BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37***(1) (2014), 173–180*

On the Total Domination Subdivision Number in Graphs

[1]O. FAVARON, [2]H. KARAMI, [3]R. KHOEILAR AND [4]S. M. SHEIKHOLESLAMI
[1]University Paris Sud and CNRS, Orsay, F-91405 France
[2,3,4]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I. R. Iran
[1]Odile.Favaron@lri.fr, [2]h.karami_math@yahoo.com, [3]ranakhoeilar@yahoo.com,
[4]s.m.sheikholeslami@azaruniv.edu

Abstract. A set $S \subseteq V$ of vertices in a graph $G=(V,E)$ without isolated vertices is a *total dominating set* if every vertex of V is adjacent to some vertex in S. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. The *total domination subdivision number* $\mathrm{sd}_{\gamma_t}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the total domination number. In this paper we prove that $\mathrm{sd}_{\gamma_t}(G) \le \alpha'(G)+1$ for some classes of graphs where $\alpha'(G)$ is the maximum cardinality of a matching of G.

2010 Mathematics Subject Classification: 05C69

Keywords and phrases: Matching, barrier, total domination number, total domination subdivision number.

1. Introduction

Let $G=(V,E)=(V(G),E(G))$ be a simple graph of order $n=|V|$. The *(open) neighborhood* of a vertex u is the set $N_G(u)=\{v|uv\in E\}$ and the *closed neighborhood* of u is the set $N[u]=N(u)\cup\{u\}$. The degree of u is $d_G(u)=|N(u)|$. The minimum degree of a vertex in V is denoted $\delta(G)$ ($d(u),\delta$ for short when no confusion on G is possible); in this paper we assume that for all graphs considered, $\delta(G)\ge 1$. For a set $S\subseteq V$, the *open neighborhood* is the set $N(S)=\cup_{x\in S}N(x)$ and the closed neighborhood is the set $N[S]=N(S)\cup S$.

A vertex y is an *S-private neighbor* of a vertex $x\in S$ if $y\in N[x]\setminus N[S\setminus\{x\}]$, and is an *S-external private neighbor* if $y\in N(x)\setminus N[S\setminus\{x\}]$. Let $d(u,v)$ denote the minimum length of a path from vertex u to vertex v and let $d_2(u)=|\{v|d(u,v)=2\}|$, $\delta_2(G)=\min\{d_2(u)|u\in V(G)\}$. The *eccentricity* of a vertex u equals $ecc(u)=\max\{d(u,v)|v\in V\}$.

A *matching* is a set $M\subseteq E$ of edges no two of which have a vertex in common. The set of vertices *covered by*, or *contained in an edge of* a matching M is denoted $V(M)$. A *perfect matching* is a matching M for which $V(M)=V(G)$. If n is odd, a *near perfect matching* leaves exactly one vertex uncovered, i.e., $|V(M)|=n-1$. A graph is *factor-critical* if the deletion of any vertex leaves a graph with a perfect matching. Note that every non-trivial factor-critical graph has odd order and minimum degree at least 2. The maximum number

---

Communicated by Sanming Zhou.
Received: March 26, 2012; *Revised:* July 19, 2012.

of edges of a matching in G is denoted by $\alpha'(G)$ (α' for short). A cycle of order n is denoted by C_n. We use [19] for terminology and notation which are not defined here.

A set S of vertices of a graph G with minimum degree $\delta(G) > 0$ is a *total dominating set* if $N(S) = V(G)$. The minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$ (γ_t for short), is called the *total domination number* of G. A $\gamma_t(G)$-*set* is a total dominating set of G of cardinality $\gamma_t(G)$. Total domination was introduced by Cockayne, Dawes and Hedetniemi [2] and is now well studied in graph theory. A survey of total domination in graphs can be found in [11]. The concept of total $\{k\}$-domination number has been introduced by Ning Li and Xinmin Hou [14] as a generalization of total domination number and has been studied by several authors (see for example [1, 17]).

The *total domination subdivision number* $\mathrm{sd}_{\gamma_t}(G)$ is the minimum number of edges of G that must be subdivided once in order to increase the total domination number. This kind of concept was first introduced for the domination number by Velammal in his Ph.D. thesis [18]. The total domination subdivision number was considered by Haynes *et al.* in [8] and since then has been studied by several authors (see for example [3–7, 9, 10, 13, 16]). Since the total domination number of the graph K_2 does not change when its unique edge is subdivided, in the study of total domination subdivision number we must assume that the graph has maximum degree at least two.

