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L^p Estimates for Higher-Order Parabolic Schrödinger Operators with Certain Nonnegative Potentials

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Abstract. Let $\partial/\partial t+(-\Delta)^2+V^2$ be a higher order parabolic Schrödinger operator on $\mathbb{R}^{n+1}$ $(n\geq 5)$, where the nonnegative potential V belongs to the reverse Hölder class $B_{q_1}(\mathbb{R}^n)$ for some $q_1>n/2$. In this paper we obtain the $L^p(\mathbb{R}^{n+1})$ estimates for the operator $\nabla^4(\partial/\partial t+(-\Delta)^2+V^2)^{-1}$.

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1. Introduction

In this paper we consider the higher order parabolic Schrödinger operator

$$\partial/\partial t+(-\Delta)^2+V^2 \text{ on } \mathbb{R}^{n+1},\ n\geq 5,$$

where $(-\Delta)^2$ is the bilaplacian on $\mathbb{R}^n$ and the nonnegative potential $V(x)$ is independent of variable t. The studies of Schrödinger operators with nonnegative potentials have attracted much attention, see for example [5, 14, 18, 9, 1, 8, 10, 17]. In recent years, on the one hand, some scholars generalize the results for Schrödinger operator to the case of higher order Schrödinger operators (cf. [13, 11, 12]). On the other hand, some study similar results for the parabolic Schrödinger operators (cf. [3, 6, 7, 15]). Motivated by the above papers, we continue this line to study the higher order parabolic Schrödinger operators and obtain the L^p boundedness of $\nabla^4(\partial/\partial t+(-\Delta)^2+V^2)^{-1}$.

Note that a nonnegative locally L^q integrable function V on $\mathbb{R}^n$ is said to belong to $B_q(\mathbb{R}^n)$ $(1<q<\infty)$ if there exists $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q\,dx\right)^{\frac{1}{q}}\leq C\left(\frac{1}{|B|}\int_B V(x)\,dx\right) \tag{1.1}$$

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holds for every ball B in $\mathbb{R}^n$. Moreover, if there exists a constant $C > 0$ such that

$$\| V \|_{L^\infty(B)} \leq C \left(\frac{1}{|B|} \int_B V(x)\, dx \right) \tag{1.2}$$

holds for every ball B in $\mathbb{R}^n$, we say $V \in B_\infty(\mathbb{R}^n)$.

It follows from [14] that the B_q class has a property of self improvement; that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$. For $1 < p < \infty$, it is easy to see that $B_\infty(\mathbb{R}^n) \subseteq B_p(\mathbb{R}^n)$. If $V \in B_\infty(\mathbb{R}^n)$, then there is a positive constant C such that $V(x) \leq Cm(x,V)^2$ a.e. on $\mathbb{R}^n$ (Remark 2.9, [14]).

We are now in a position to give the main results in this paper.

Theorem 1.1. *Suppose $V(x) \in B_{q_1}(\mathbb{R}^n)$, $q_1 > n/2$. Then, for $1 < p \leq q_1/2$,*

$$\| \nabla^4 (\partial/\partial t + (-\Delta)^2 + V^2)^{-1} f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})},$$

where $\nabla^4 = \partial^4 / \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\alpha_1 + \cdots + \alpha_n = 4$.

If the potential V satisfies stronger condition, we can get the following result which removes the restriction of the range of p.

Corollary 1.1. *Suppose $V(x) \in B_\infty(\mathbb{R}^n)$. Then, for $1 < p < \infty$,*

$$\| \nabla^4 (\partial/\partial t + (-\Delta)^2 + V^2)^{-1} f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})},$$

where $\nabla^4 = \partial^4 / \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\alpha_1 + \cdots + \alpha_n = 4$.

We also obtain the L^p boundedness of the operator $V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha}$ for $0 < \alpha < 1$. See Theorem 5.1 in the last section.

This paper is organized as follows. In Section 2 we recall some basic facts for the auxiliary function $m(x,V)$ and give some estimates on the fundamental solution to $\partial u/\partial t + (-\Delta)^2 u + V^2(x) u = 0$ in $\mathbb{R}^{n+1}$. In Section 3 we recall some basic facts for $L^p(\mathbb{R}^{n+1})$ multipliers. Section 4 shows that Theorem 1.1 holds true. In the last section we prove the L^p boundedness of the operator $V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha}$ for $0 < \alpha < 1$.

Throughout this paper the letter C stands for a constant and is not necessarily the same at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $1/C \leq B_1/B_2 \leq C$.

2. The auxiliary function $m(x,V)$ and estimates of fundamental solutions

In the first part of this section we recall the definition of the auxiliary function $m(x,V)$ and some lemmas about the auxiliary function $m(x,V)$ which have been proved in [14]. We always assume $V \in B_{q_1}$ for $q_1 > n/2$ throughout this section.

The auxiliary function $m(x,V)$ is defined by

$$\frac{1}{m(x,V)} \doteq \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\, dy \leq 1 \right\}, \qquad x \in \mathbb{R}^n.$$

Lemma 2.1. *The measure $V(x)\,dx$ satisfies the doubling condition, that is, there exists a constant $C > 0$ such that*

$$\int_{B(x,2r)} V(y)\, dy \leq C \int_{B(x,r)} V(y)\, dy$$

holds for all balls $B(x,r)$ in $\mathbb{R}^n$.

