsilon_Y = g(Y, Y) \in \{\pm 1\}$ .

Using Proposition 4.1, we have the following result.

**Proposition 4.2.** *Let  $\pi : M \rightarrow N$  a para-contact para-complex semi-Riemannian submersion from an almost para-contact metric manifold  $M$  onto an almost para-Hermitian manifold  $N$ . Let  $U$  be nonzero nonlightlike unit vertical vector, and  $X$  nonzero nonlightlike unit horizontal vector orthogonal to  $\xi$ . Then, one has:*

- (a)  $H(U) = \hat{H}(U) + \|T_U \varphi U\|^2 - g(T_{\varphi U} \varphi U, T_U U);$
- (b)  $H(X) = H'(X') \circ \pi - 3\|A_X \varphi X\|^2.$

If the total manifold is a para-cosymplectic manifold, then we have the following result for curvature relations between  $M, N$  and  $\pi^{-1}(x)$ .

**Theorem 4.1.** *Let  $\pi : M \rightarrow N$  be a para-contact para-complex semi-Riemannian submersion from a para-cosymplectic manifold  $M$  onto an almost para-Hermitian manifold  $N$ . Let  $U$  and  $V$  be nonzero nonlightlike unit vertical vectors, and  $X$  and  $Y$  nonzero nonlightlike unit horizontal vectors orthogonal to  $\xi$ . Then, we have:*

- (a)  $B(U, V) = \hat{B}(U, V) - \varepsilon_U \varepsilon_V 2\|T_U V\|^2;$
- (b)  $B(X, Y) = B'(X', Y') \circ \pi;$
- (c)  $B(X, U) = -\varepsilon_U \varepsilon_X [2\|T_U X\|^2 - 2\eta(T_U X)^2].$

*Proof.* (a) From Proposition 4.1(a), we have

$$B(U, V) = \hat{B}(U, V) - \varepsilon_U \varepsilon_V [g(T_U V, T_{\varphi U} \varphi V) - g(T_{\varphi U} V, T_U \varphi V)].$$

Using Lemma 3.2, we get

$$(4.1) \quad \begin{aligned} g(T_U \varphi V, T_{\varphi U} V) &= g(\varphi T_U V, \varphi T_U V) \\ &= -g(T_U V, T_U V) + \eta(T_U V)\eta(T_U V) = -\|T_U V\|^2. \end{aligned}$$

Using again Lemma 3.2, we get

$$(4.2) \quad \begin{aligned} g(T_{\varphi U} \varphi V, T_U V) &= g(\varphi^2 T_U V, T_U V) \\ &= g(T_U V - \eta(T_U V)\xi, T_U V) = \|T_U V\|^2. \end{aligned}$$

From (4.1) and (4.2), we have (a).

(b) Since  $M$  is a para-cosymplectic manifold and the distribution  $\mathcal{H}$  is integrable we have  $A = 0$ . Then using Proposition 4.1(c), we have  $B(X, Y) = B'(X', Y') \circ \pi$ .

(c) Since  $M$  is a para-cosymplectic manifold and  $A = 0$ , then using Proposition 4.1(b) we have

$$B(X, U) = -\varepsilon_U \varepsilon_X [g(T_U X, T_{\varphi U} \varphi X) - g(T_{\varphi U} X, T_U \varphi X)].$$

On the other hand, using Lemma 3.2, we have

$$(4.3) \quad g(T_U X, T_{\varphi U} \varphi X) = g(T_U X, T_U X) - \eta(T_U X)^2$$

and

$$(4.4) \quad g(T_{\varphi U} X, T_U \varphi X) = -g(T_U X, T_U X) + \eta(T_U X)^2$$

From (4.3) and (4.4), we have (c). ■

As a result of Theorem 4.1, we have the following result.

**Corollary 4.1.** *Let  $\pi : M \rightarrow N$  be a para-contact para-complex semi-Riemannian submersion from a para-cosymplectic manifold  $M$  onto an almost para-Hermitian manifold  $N$ . Let  $U$  be nonzero nonlightlike unit vertical vector, and  $X$  nonzero nonlightlike unit horizontal vector orthogonal to  $\xi$ . Then, one has*

$$(a) \quad H(U) = \hat{H}(U) - 2\|T_U U\|^2;$$

$$(b) \quad H(X) = H'(X') \circ \pi.$$

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