In some classes of graphs, $\mathrm{sd}_{\gamma_t}(G)$ is bounded above by a constant c. For instance, $c = 3$ for cycles, trees and 2-trees or maximal outerplanar graphs, $c = 4$ for $r \times s$-grids, $c = 2k-1$ for k-regular graphs [8]. But this is not always the case and it is known that the parameter sd_{γ_t} can take arbitrarily large values [9]. An interesting problem is to find good bounds on $\mathrm{sd}_{\gamma_t}(G)$ in terms of other parameters of G. For instance it has been proved that for any graph G of order n, $\mathrm{sd}_{\gamma_t}(G) \le n - \gamma_t(G) + 1$ [6], $\mathrm{sd}_{\gamma_t}(G) \le \lfloor 2n/3 \rfloor$ [7], $\mathrm{sd}_{\gamma_t}(G) \le n - \Delta + 2$ [3] and $\mathrm{sd}_{\gamma_t}(G) \le 2\alpha'(G)$ when $\delta(G) \ge 2$ [16]. The first bound is only sharp for P_3, C_3, K_4, P_6, C_6, the second bound is sharp only for $P_3, C_3, K_{1,3}, K_{1,3}+e, K_4-e, K_5$ or $K_2^c \vee K_3$ where $\vee$ denotes the join operation, the third bound is only sharp for K_n $(n \ge 4)$ and the last bound is sharp for C_3.

As mentioned in Conjectures 2.1 and 2.2 at the end of the paper, we think that every connected graph G of order $n \ge 3$ satisfies $\mathrm{sd}_{\gamma_t}(G) \le \gamma_t(G) + 1$ and $\mathrm{sd}_{\gamma_t}(G) \le \alpha'(G) + 1$. These inequalities are true when $\gamma_t(G) \le \alpha'(G)$ by Theorem 1.2 below. Hence we are interested in proving them when $\gamma_t(G) > \alpha'(G)$. In Section 2, we show that $\mathrm{sd}_{\gamma_t}(G) \le \alpha'(G) + 1$ if G belongs to some particular classes of graphs, in particular if G is in the class $\mathscr{C}$ of graphs such that no vertex belongs to three induced C_4 nor to two induced C_4 and one induced C_6. Theorem 2.2 can be compared to the main result of [5] which says that $\mathrm{sd}_{\gamma_t}(G) \le \gamma_t(G) + 1$ if G belongs to the larger class $\mathscr{C}'$ of graphs such that no vertex belongs to four induced C_4. None of the two results implies the other one when $\gamma_t(G) > \alpha'(G)$.

We will use the following results on $\alpha'(G)$, $\gamma_t(G)$ and $\mathrm{sd}_{\gamma_t}(G)$.

Theorem 1.1. [3] *Let G be a simple connected graph. If $v \in V(G)$ is a support vertex contained in a triangle, then* $\mathrm{sd}_{\gamma_t}(G) \le 2$.

Theorem 1.2. [4] *For any connected graph G with order $n \ge 3$ and $\gamma_t(G) \le \alpha'(G)$,*

$$\mathrm{sd}_{\gamma_t}(G) \le \gamma_t(G) + 1.$$

Theorem 1.3. [4] *If G is a connected graph such that $\delta(G) \ge 3$, $\gamma_t(G) = \delta(G) + 1$ and G contains a vertex v with $d(v) = \delta(G)$ and $ecc(v) = 2$, then $\gamma_t(G) \le \alpha'(G)$.*

Theorem 1.4. [8] *For any graph G having a vertex of degree two which is contained in a triangle,* $1 \leq \mathrm{sd}_{\gamma_t}(G) \leq 3$.

Theorem 1.5. [9] *For any connected graph G with adjacent vertices u and v, each of degree at least two,*

$$\mathrm{sd}_{\gamma_t}(G) \leq d(u) + d(v) - |N(u) \cap N(v)| - 1 = |N(u) \cup N(v)| - 1.$$

Theorem 1.6. [3] *Let G be a connected graph of minimum degree at least 2. Then* $sd_{\gamma_t}(G) \leq \delta_2(G) + 3$.

Theorem 1.7. [12] *For any claw-free graph G with* $\delta(G) \geq 3$, $\gamma_t(G) \leq \alpha'(G)$.

Theorem 1.8. [12] *For every k-regular graph G with* $k \geq 3$, $\gamma_t(G) \leq \alpha'(G)$.