Lemma 2.2. *There exists a constant $C > 0$ such that, for $0 < r < R < \infty$,*

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \le C\left(\frac{r}{R}\right)^{2-n/q_1}\frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\,dy.$$

Lemma 2.3. *If $r = \frac{1}{m(x,V)}$, then*

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy = 1.$$

Moreover,

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \sim 1 \qquad \text{if and only if} \qquad r \sim \frac{1}{m(x,V)}.$$

Lemma 2.4. *There exists $l_0 > 0$ such that, for any x and y in $\mathbb{R}^n$,*

$$\frac{1}{C}\Big(1 + m(x,V)\,|\,x-y\,|\Big)^{-l_0} \le \frac{m(x,V)}{m(y,V)} \le C\Big(1 + m(x,V)|\,x-y\,|\Big)^{l_0/l_0+1}.$$

In particular, $m(x,V) \sim m(y,V)$ if $|\,x-y\,| < \frac{C}{m(x,V)}$.

Lemma 2.5. *There exists $l_1 > 0$ such that*

$$\int_{B(x,R)} \frac{V(y)}{|\,x-y\,|^{n-2}}\,dy \le \frac{C}{R^{n-2}}\int_{B(x,R)} V(y)\,dy \le C\Big(1 + Rm(x,V)\Big)^{l_1}.$$

The next lemma has been proved by Tao and Wang in [16].

Lemma 2.6. *Let $q > s \ge 0$, $q \ge \max(1, sn/\alpha)$, $\alpha > 0$, and k sufficiently large, then there are positive constants k_0, C and C_k such that*

$$\int_{|x-y|<r} \frac{V(y)^s}{|x-y|^{n-\alpha}}\,dy \le Cr^{\alpha-2s}(1 + rm(x,V))^{sk_0} \tag{2.1}$$

and

$$\int_{\mathbb{R}^n} \frac{V(y)^s}{(1+m(x,V)|x-y|)^k|x-y|^{n-\alpha}}\,dy \le C_k m(x,V)^{2s-\alpha} \tag{2.2}$$

for any $r > 0, x \in \mathbb{R}^n$ and $V \in B_q(\mathbb{R}^n)$.

Next we recall some fundamental properties of functions in the reverse Hölder class (cf. [19]).

Lemma 2.7. *If $V(x) \in B_q (1 < q \le \infty)$, λ is a nonnegative constant, then $V(x) + \lambda \in B_q$.*

Lemma 2.8. *If $V(x) \in B_q (q \ge n/2)$, λ is a nonnegative constant, then $m(x,V) \le m(x, V + \lambda)$.*

Similar to the proof of above lemmas, we easily obtain the following lemma.

Lemma 2.9. *If $V(x) \in B_q (1 < q \le \infty)$, λ is a nonnegative constant, then $\sqrt{V^2(x) + \lambda} \in B_q$.*

Lemma 2.10. *If $V(x) \in B_q (q \ge n/2)$, λ is a nonnegative constant, then*

$$m(x,V) \le m(x, \sqrt{V^2 + \lambda}).$$

Remark 2.1. It is not difficult to check that if λ is a nonnegative constant, then $m(x,\lambda) = C\sqrt{\lambda}$, where C is a positive constant and is independent of λ.

In this paper we endow the space $\mathbb{R}^{n+1}$ with the following parabolic metric which is different from the usual Euclidean metric:

$$d((x,t),(y,s)) = \max(|x-y|, |t-s|^{1/4}), \tag{2.3}$$

for any $(x,t),(y,s) \in \mathbb{R}^{n+1}$.

Next we give some estimates on the fundamental solution of higher order parabolic Schrödinger operator.

Let $\Gamma(x,t;y,s;\lambda)$ be the fundamental solution to $\partial u/\partial t + (-\Delta)^2 u + V^2(x)u + \lambda u = 0$ in $\mathbb{R}^{n+1}$, where $\lambda \in [0,\infty)$. Especially, we denote $\Gamma(x,t;y,s;0) \doteq \Gamma(x,t;y,s)$. By Lemma 2.5 in [2] we easily get the following lemma.

Lemma 2.11. *Let $V \in B_{q_1}$ for $q_1 > n/2$. For every $N \in \mathbb{N}$, there exist positive constants C_N and $\tilde{C}$ such that for all $(x,t),(y,s) \in \mathbb{R}^{n+1}$ and $t > s$,*

$$| \Gamma(x,t;y,s) | \le \frac{C_N}{[1+(t-s)^{1/2}m^2(x,V)]^N}(t-s)^{-n/4}\exp\left(-\tilde{C}\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right). \tag{2.4}$$

Lemma 2.12. *Let $V \in B_{q_1}$ for $q_1 > n/2$. For every $N \in \mathbb{N}$, there exist positive constants C_N and $\tilde{C}_1$ such that for all $(x,t),(y,s) \in \mathbb{R}^{n+1}$ and $t > s$,*

$$| \Gamma(x,t;y,s) | \le \frac{C_N}{[1+|x-y|^2 m^2(x,V)]^N}(t-s)^{-n/4}\exp\left(-\tilde{C}_1\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right). \tag{2.5}$$