Theorem 1.9. [16] *For any connected graph G of order* $n \geq 3$ *with* $\alpha'(G) = 1$ or 2, $\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G) + 1$.

In the proof of Theorem 2.2 below, we use the concept of barrier. If S is a *separator* of a connected graph G, $o(G - S)$ denotes the number of odd components of $G - S$, i.e., components of odd order. Tutte's Theorem says that a connected graph admits a matching covering all its vertices if and only if $o(G - S) \leq |S|$ for every $S \subseteq V(G)$. A *barrier* of G is a separator S such that $o(G - S) = |S| + t$ where $t = n - 2\alpha'(G)$ is the number of vertices of G which are not covered by a maximum matching. By Berge's Formula, every connected graph admits barriers. Moreover (see for example exercise 3.3.18 in [15]) if S is an inclusion-wise maximal barrier, then $G - S$ admits $|S| + t$ components G_i which are all factor-critical (hence odd), and every maximum matching of G is formed by a matching pairing S with $|S|$ different components of $G - S$ and a near perfect matching in each component. Therefore, with the notation $|S| + t = \ell$ and $|V(G_i)| = n_i$,

$$\alpha'(G) = |S| + \sum_{i=1}^{\ell} \frac{n_i - 1}{2}. \tag{1.1}$$

The reader can find more details on factor-critical graphs, Berge's Formula and barriers in Sections 3.1 and 3.3 of [15].

2. Main results

In this section, we determine some classes of graphs such that $\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G) + 1$. The first two corollaries are immediate consequences of Theorems 1.2 and 1.8.

Corollary 2.1. *For any connected graph G with order* $n \geq 3$ *and* $\gamma_t(G) \leq \alpha'(G)$,

$$\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G) + 1.$$

The bound is sharp for K_4 *and* K_5.

Corollary 2.2. *For every k-regular graph G with* $k \geq 3$,

$$\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G) + 1.$$

The bound is sharp for K_4 *and* K_5.

The beginning of the proof of the following theorem is nearly the same as the proof of Theorem 2 in [4] where it is shown that $n \geq 3$ and $\delta = 1$ imply $\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G)$.

Theorem 2.1. *Every connected graph G of order $n \geq 3$ with $\delta = 1$ satisfies*

$$\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G) + 1.$$

This bound is sharp.

Proof. Let $v \in V$ be a vertex of degree one, $uv \in E(G)$ and $N(u) \setminus \{v\} = \{u_1, \ldots, u_k\}$. If $u_i u_j \in E(G)$ for some i and j, then u is a support vertex contained in a triangle and $\mathrm{sd}_{\gamma_t}(G) \leq 2$ by Theorem 1.1. Now let $N(u) \setminus \{v\}$ be an independent set. If $N(u_i) \setminus \{u\} = \emptyset$ for every $1 \leq i \leq k$, then G is a star, $\mathrm{sd}_{\gamma_t}(G) = 2$ and the result follows. Assume $N(u_1) \setminus \{u\} = \{w_1, \ldots, w_r\}$. We claim that subdividing the edges uv, uu_1 and $u_1 w_i$ for $1 \leq i \leq r$ increases $\gamma_t(G)$. Let G' be obtained from G by subdividing the edge uv with a vertex x_1, the edge uu_1 with a vertex x_2, and the edge $u_1 w_i$ with a vertex z_i for $1 \leq i \leq r$. Let Z be the set of the $r+2$ subdividing vertices and let D be a $\gamma_t(G')$-set. Without loss of generality we may assume $u, x_1 \in D$. If $u_1 \in D$, then obviously $D \setminus Z$ is a TDS of G smaller than D. Let $u_1 \notin D$. In order to dominate u_1, we must have $D \cap (Z \setminus \{x_1\}) \neq \emptyset$. Then $(D \setminus Z) \cup \{u_1\}$ is a TDS of G smaller than D and this proves the claim. Let T be a smallest set of edges of $\{u_1 w_i \mid 1 \leq i \leq r\}$ such that subdividing the edges uv, uu_1 and $u_1 w$ for each $u_1 w \in T$ increases the total domination number of G. Without loss of generality, we assume $T = \{u_1 w_i \mid 1 \leq i \leq s\}$. By the definition of T, $\mathrm{sd}_{\gamma_t}(G) \leq s + 2$. We may assume $s \geq 2$ for otherwise $\mathrm{sd}_{\gamma_t}(G) \leq 3 \leq \alpha'(G) + 1$. Let G_1 be the graph obtained from G by subdividing the edge uv with vertex x, the edge uu_1 with vertex y and the edge $u_1 w_i$ with vertex z_i for $i = 1, \ldots, s-1$. By the definition of T, $\gamma_t(G_1) = \gamma_t(G)$. Let S be a $\gamma_t(G_1)$-set. We may assume $u, x \in S$. Since $\gamma_t(G_1) = \gamma_t(G)$, we have $u_1 \notin S$ and $y \notin S$ for otherwise $(S \setminus \{x, y, z_1, \ldots, z_{s-1}\}) \cup \{u_1\}$ is a total dominating set of order at most $|S| - 1$ of G. If $S \cap \{z_1, \ldots, z_{s-1}\} \neq \emptyset$, then $(S \setminus \{z_1, \ldots, z_{s-1}, x\}) \cup \{u_1\}$ is a total dominating set of G of order at most $|S| - 1$, a contradiction. Suppose that $S \cap \{z_1, \ldots, z_{s-1}\} = \emptyset$. Thus $w_i \in S$ for $1 \leq i \leq s-1$ to dominate z_i and $S \cap \{w_s, \ldots, w_r\} \neq \emptyset$ to dominate u_1. Without loss of generality, $w_s \in S$. The set $S' = (S \setminus \{x\}) \cup \{u_1\}$ is a $\gamma_t(G)$-set and for $1 \leq i \leq s$, the vertex w_i, which is not isolated in S', admits an external S'-private neighbor t_i. Now $M = \{uv, w_i t_i \mid 1 \leq i \leq s\}$ is a matching of G. Hence $\alpha' \geq s + 1$ and $\mathrm{sd}_{\gamma_t}(G) \leq s + 2 \leq \alpha'(G) + 1$. Stars show that the bound is attained. ∎