Proof. For any $(x,t),(y,s) \in \mathbb{R}^{n+1}$ and $t > s$, it is easy to deduce that the inequality (2.5) holds true when $(t-s)^{1/2} > |x-y|^2$. Now, we assume that $(t-s)^{1/2} \le |x-y|^2$. $\forall N > 0$, by (2.4) we have

$$\begin{aligned}
[|x-y|^2 m^2(x,V)]^N \mid \Gamma(x,t;y,s) \mid &\le \frac{C_N[|x-y|^2 m^2(x,V)]^N}{[1+(t-s)^{1/2}m^2(x,V)]^N}(t-s)^{-n/4}\exp\left(-\tilde{C}\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)\\
&\le C_N\left(\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)^{3N/2}(t-s)^{-n/4}\exp\left(-\tilde{C}\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)\\
&\le C_N(t-s)^{-n/4}\exp\left(\varepsilon\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)\exp\left(-\tilde{C}\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)\\
&= C_N(t-s)^{-n/4}\exp\left(-\tilde{C}_1\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right),
\end{aligned}$$

where $\tilde{C}_1 = \tilde{C} - \varepsilon$ and $0 < \varepsilon < \tilde{C}$. Therefore,

$$[|x-y|^2 m^2(x,V)] \mid \Gamma(x,t;y,s) \mid^{1/N} \le \left(C_N(t-s)^{-n/4}\exp\left(-\tilde{C}_1\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)\right)^{1/N}.$$

Furthermore, applying (2.4) again we have

$$\mid \Gamma(x,t;y,s) \mid \le C_N(t-s)^{-n/4}\exp\left(-\tilde{C}_1\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right).$$

Combining the above inequalities, we deduce that (2.5) is valid. ∎

From Lemma 2.10, Lemma 2.11, Lemma 2.12 and Remark 1, we deduce the following corollary.

Corollary 2.1. *Let $V \in B_{q_1}$ for $q_1 > n/2$. For every $N \in \mathbb{N}$, there exist positive constants C_N and $\tilde{C}_1$ such that for all $(x,t),(y,s) \in \mathbb{R}^{n+1}$ and $t > s$,*

(2.6)

$$| \Gamma(x,t;y,s;\lambda) | \leq \frac{C_N}{[1+\lambda^{1/4}d((x,t),(y,s))]^N[1+d((x,t),(y,s))m(x,V)]^N} \frac{\exp\left(-\tilde{C}_1 \frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)}{(t-s)^{n/4}}.$$

3. $L^p(\mathbb{R}^{n+1})$ Multipliers

In this section we recall some results for $L^p(\mathbb{R}^{n+1})$ multipliers in order to prove Theorem 1.1 (cf. [4]).

Let $a = (1,\dots,1,4)$. For $\beta = (\beta_1,\dots,\beta_{n+1})$ define

$$(a,\beta) = \sum_{j=1}^{n+1} a_j\beta_j = \sum_{j=1}^{n} \beta_j + 4\beta_{n+1} \text{ and } |\beta| = \sum_{j=1}^{n+1} \beta_j.$$

For $x' = (x,t) = (x_1,\dots,x_n,t)$ define

$$\lambda^a x' = (\lambda x_1,\dots,\lambda x_n,\lambda^4 t) \text{ and } (x')^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n} x_{n+1}^{\beta_{n+1}}.$$

For a fixed $x' \in \mathbb{R}^{n+1}$, defined $\rho(x')$ as the unique solution of $F(x',\rho) = \sum_{j=1}^n x_j^2/\rho^2 + t^2/\rho^8 = 1$. It follow from [4] that $\rho(x')$ is a non-isotropic norm on $\mathbb{R}^{n+1}$ and has a dilation invariance property $\rho(\lambda^a x') = \lambda\rho(x')$. Note that the metric induced by $\rho(x')$ is equivalent to the parabolic metric introduced in (2.3).

The function $h(x)$ is said to be a multiplier when

$$\| T\varphi \|_{L^p(\mathbb{R}^{n+1})} \leq A_p \| \varphi \|_{L^p(\mathbb{R}^{n+1})} \text{ for every } p, 1 < p < \infty,$$

where $T\varphi = F^{-1}(hF(\varphi))$ and F is the Fourier transform operator.

The following proposition has been proved in [4].

Proposition 3.1. *Let $h(x,t) \in L^\infty(\mathbb{R}^{n+1})$, and assume $h(x,t)$ is N times continuously differentiable, where $N > |a|/2 = (n+4)/2$; moreover, assume that*

$$\int_{R/2\leq\rho(x,t)\leq 3R} |R^{(a,\beta)}(\frac{\partial}{\partial x'})^\beta h(x,t)|^2 \frac{dxdt}{R^{|a|}} \leq C, \quad |h(x,t)| \leq C \text{ a.e.}, \tag{3.1}$$

where C is independent of R, say $C \geq 1$ and $(\frac{\partial}{\partial x'})^\beta = (\frac{\partial}{\partial x_1})^{\beta_1}\cdots(\frac{\partial}{\partial x_n})^{\beta_n}(\frac{\partial}{\partial t})^{\beta_{n+1}}$. Then

$$\| T\varphi \|_{L^p(\mathbb{R}^{n+1})} = \| F^{-1}(hF(\varphi)) \|_{L^p(\mathbb{R}^{n+1})} \leq A_pC \| \varphi \|_{L^p(\mathbb{R}^{n+1})}, \varphi \in C_0^\infty(\mathbb{R}^{n+1}),$$

where A_p depends only on a and p.

We define an operator T_j by

$$F(T_jf)(x,t) = \frac{ix_j}{(it+|x|^4)^{1/4}} F(f)(x,t), 1 \leq j \leq n, f \in C_0^\infty(\mathbb{R}^{n+1}).$$

By simple computation we conclude that the function $h(x,t) = \frac{ix_j}{(it+|x|^4)^{1/4}}$ satisfies the condition (3.1) in Proposition 1. Therefore,

$$\| T_jf \|_{L^p(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})}, j = 1,2,\dots,n.$$

Then for multi-index $\alpha=(\alpha_1,\ldots,\alpha_n)$ with $|\alpha|=4$,

$$\begin{aligned}F(\nabla^4 f)(x,t)&=(ix)^\alpha F(f)(x,t)=\frac{(ix)^\alpha}{it+|x|^4}(it+|x|^4)F(f)(x,t)\\&=\frac{ix_1^{\alpha_1}ix_2^{\alpha_2}\cdots ix_n^{\alpha_n}}{it+|x|^4}(it+|x|^4)F(f)(x,t)\\&=\frac{ix_1^{\alpha_1}}{(it+|x|^4)^{\alpha_1/4}}\cdots\frac{ix_n^{\alpha_n}}{(it+|x|^4)^{\alpha_n/4}}(it+|x|^4)F(f)(x,t)\\&=F\left(T_1^{\alpha_1}\cdots T_n^{\alpha_n}(\partial/\partial t+(-\Delta)^2)f\right)(x,t).\end{aligned}$$

By the L^p boundedness of T_j, we have

$$\begin{aligned}\|\nabla^4 f\|_{L^p(\mathbb{R}^{n+1})}&\le C\,\|T_1^{\alpha_1}\cdots T_n^{\alpha_n}(\partial/\partial t+(-\Delta)^2)f\|_{L^p(\mathbb{R}^{n+1})}\\&\le C\,\|(\partial/\partial t+(-\Delta)^2)f\|_{L^p(\mathbb{R}^{n+1})}.\end{aligned}$$

Therefore,

$$\|\nabla^4(\partial/\partial t+(-\Delta)^2)^{-1}f\|_{L^p(\mathbb{R}^{n+1})}\le C\,\|f\|_{L^p(\mathbb{R}^{n+1})}. \tag{3.2}$$

4. The proof of Theorem 1.1

In this section we devote to the proof of Theorem 1.1. Before completing the proof of Theorem 1.1, we first give the following theorem.

Theorem 4.1. *Suppose $V(x)\in B_{q_1}(\mathbb{R}^n)$, $q_1>n/2$. Then, for $1<p\le q_1/2$,*

$$\|V^2(\frac{\partial}{\partial t}+(-\Delta)^2+V^2)^{-1}f\|_{L^p(\mathbb{R}^{n+1})}\le C\,\|f\|_{L^p(\mathbb{R}^{n+1})}.$$

Proof. Let $f\in L^p(\mathbb{R}^{n+1})$ for $1<p\le q_1/2$ and

$$u(x,t)=(\frac{\partial}{\partial t}+(-\Delta)^2+V^2)^{-1}f(x,t)=\int_{-\infty}^t\int_{\mathbb{R}^n}\Gamma(x,t;y,s)f(y,s)dyds.$$

Write

$$\begin{aligned}u(x,t)&=\int_{-\infty}^t\int_{|x-y|<r}\Gamma(x,t;y,s)f(y,s)dyds+\int_{-\infty}^t\int_{|x-y|\ge r}\Gamma(x,t;y,s)f(y,s)dyds\\&=u_1(x,t)+u_2(x,t),\end{aligned}$$

where $r=1/m(x,V)$.

Because of self improvement of the B_{q_1} class, $V\in B_{q_0}$ for some $q_0>q_1$. For convenience, we denote

$$G_1(x,t;y,s)=\frac{C_N}{[1+(t-s)^{1/2}m^2(x,V)]^N[1+|x-y|m(x,V)]^N}$$

and

$$G_2(x,t;y,s)=(t-s)^{-n/4}\exp\left(-\tilde{C}_1\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right).$$

At first, we have

$$\left[\int_{-\infty}^t\int_{|x-y|<r}|G_2(x,t;y,s)|^q dyds\right]^{1/q}$$

$$\leq \left[\int_{|x-y|<r}\int_0^\infty t^{-qn/4}\exp\left(-q\tilde{C}_1\frac{|x-y|^{4/3}}{t^{1/3}}\right)dtdy\right]^{1/q}$$
$$\leq \left[\int_{|x-y|<r}\frac{1}{|x-y|^{qn-4}}\int_0^\infty s^{3qn/4-4}e^{-q\tilde{C}_1 s}dsdy\right]^{1/q}$$
$$\leq C(\int_0^r t^{-qn+4+n-1}dt)^{1/q} = Cr^{n/q-n+4/q} = Cr^{-2n/q_0+4/q},$$

where $1/q+2/q_0=1$.