By Theorems 2.1, 1.5, 1.6, 1.9 and Corollary 2.1, the inequality $\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G) + 1$ is satisfied if $\delta(G) = 1$, or if $\alpha'(G) \leq 2$, or if $\gamma_t(G) \leq \alpha'(G)$, or if $|N(u) \cup N(v)| \leq \alpha'(G) + 2$ for some edge uv of G joining two vertices of degree at least 2, or if $\delta(G) \geq 2$ and $\delta_2(G) \leq \alpha'(G) - 2$. Hence we assume from now that $\delta(G) \geq 2$ and

$$3 \leq \alpha'(G) \leq \gamma_t(G) - 1, \tag{2.1}$$

$$|N(u) \cup N(v)| \geq \alpha'(G) + 3 \quad \text{for each edge } uv \text{ of } G, \tag{2.2}$$

$$\delta_2(G) \geq \alpha'(G) - 1. \tag{2.3}$$

Theorem 2.2. *Every connected graph G of order $n \geq 3$ such that no vertex belongs to three induced C_4 nor to two induced C_4 and one induced C_6 satisfies*

$$\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G) + 1.$$

This bound is sharp.

Proof. Let S be a maximal barrier of G and $G_1, G_2, \ldots, G_\ell$ the components of $G-S$ with $|V(G_i)| = n_i$ and $n_1 \geq n_2 \geq \cdots \geq n_\ell$. Let S_1 be the set of the isolated vertices of $G[S]$.

Case 1. G_1 and G_2 are not trivial. Then by (1.1), $\alpha' \geq |S| + (n_1-1)/2 + (n_2-1)/2$. Let uv be an edge of G_2. Then

$$|N(u) \cup N(v)| \leq n_2 + |S| \leq \frac{n_1+n_2}{2} + |S| \leq \alpha' + 1,$$

contrary to (2.2). Therefore Case 1 is impossible.

Case 2. All the components G_i are trivial. Then $\alpha' = |S|$.

Let $V(G_i) = \{y_i\}$ for $1 \leq i \leq \ell$ and $Y = \{y_1, \ldots, y_\ell\}$. If $S_1 = \emptyset$, then S is a total dominating set of order α' of G. If $S_1 = \{x\}$, let w be a neighbor of x. Since $\delta \geq 2$, every vertex of Y has a neighbor in $S \setminus \{x\}$. Thus $(S \setminus \{x\}) \cup \{w\}$ is a total dominating set of order α' of G. If $S_1 = \{x, z\}$, then every vertex of $N(x) \setminus N(z)$ and of $N(z) \setminus N(x)$ has a neighbor in $S \setminus S_1$. If $y \in N(x) \cap N(z)$, then $(S \setminus \{z\}) \cup \{y\}$ is a total dominating set of order α' of G. If $N(x) \cap N(z) = \emptyset$, then $(S \setminus \{x,z\}) \cup \{x', z'\}$, where $x' \in N(x)$ and $z' \in N(z)$ is a total dominating set of order α' of G. All cases contradict (2.1). Therefore $|S_1| \geq 3$.