Then using Hölder inequality,

$$|u_1(x,t)| \leq \left[\int_{-\infty}^t\int_{|x-y|<r}|G_1(x,t;y,s)||f(y,s)|^{q_0/2}dyds\right]^{2/q_0}\left[\int_{-\infty}^t\int_{|x-y|<r}|G_2(x,t;y,s)|^q dyds\right]^{\frac{1}{q}}$$
$$\leq Cm(x,V)^{2n/q_0-4/q}\left[\int_{-\infty}^t\int_{|x-y|<r}|G_1(x,t;y,s)||f(y,s)|^{q_0/2}dyds\right]^{2/q_0}.$$

Note that $|x-y|<1/m(x,V)$ and $m(x,V)\sim m(y,V)$. Denote $R=1/m(y,V)$. Therefore, by Lemma 2.5,

$$\int_{\mathbb{R}^{n+1}}(V^2(x)|u_1(x,t)|)^{q_0/2}dxdt$$
$$\leq C\int_{\mathbb{R}^{n+1}}V^{q_0}(x)[m(x,V)]^{n-2q_0/q}\left[\int_{-\infty}^t\int_{|x-y|<r}|G_1(x,t;y,s)||f(y,s)|^{q_0/2}dyds\right]dxdt$$
$$\leq \int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}\left[\int_{|x-y|<C_1R}\int_s^\infty V^{q_0}(x)[m(y,V)]^{n-2q_0/q}|G_1(x,t;y,s)|dxdt\right]dyds$$
$$\leq \int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}\left[\int_{|x-y|<C_1R}\int_s^\infty \frac{C_NV^{q_0}(x)[m(y,V)]^{n-2q_0/q}}{[1+|x-y|m(y,V)]^N}\frac{1}{[1+(t-s)^{\frac{1}{2}}m^2(x,V)]^N}dxdt\right]dyds$$
$$\leq \int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}\int_{|x-y|<C_1R}\frac{C_NV^{q_0}(x)[m(y,V)]^{n-2q_0/q}}{[1+|x-y|m(y,V)]^N}\int_s^\infty\frac{1}{[1+(t-s)^{1/2}m^2(x,V)]^N}dxdtdyds$$
$$\leq CC_N\int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}\left[\int_{|x-y|<C_1R}\frac{V^{q_0}(x)[m(y,V)]^{n-2q_0/q-4}}{[1+|x-y|m(y,V)]^N}dx\right]dyds$$
$$\leq CC_N\int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}(m(y,V))^{n-2q_0}\left[\int_{|x-y|<C_1R}V^{q_0}(x)dx\right]dyds$$
$$\leq CC_N\int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}(m(y,V))^{n-2q_0}R^n\left[\frac{1}{R^n}\int_{|x-y|<C_1R}V^{q_0}(x)dx\right]dyds$$
$$\leq CC_N\int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}(m(y,V))^{n-2q_0}R^n\left[\frac{1}{R^n}\int_{|x-y|<C_1R}V(x)dx\right]^{q_0}dyds$$
$$\leq CC_N\int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}(m(y,V))^{n-2q_0}R^{n-2q_0}\left[\frac{1}{R^{n-2}}\int_{|x-y|<C_1R}V(x)dx\right]^{q_0}dyds$$
$$\leq CC_N\int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}(m(y,V))^{n-2q_0}R^{n-2q_0}dyds$$
$$\leq CC_{k_0}\int_{\mathbb{R}^{n+1}}|f(y,s)|^{q_0/2}dyds.$$

Moreover, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} V^2(x)|u_1(x,t)|dxdt\\
&\le C\int_{\mathbb{R}^{n+1}} V^2(x)\left[\int_{-\infty}^{t}\int_{|x-y|<r}|\Gamma(x,t;y,s)||f(y,s)|dyds\right]dxdt\\
&\le \int_{\mathbb{R}^{n+1}}|f(y,s)|\left[\int_{|x-y|<C_1R}\int_s^\infty V^2(x)|\Gamma(x,t;y,s)|dxdt\right]dyds\\
&\le \int_{\mathbb{R}^{n+1}}|f(y,s)|\left[\int_{|x-y|<C_1R}\int_s^\infty \frac{C_NV^2(x)}{[1+|x-y|m(x,V)]^N}\frac{\exp\left(-\tilde{C}_1\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)}{(t-s)^{-n/4}}dxdt\right]dyds\\
&\le \int_{\mathbb{R}^{n+1}}|f(y,s)|\int_{|x-y|<C_1R}\frac{C_NV^2(x)}{[1+|x-y|m(y,V)]^N}\int_0^\infty t^{-n/4}\exp\left(-\tilde{C}_1\frac{|x-y|^{4/3}}{t^{1/3}}\right)dxdtdyds\\
&\le C\int_{\mathbb{R}^{n+1}}|f(y,s)|\left[\int_{|x-y|<C_1R}\frac{V^2(x)}{[1+|x-y|m(y,V)]^N|x-y|^{n-4}}dx\right]dyds\\
&\le CC_{k_0}\int_{\mathbb{R}^{n+1}}|f(y,s)|dyds,
\end{aligned}$$

where we use (2.1) in Lemma 2.6 in the last inequality.