Let $x \in S_1$, $N(x) = \{w_1, w_2, \ldots, w_r\} \subseteq Y$ with $d(w_{i_0}) \leq d(w_i)$ for $1 \leq i \leq r$, and $W_i = N(w_i) \setminus \{x\}$. Note that w_{i_0} is a vertex of $N(x)$ with least degree. Since x is contained in at most two induced C_4, $W_i \cap W_j \cap W_k = \emptyset$ for each triple i, j, k and $|W_i \cap W_j| \leq 1$ for each pair i, j. Moreover $|W_i \cap W_j| = 1$ for at most two pairs of indices. Hence $\sum_{1 \leq i < j \leq r} |W_i \cap W_j| \leq 2$. Therefore, by the inclusion-exclusion principle and since $|W_{i_0}| \geq \delta - 1 \geq 1$, we have

$$r \leq r|W_{i_0}| \leq \sum_{i=1}^{r} |W_i| = \left| \bigcup_{1 \leq i \leq r} W_i \right| + \sum_{1 \leq i < j \leq r} |W_i \cap W_j| \leq |S| + 1. \tag{2.4}$$

Hence $|S| \geq r-1$ and

$$|W_{i_0}| \leq \left\lfloor \frac{|S|+1}{r} \right\rfloor.$$

(i) If $|S| \geq r+1$ then, since $r = d(x) \geq 2$,

$$|N(x) \cup N(w_{i_0})| = d(x) + d(w_{i_0}) = r + |W_{i_0}| + 1 \leq r + \frac{|S|+1}{r} + 1 \leq |S| + 2 = \alpha' + 2$$

contrary to (2.2).

(ii) If $|S| = r$ then $\lfloor (|S|+1)/r \rfloor = 1$ and $d(x) + d(w_{i_0}) = r + |W_{i_0}| + 1 = r + 2 = \alpha' + 2$ contrary to (2.2).

(iii) If $|S| = r-1$, then all inequalities in (2.4) become equalities and $|W_i| = 1$ for all i. Let y, z be two other vertices of S_1. By (2.3), every vertex of S_1 is at distance two from each other vertex of S and the three vertices x, y, z are mutually at distance two. Therefore there exist three internally disjoint paths of length two, say xay, ybz, zcx, joining them. Hence x belongs to the induced C_6 $xaybzcx$. Equalities in (2.4) also imply that either $|W_i \cap W_j| = 1$ for two pairs of indices or $|W_i \cap W_j| = 2$ for one pair. Therefore x also belongs to two induced C_4, which contradicts the hypothesis. Whence Case 2 is impossible.

Case 3 G_1 is the unique non-trivial component of $G-S$. Then $\alpha' = |S| + (n_1-1)/2$.
The proof of Case 3 is quite similar to the proof of Theorem 1 in [5]. We repeat the main arguments for the sake of self-containedness. Let $V(G_i) = \{y_i\}$ for $2 \leq i \leq \ell$ and $Y =$

$\{y_2,\ldots,y_\ell\}$. The component G_1 has order at least 5 for otherwise $n_1 = 3$ and each edge uv of G_1 satisfies

$$|N(u)\cup N(v)| \le |S| + n_1 = |S| + \frac{n_1-1}{2} + 2 = \alpha' + 2$$

contrary to (2.2). It is proved in [5] that G_1 admits a total dominating set X of order $(n_1-1)/2$. If some isolated vertex x of $G[S]$ has no neighbor in G_1, then $\delta_2(G) \le |S|-1 < \alpha'-1$ contrary to (2.3). Hence every vertex of S_1 has a neighbor in G_1.

If $S_1 = \emptyset$ or if every vertex of S_1 has a neighbor in X, then $S\cup X$ is a total dominating set of G and thus $\gamma_t(G) \le |S|+|X| = \alpha'(G)$, contrary to (2.1). Hence the set S_2 of the isolated vertices of $G[S]$ with no neighbor in X is not empty.

If $N(y_i) \not\subseteq S_2$ for each i with $2 \le i \le \ell$, we associate to each vertex x of S_2 one of its neighbors $f(x)$ in $V(G_1)\setminus X$ (recall that each vertex of S_1 has at least one neighbor in G_1) and we let $S_2' = \{f(x) \mid x \in S_2\}$. Clearly $|S_2'| \le |S_2|$, S_2' dominates S_2, and $X\cup S_2'$ is a total dominating set of $V(G_1)\cup S_1$. Therefore $(S\setminus S_2)\cup X\cup S_2'$ is a total dominating set of G and $\gamma_t(G) \le |S|+|X| = \alpha'(G)$, contrary to (2.1).

Hence some vertex y_i of Y, say y_2, has all its neighbors in S_2 and $|S_2| \ge 2$ since $\delta(G) \ge 2$. If uv is an edge of $G_1[X]$, then $N(u)$ and $N(v)$ are contained in $V(G_1)\cup(S\setminus S_2)$. By (2.2) we have

$$|S| + \frac{n_1-1}{2} = \alpha' \le |N(u)\cup N(v)| - 3 \le n_1 + |S| - |S_2| - 3 \le n_1 + |S| - 5.$$

Therefore $n_1 \ge 9$.

Let z_1 and z_2 be two neighbors of y_2 with $d_{G_1}(z_1) \le d_{G_1}(z_2)$. The neighborhoods $N_{G_1}(z_1)$ and $N_{G_1}(z_2)$ are contained in $V(G_1)\setminus X$. Each vertex of $N_{G_1}(z_1)\cap N_{G_1}(z_2)$ induces with y_2, z_1, z_2 an induced cycle C_4. Therefore $|N_{G_1}(z_1)\cap N_{G_1}(z_2)| \le 2$. Then

$$(2.5)\qquad |N_{G_1}(z_1)| \le \left\lfloor \frac{|N_{G_1}(z_1)| + |N_{G_1}(z_2)|}{2} \right\rfloor \le \left\lfloor \frac{1}{2}\left(\frac{n_1+1}{2}+2\right) \right\rfloor = \left\lfloor \frac{n_1+5}{4} \right\rfloor.$$

Let $A = N_Y(z_1)\setminus\{y_2\}$ and $B = N(y_2)\setminus\{z_1\}$ $(\subseteq S_2)$. For each $a \in A$, let a' be one of its neighbors in $S\setminus\{z_1\}$ (a' exists since $\delta(G) \ge 2$) and let $A' = \{a' \mid a \in A\}$. Then $|A'| \le |A|$ and $|A|-|A'|$ is at most the number of pairs a_i, a_j of vertices of A such that $a_i' = a_j'$. Note that if $a_i' = a_j'$, then $\{a_i', a_i, a_j, z_1\}$ induces a C_4 not containing y_2. Since the set $B\cup A'$ is contained in $S\setminus\{z_1\}$, $|B\cup A'| \le |S|-1$. Each vertex a' of $B\cap A'$ corresponds to at least one induced C_4 of the form $z_1y_2a'az_1$ (possibly more if a' is associated to several vertices of A). Since z_1 belongs to at most two induced C_4, $|A|-|A'|+|B\cap A'| \le 2$. Therefore

$$\begin{aligned}|N_Y(z_1)| + |N(y_2)| &= |A|+1+|B|+1 = |A|-|A'|+|A'|+|B|+2\\ &= |A|-|A'|+|A'\cup B|+|A'\cap B|+2\\ &\le |A'\cup B|+4 \le |S|+3.\end{aligned}$$

Since $N(z_1)\cap N(y_2) = \emptyset$ and n_1 is odd ≥ 9, and by Theorem 1.5 and (2.5), we get

$$\begin{aligned}\mathrm{sd}_{\gamma_t}(G) &\le |N(z_1)|+|N(y_2)|-1 \le |N_{G_1}(z_1)|+|N_Y(z_1)|+|N(y_2)|-1\\ &\le \left\lfloor \frac{n_1+5}{4} \right\rfloor + (|S|+3) - 1 \le |S| + \frac{n_1-1}{2} + 1 \le \alpha'(G)+1.\end{aligned}$$

This completes the proof of Theorem 2.2. Stars show that the bound is attained. ∎

The bound is also attained by stars in the following two corollaries.

Corollary 2.3. *For any connected graph G of order $n \geq 3$ with no induced C_4,* $\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G)+1$.

Corollary 2.4. *For any connected chordal graph G of order $n \geq 3$,* $\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G)+1$.

Theorem 2.3. *For any connected claw-free graph G of order $n \geq 3$,*

$$\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G)+1.$$

Furthermore, this bound is sharp for K_4 and K_5.