Therefore, by using interpolation,

$$\| V^2u_1 \|_{L^p(\mathbb{R}^{n+1})}\le C\| f \|_{L^p(\mathbb{R}^{n+1})} \qquad \text{for } 1\le p\le q_0/2.$$

To finish the proof, using (2.2) in Lemma 2.6, we first have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} | \Gamma(x,t;y,s) | dyds\\
&\le \int_{-\infty}^{t}\int_{\mathbb{R}^n}\frac{C_N}{[1+|x-y|m(x,V)]^N}(t-s)^{-n/4}\exp\left(-\tilde{C}_1\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)dyds\\
&\le \int_{\mathbb{R}^n}\frac{C}{[1+|x-y|m(x,V)]^N|x-y|^{n-4}}\int_0^\infty s^{3n/4-4}e^{-C_0s}ds\\
&\le Cm(x,V)^{-4}.
\end{aligned}$$

For $1<p\le q_0/2$, we obtain

$$\begin{aligned}
|u_2(x,t)|&\le\left[\int_{-\infty}^{t}\int_{|x-y|<r}|\Gamma(x,t;y,s)||f(y,s)|^pdyds\right]^{1/p}\left[\int_{-\infty}^{t}\int_{|x-y|<r}|\Gamma(x,t;y,s)|dyds\right]^{1/p'}\\
&\le Cm(x,V)^{-4/p'}\left[\int_{-\infty}^{t}\int_{|x-y|<r}|\Gamma(x,t;y,s)||f(y,s)|^pdyds\right]^{1/p},
\end{aligned}$$

where $1/p+1/p'=1$.

Let $R=1/m(y,V)$. By Lemma 2.4,

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}}(V^2(x)|u_2(x,t)|)^pdxdt\\
&\le C\int_{\mathbb{R}^{n+1}}V^{2p}(x)[m(x,V)]^{-4p/p'}\left[\int_{-\infty}^{t}\int_{|x-y|\ge r}|\Gamma(x,t;y,s)||f(y,s)|^pdyds\right]dxdt
\end{aligned}$$

$$\leq \int_{\mathbb{R}^{n+1}} |f(y,s)|^p \left[\int_{|x-y|\geq r} \int_s^\infty V^{2p}(x)[m(y,V)]^{-\frac{4p}{p'}} |\Gamma(x,t;y,s)| dxdt \right] dyds$$

$$\leq \int_{\mathbb{R}^{n+1}} |f(y,s)|^p \left[\int_{|x-y|\geq C_1 R} \int_s^\infty \frac{C_N V^{2p}(x)[m(y,V)]^{-4p/p'}}{[1+|x-y|m(y,V)]^N} \frac{\exp\left(-\tilde{C}_1 \frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)}{(t-s)^{-n/4}} dxdt \right] dyds$$

$$\leq \int_{\mathbb{R}^{n+1}} |f(y,s)|^p \int_{|x-y|\geq C_1 R} \frac{C_N V^{2p}(x)[m(y,V)]^{-4p/p'}}{[1+|x-y|m(y,V)]^N} \int_0^\infty t^{-n/4} \exp\left(-\tilde{C}_1 \frac{|x-y|^{4/3}}{t^{1/3}}\right) dxdtdyds$$

$$\leq C \int_{\mathbb{R}^{n+1}} |f(y,s)|^p \left[\int_{|x-y|\geq C_1 R} \frac{V^{2p}(x)[m(y,V)]^{-4p/p'}}{[1+|x-y|m(y,V)]^N |x-y|^{n-4}} dx \right] dyds$$

$$\leq C \int_{\mathbb{R}^{n+1}} |f(y,s)|^p [m(y,V)]^{-4p/p'} \left[\sum_{j=0}^\infty \int_{2^j R\leq |x-y|<2^{j+1}R} \frac{V^{2p}(x)}{[1+|x-y|m(y,V)]^N |x-y|^{n-4}} dx \right] dyds$$

$$\leq C \int_{\mathbb{R}^{n+1}} |f(y,s)|^p [m(y,V)]^{-4p/p'} \left[\sum_{j=0}^\infty \frac{1}{[1+2^j]^N (2^j R)^{n-4}} \int_{|x-y|<2^{j+1}R} V^{2p}(x) dx \right] dyds$$

$$\leq C \int_{\mathbb{R}^{n+1}} |f(y,s)|^p [m(y,V)]^{-4p/p'} \left[\sum_{j=0}^\infty \frac{(2^j R)^4}{[1+2^j]^N} \Big(\frac{1}{(2^j R)^n} \int_{|x-y|<2^{j+1}R} V^{2p}(x) dx \Big) \right] dyds$$

$$\leq C \int_{\mathbb{R}^{n+1}} |f(y,s)|^p [m(y,V)]^{-4p/p'} \left[\sum_{j=0}^\infty \frac{(2^j R)^4}{[1+2^j]^N} \Big(\frac{1}{(2^j R)^n} \int_{|x-y|<2^{j+1}R} V(x) dx \Big)^{2p} \right] dyds$$

$$\leq C \int_{\mathbb{R}^{n+1}} |f(y,s)|^p [m(y,V)]^{-4p/p'} \left[\sum_{j=0}^\infty \frac{(2^j R)^{4-2p}}{[1+2^j]^N} \Big(\frac{1}{(2^j R)^{n-2}} \int_{|x-y|<2^{j+1}R} V(x) dx \Big)^{2p} \right] dyds$$

$$\leq C \int_{\mathbb{R}^{n+1}} |f(y,s)|^p [m(y,V)]^{-4p/p'} R^{4-2p} \left[\sum_{j=0}^\infty \frac{2^j}{[1+2^j]^{N-2pl_1}} \right] dyds$$

$$\leq C \int_{\mathbb{R}^{n+1}} |f(y,s)|^p dyds,$$

where we choose N sufficiently large.

Hence,

$$\int_{\mathbb{R}^{n+1}} |V^2(x) u_2(x,t)|^p \, dxdt \leq \int_{\mathbb{R}^{n+1}} |f(x,t)|^p \, dxdt \qquad \text{for } 1 \leq p \leq q_0/2.$$

Thus the theorem is proved. ∎

Proof of Theorem 1.1. By Theorem 5.1, we have

$$\| V^2 (\partial/\partial t + (-\Delta)^2 + V^2)^{-1} f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})} \text{ for } 1 \leq p \leq q_0/2.$$

It follows from (3.2) that

$$\begin{aligned}
&\| \nabla^4 (\partial/\partial t + (-\Delta)^2 + V^2)^{-1} f \|_{L^p(\mathbb{R}^{n+1})} \\
&\leq C \| \nabla^4 (\partial/\partial t + (-\Delta)^2)^{-1} (\partial/\partial t + (-\Delta)^2)(\partial/\partial t + (-\Delta)^2 + V^2)^{-1} f \|_{L^p(\mathbb{R}^{n+1})} \\
&\leq \| (\partial/\partial t + (-\Delta)^2)(\partial/\partial t + (-\Delta)^2 + V^2)^{-1} f \|_{L^p(\mathbb{R}^{n+1})} \\
&\leq \| f \|_{L^p(\mathbb{R}^{n+1})} + \| V^2 (\partial/\partial t + (-\Delta)^2 + V^2)^{-1} f \|_{L^p(\mathbb{R}^{n+1})}
\end{aligned}$$

$$\leq C \parallel f \parallel_{L^p(\mathbb{R}^{n+1})} .$$

The proof of Theorem 1.1 is finished. ∎

5. The L^p boundedness of the operator $V^{2\alpha}(\partial/\partial t+(-\Delta)^2+V^2)^{-\alpha}$

Theorem 5.1. *Suppose* $V(x) \in B_{q_1}(\mathbb{R}^n)$, $q_1 > n/2$. *Then, for* $1 < p \leq q_1/2$,

$$\parallel V^{2\alpha}(\partial/\partial t+(-\Delta)^2+V^2)^{-\alpha} f \parallel_{L^p(\mathbb{R}^{n+1})} \leq C \parallel f \parallel_{L^p(\mathbb{R}^{n+1})},$$

Proof. By the functional calculus, we may write, for all $0<\alpha<1$,

$$(\partial/\partial t+(-\Delta)^2+V^2)^{-\alpha} = \frac{1}{\pi}\int_0^\infty \lambda^{-\alpha}(\partial/\partial t+(-\Delta)^2+V^2)^{-1}d\lambda.$$

Let $f \in C_0^\infty(\mathbb{R}^{n+1})$. From $(\partial/\partial t+(-\Delta)^2+V^2)^{-1}f(x,t) = \int_{\mathbb{R}^{n+1}}\Gamma(x,t;y,s;\lambda)f(y,s)dyds$, it follows that

$$V^{2\alpha}(\partial/\partial t+(-\Delta)^2+V^2)^{-\alpha}f(x,t) = \int_{\mathbb{R}^{n+1}} K(x,t;y,s)V(x)^{2\alpha}f(y,s)dyds,$$

where

$$K(x,t;y,s) = \frac{1}{\pi}\int_0^\infty \lambda^{-\alpha}\Gamma(x,t;y,s;\lambda)d\lambda \quad \text{for } \ 0<\alpha<1,$$

Let $f \in C_0^\infty(\mathbb{R}^{n+1})$. The adjoint of $V^{2\alpha}(\partial/\partial t+(-\Delta)^2+V^2)^{-\alpha}$ is given by

$$(V^{2\alpha}(\partial/\partial t+(-\Delta)^2+V^2)^{-\alpha})^*f(x,t) = \int_{\mathbb{R}^{n+1}} K(y,s;x,t)V(y)^{2\alpha}f(y,s)dyds.$$