Proof. Since $\delta(G) \geq 2$ and by Theorem 1.7 and Corollary 2.1, we can assume $\delta = 2$. Let v be a vertex of degree 2 and $N(v) = \{v_1, v_2\}$. By Theorem 1.4 and since $\alpha' \geq 3$ by (2.1), we can also assume $v_1v_2 \notin E(G)$. From (2.2) applied to the edge vv_i, $1 \leq i \leq 2$, we get $d(v_1) \geq 4$ and $d(v_2) \geq 4$. Since G is claw-free, $N(v_1)\setminus\{v\}$ and $N(v_2)\setminus\{v\}$ induce cliques of order at least 3 in G. Let y_1, y_2 be two vertices in $N(v_1)\setminus\{v\}$ and z_1, z_2 two vertices in $N(v_2)\setminus\{v\}$ such that the two edges y_1y_2 and z_1z_2 are distinct. Let G' be the graph obtained from G by subdividing the four edges vv_1, vv_2, y_1y_2 and z_1z_2 respectively by vertices w_1, w_2, t_1, t_2, and let G'' be obtained from G by uniquely subdividing y_1y_2 and z_1z_2. Let S be a $\gamma_t(G')$-set. To dominate t_1, t_2 and v, we have $S \cap \{y_1, y_2\} \neq \emptyset$, $S \cap \{z_1, z_2\} \neq \emptyset$ and $S \cap \{w_1, w_2\} \neq \emptyset$. Suppose $\{y_1, z_1, w_1\} \subseteq S$ (the two vertices y_1, z_1 are not necessarily distinct). If $\{v_1, v_2\} \subseteq S$ then $S \setminus \{w_1, w_2\}$ is a total dominating set of G'' smaller than S. If $v_k \notin S$ for some $k \in \{1,2\}$, then $v \in S$ and $(S\setminus\{v, w_1, w_2\}) \cup \{v_k\}$ is a total dominating set of G'' smaller than S. Therefore $\gamma_t(G) \leq \gamma_t(G'') < \gamma_t(G')$ and $\mathrm{sd}_{\gamma_t}(G) \leq 4 \leq \alpha'(G)+1$. The proof is complete. ∎

Theorem 2.4. *If G is a connected graph with a vertex v of degree δ and eccentricity 2, then* $\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G)+1$.

Proof. Let $N(v) = \{v_1, \ldots, v_\delta\}$. The set $N[v]$ is a total dominating set of G of order $\delta+1$. Hence by (2.1), $4 \leq \alpha'(G)+1 \leq \gamma_t(G) \leq \delta+1$ and $\delta \geq 3$. If $\gamma_t(G) = \delta+1$, then the result follows from Theorem 1.3 and Corollary 2.1. Therefore we can suppose

$$4 \leq \alpha'(G)+1 \leq \gamma_t(G) \leq \delta. \tag{2.6}$$

Let D be a smallest subset of $N[v]$ such that D is a total dominating set of G. If possible, we choose D containing v. Let $D\setminus\{v\} = \{v_1, \ldots, v_r\}$. If v_i does not have a D-external private neighbor for some i, then the set $(D\setminus\{v_i\}) \cup \{v\}$ is a total dominating set contradicting the choice of D. Thus we may assume v_i has a D-external private neighbor w_i for each i with $1 \leq i \leq r$. The edges $v_1w_1, \ldots, v_rw_r$ form a matching M of G of size at least $|D|-1 \geq \gamma_t(G)-1$. Hence $\alpha'(G) \geq \gamma_t(G)-1$. By (2.6), $\alpha'(G) = \gamma_t(G)-1$, which shows that M is a maximum matching. Therefore $r = \delta$ for otherwise $M \cup \{vv_\delta\}$ is a matching greater than M. Let $Y = V(G)\setminus(D \cup \{w_1, \ldots, w_\delta\})$. Since M is a maximum matching, Y is an independent set of G and there is no edge joining a vertex of Y to a vertex w_i for $1 \leq i \leq \delta$. Since $\deg(w_1) \geq \delta$ and since w_1 is adjacent to only v_1 and possibly w_i for $2 \leq i \leq \delta$, the vertex w_1 dominates $\{w_1, \ldots, w_\delta\}$. On the other hand, since $\deg(y) \geq \delta$ for each $y \in Y$, every vertex y of Y, if any, is adjacent to every v_i. Hence $\{v, v_1, w_1\}$ is a total dominating set of G, which contradicts $\gamma_t \geq 4$. ∎

We conclude this paper with the following conjectures.