Note that for all $(x,t),(y,s) \in \mathbb{R}^{n+1}$ and $t>s$,

$$(t-s)^{-n/4}\exp\left(-\tilde{C}_1\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right) \leq \frac{1}{d((x,t),(y,s))^n}.$$

By Corollary 2.1 we conclude that for every $N \in \mathbb{N}$, there exists positive constants C_N and $\tilde{C}_1$ such that for all $(x,t),(y,s) \in \mathbb{R}^{n+1}$ and $s>t$,

$$(5.1) \quad |\bar{K}(y,s;x,t)| \leq \frac{C_N}{[d((x,t),(y,s))]^{4-4\alpha}[1+d((x,t),(y,s))m(x,V)]^N}\frac{1}{d((x,t),(y,s))^n}.$$

Let $r=\frac{1}{m(x,V)}$. It follows from Hölder's inequality and (5.1) that

$$\begin{aligned}
&\left|(V^{2\alpha}(\frac{\partial}{\partial t}+(-\Delta)^2+V^2)^{-\alpha})^*f(x,t)\right| \\
&\leq \int_{\mathbb{R}^{n+1}} \frac{C_N}{[d((x,t),(y,s))]^{4-4\alpha}[1+d((x,t),(y,s))m(x,V)]^N}\frac{V(y)^{2\alpha}}{d((x,t),(y,s))^n}|f(y,s)|dyds \\
&\leq C_N\sum_{j=-\infty}^{\infty}\int_{2^{j-1}r<d((x,t),(y,s))\leq 2^jr}\frac{1}{\left(1+2^{j-1}\right)^N}\frac{(2^{j-1}r)^{4\alpha}}{(2^{j-1}r)^{n+4}}V(y)^{\alpha}|f(y,s)|dyds \\
&\leq CC_N\sum_{j=-\infty}^{\infty}\frac{(2^jr)^{4\alpha}}{(1+2^{j-1})^N}\Big(\frac{1}{(2^{j-1}r)^n}\int_{|x-y|\leq 2^jr}V(y)^{q_1}dy\Big)^{2\alpha/q_1} \\
&\times\Big(\frac{1}{(2^{j-1}r)^{n+4}}\int_{d((x,t),(y,s))\leq 2^jr}|f(y,s)|^{q_2}dyds\Big)^{1/q_2}
\end{aligned}$$

$$
\begin{aligned}
&\le CC_N \sum_{j=-\infty}^{\infty} \frac{(2^j r)^{4\alpha}}{(1+2^{j-1})^N} \Big(\frac{1}{(2^{j-1}r)^n} \int_{|x-y|\le 2^j r} V(y)dy \Big)^{2\alpha} \\
&\times \Big(\frac{1}{(2^{j-1}r)^{n+4}} \int_{d((x,t),(y,s))\le 2^j r} |f(y,s)|^{q_2} dyds \Big)^{1/q_2} \\
&= CC_N \sum_{j=-\infty}^{\infty} \frac{1}{(1+2^{j-1})^N} \Big(\frac{1}{(2^{j-1}r)^{n-2}} \int_{|x-y|\le 2^j r} V(y)dy \Big)^{2\alpha} \\
&\times \Big(\frac{1}{(2^{j-1}r)^{n+4}} \int_{d((x,t),(y,s))\le 2^j r} |f(y,s)|^{q_2} dyds \Big)^{1/q_2}.
\end{aligned}
$$

By Lemma 2.5 we conclude that for the case $j \ge 1$ there exists a constant C_0 such that

$$\frac{(2^j r)^2}{|(x, 2^j r)^n|} \int_{|x-y|\le 2^j r} V(y)dy \le C_0 (2^j)^{l_1}.$$

For the case $j \le 0$, by using Lemma 2.2 we see that

$$
\begin{aligned}
\frac{(2^j r)^2}{|(x, 2^j r)^n|} \int_{|x-y|\le 2^j r} V(y)dy &\le C\Big(\frac{r}{2^j r}\Big)^{n/q_1 - 2} \Big(\frac{1}{r^{n-2}} \int_{|x-y|\le r} V(y)dy \Big) \\
&= C(2^j)^{2-n/q_1}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\big| (V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha})^* f(x,t) \big| \\
&\le CC_N (\mathscr{M}(|f|^{q_2})(x,t))^{1/q_2} \sum_{j=-\infty}^{\infty} \left(\frac{(2^j)^{l_1}}{(1+2^{j-1})^N} + (2^j)^{2-n/q_1} \right) \\
&\le C(\mathscr{M}(|f|^{q_2})(x,t))^{1/q_2},
\end{aligned}
$$

where $\mathscr{M}$ is the Hardy-Littlewood maximal operator on $\mathbb{R}^{n+1}$ and we take N sufficiently large.

Then

$$\| (V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha})^* f \|_{L^p(\mathbb{R}^{n+1})} \le C \| f \|_{L^p(\mathbb{R}^{n+1})}, \ (q_1/2\alpha)' \le p < \infty.$$

Therefore,

$$\| (V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha}) f \|_{L^p(\mathbb{R}^{n+1})} \le C \| f \|_{L^p(\mathbb{R}^{n+1})}, \ 1 < p \le q_1/2\alpha. \qquad \blacksquare$$

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