Conjecture 2.1. *For any connected graph G of order $n \geq 3$,*

$$\mathrm{sd}_{\gamma_t}(G) \leq \alpha'(G)+1.$$

Conjecture 2.2. [4] *For any connected graph G of order $n \geq 3$,*

$$\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1.$$

Conjecture 2.3. [4] *For any connected graph G of order $n \geq 3$,*

$$\mathrm{sd}_{\gamma_t}(G) \leq \frac{n+1}{2}.$$

Conjecture 2.4. [4] *For any connected claw-free graph G of order $n \geq 3$,*

$$\mathrm{sd}_{\gamma_t}(G) \leq \frac{\gamma_t(G)}{2} + 2.$$

It follows from Theorem 1.2 that Conjecture 2.1 implies Conjecture 2.2. Also note that for any connected graph G of odd order $n \geq 3$ we have $\alpha'(G) \leq (n-1)/2$ and hence Conjecture 2.1 implies Conjecture 2.3 for connected graphs of odd order.

References

[1] H. Aram, S. M. Sheikholeslami and L. Volkmann, On the total $\{k\}$-domination and total $\{k\}$-domatic number of graphs, *Bull. Malays. Math. Sci. Soc. (2)* **36** (2013), no. 1, 39–47.

[2] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, Total domination in graphs, *Networks* **10** (1980), no. 3, 211–219.

[3] O. Favaron, H. Karami, R. Khoeilar and S.M. Sheikholeslami, A new bound on the total domination subdivision number, *Graphs Combin.* **25** (2009), no. 1, 41–47.

[4] O. Favaron, H. Karami, R. Khoeilar and S.M. Sheikholeslami, On the total domination subdivision number in some classes of graphs, *J. Comb. Optim.* **20** (2010), no. 1, 76–84.

[5] O. Favaron, H. Karami, R. Khoeilar and S. M. Sheikholeslami, Matchings and total domination subdivision number in graphs with few induced 4-cycles, *Discuss. Math. Graph Theory* **30** (2010), no. 4, 611–618.

[6] O. Favaron, H. Karami and S. M. Sheikholeslami, Total domination and total domination subdivision numbers, *Australas. J. Combin.* **38** (2007), 229–235.

[7] O. Favaron, H. Karami and S. M. Sheikholeslami, Bounding the total domination subdivision number of a graph in terms of its order, *J. Comb. Optim.* **21** (2011), no. 2, 209–218.

[8] T. W. Haynes, S. T. Hedetniemi and L. C. van der Merwe, Total domination subdivision numbers, *J. Combin. Math. Combin. Comput.* **44** (2003), 115–128.

[9] T. W. Haynes, M. A. Henning and L. S. Hopkins, Total domination subdivision numbers of graphs, *Discuss. Math. Graph Theory* **24** (2004), no. 3, 457–467.

[10] T. W. Haynes, M. A. Henning and L. Hopkins, Total domination subdivision numbers of trees, *Discrete Math.* **286** (2004), no. 3, 195–202.

[11] M. A. Henning, A survey of selected recent results on total domination in graphs, *Discrete Math.* **309** (2009), no. 1, 32–63.

[12] M. A. Henning, L. Kang, E. Shan and A. Yeo, On matching and total domination in graphs, *Discrete Math.* **308** (2008), no. 11, 2313–2318.

[13] H. Karami, A. Khodkar and S. M. Sheikholeslami, An upper bound for total domination subdivision numbers, *Ars Combin.* **102** (2011), 321–331.

[14] N. Li and X. Hou, On the total $\{k\}$-domination number of Cartesian products of graphs, *J. Comb. Optim.* **18** (2009), no. 2, 173–178.

[15] L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland Mathematics Studies, 121, North-Holland, Amsterdam, 1986.

[16] S. M. Sheikholeslami, On the total domination subdivision numbers in graphs, *Cent. Eur. J. Math.* **8** (2010), no. 3, 468–473.

[17] S. M. Sheikholeslami and L. Volkmann, The total $\{k\}$-domatic number of a graph, *J. Comb. Optim.* **23** (2012), no. 2, 252–260.

[18] S. Velammal, *Studies in Graph Theory: Covering, Independence, Domination and Related Topics*, Ph.D Thesis, Manonmaniam Sundaranar University, Tirunelveli, 1997.

[19] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, NJ, 1996.