Travelling Wave Solution of the Kolmogorov-Petrovskii-Piskunov Equation by the First Integral Method

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Abstract. In this paper, the first integral method is proposed to solve the Kolmogorov-Petrovskii-Piskunov equation. New exact travelling wave solutions of the Kolmogorov-Petrovskii-Piskunov equation are obtained that illustrate the efficiency of the method.

1. Introduction

The nonlinear reaction-diffusion equations play fundamental role in a great number of various models of reaction-diffusion processes, mathematical biology, chemistry, genetics and so on. Thus, one of this diffusion equation is the Kolmogorov-Petrovskii-Piskunov (KPP) equation [14]. The Cauchy problem of the Kolmogorov-Petrovskii-Piskunov

$$(1.1) \qquad u_{xx} - u_t = f(u), \quad f \text{ nonlinear, } f(0) = 0,$$

has been extensively investigated both by analytic techniques [3, 12], and by probabilistic methods [4, 17], and the existence of traveling wave solutions with various velocities has been also proved.

In this paper, we consider the following Kolmogorov-Petrovskii-Piskunov equation [14, 16]

$$(1.2) \qquad u_{xx} - u_t + \alpha u + \beta u^2 + \gamma u^3 = 0,$$

where α, β, γ are three real constants. Notice that various equation belongs to the Kolmogorov-Petrovskii-Piskunov type. The Kolmogorov-Petrovskii-Piskunov equation (1.2) contains the various equations with $\alpha + \beta + \gamma = 0$, for which there is always the condition $\Delta = \beta^2 - 4\alpha\gamma = (\alpha - \gamma)^2 \geq 0$ and thus has five explicit solutions, such as the Fisher equation. The nonintegrable Newell-Whitehead equation and the FitzHugh-Nagumo equation (see for instance [2, 5, 10, 11, 13, 15, 18, 19]). Our purpose is to look for new exact travelling wave solutions for the general case of (1.2) by the first integral method.

The first integral method was first proposed by Feng [7] in solving Burgers-KdV equation. Recently, this useful method is widely used by many such as in [1, 8, 9, 20] and by the reference therein.

The remaining structure of this article is organized as follows: Section 2 is a brief introduction to the first integral method for finding exact travelling wave solutions of nonlinear equations. In Section 3, we illustrate this method in detail with the KPP equation. In Section 4, some conclusions are given.

2. The first integral method

In [20], the first integral method is summarized by Raslan as follows.

   Step 1: Consider a general nonlinear PDE in the form

$$(2.1) \qquad\qquad P(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, u_{xxx}, ...) = 0.$$

Using the wave variable $\xi = x - ct$, we can transform Eq. (2.1) into the following ordinary differential equation (ODE)

$$(2.2) \qquad\qquad Q(U, U', U'', U''', ...) = 0.$$

where the prime denotes the derivative with respect to ξ.

   Step 2: Assume that the solution of ODE (2.2) can be written as follows

$$(2.3) \qquad\qquad u(x,t) = f(\xi).$$

   Step 3: We introduce new independent variables

$$(2.4) \qquad\qquad X(\xi) = f(\xi), \qquad Y(\xi) = f_\xi(\xi),$$

which change (2.2) to a system of ODEs

$$(2.5) \qquad\qquad \begin{cases} X(\xi) = f(\xi), \\ Y(\xi) = F(X(\xi), Y(\xi)). \end{cases}$$

   Step 4: By the qualitative theory of ODEs [6], if we can find the integrals to (2.5) under the same conditions, then the general solutions to (2.5) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first integral to (2.5) which reduces (2.2) to a first order integrable ordinary differential equation. An exact solution to (2.1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Theorem 2.1. *Suppose that $K(w,z)$, $H(w,z)$ are polynomials in $\mathbb{C}[w,z]$ and $K(w,z)$ is irreducible in $\mathbb{C}[w,z]$. If $H(w,z)$ vanishes at all zero points of $K(w,z)$, then there exists a polynomial $G(w,z)$ in $\mathbb{C}[w,z]$ such that*

$$(2.6) \qquad\qquad H(w,z) = K(w,z)G(w,z).$$

3. Exact solutions for KPP equation

Let us consider the KPP equation as the following

(3.1) $$u_{xx} - u_t + \alpha u + \beta u^2 + \gamma u^3 = 0.$$

Using the wave variable $\xi = x - ct$ carries (3.1) into an ODE as follows

(3.2) $$u'' + cu' + \alpha u + \beta u^2 + \gamma u^3 = 0,$$

where prime denotes the derivative with respect to the same variable ξ.

By selection in (2.4), equation (3.2) is transformed to the following system of ODEs

(3.3) $$\begin{cases} X'(\xi) = Y(\xi), \\ Y'(\xi) = -cY(\xi) - \alpha X(\xi) - \beta X(\xi)^2 - \gamma X(\xi)^3. \end{cases}$$

Now, we apply the Division Theorem to look for the first integral to (3.3). Suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions to (3.3), and

(3.4) $$p(X,Y) = \sum_{j=0}^{m} a_j(X)Y^j,$$

is an irreducible polynomial in $\mathbb{C}[X,Y]$ such that

(3.5) $$p(X(\xi),Y(\xi)) = \sum_{j=0}^{m} a_j(X(\xi))Y(\xi)^j = 0,$$

where $a_j(X)$, $(j = 0,1,\ldots,m)$ are polynomials of X and all relatively prime in $\mathbb{C}[X,Y]$, $a_m(X) \neq 0$. Eq. (3.5) is also called the first integral of (3.3). Note that $p(X(\xi),Y(\xi))$ is a polynomial in X and Y, and $dp/d\xi$ implies $(dp/d\xi)|_{(3.4)} = 0$. By the Division Theorem, there exists a polynomial $H(X,Y) = (h(X) + g(X)Y)$ in $\mathbb{C}[X,Y]$ such that

(3.6) $$\frac{dp}{d\xi}\big|_{(3.4)} = \left(\frac{dp}{dX}\frac{dX}{d\xi} + \frac{dp}{dY}\frac{dY}{d\xi} \right)\big|_{(3.4)} = (h(X) + g(X)Y)(\sum_{j=0}^{m} a_j(X)Y^j).$$

Case I: Assume the $m = 1$, form (3.6) we have

(3.7) $$\sum_{j=0}^{1} a'_j(X)Y^{j+1} + \sum_{j=0}^{1} j a_j(X)Y^{j-1}(-cY(\xi) - \alpha X(\xi) - \beta X(\xi)^2 - \gamma X(\xi)^3)$$
$$= (h(X) + g(X)Y)(a_0(X) + a_1(X)Y),$$

and by equating the coefficients of Y^j $(j = 0,1,2)$ on both sides of Equation (3.7), we obtain

(3.8) $$a'_1(X) = g(X)a_1(X),$$

(3.9) $$a'_0(X) - ca_1(X) = h(X)a_1(X) + g(X)a_0(X),$$

(3.10) $$a_1(X)(-\alpha X(\xi) - \beta X(\xi)^2 - \gamma X(\xi)^3) = h(X)a_0(X).$$

Since $a_j(X),(j = 0,1)$ are polynomials, then from (3.8) we conclude that $a_1(X)$ is constant and $g(X) = 0$. For simplicity, let us take $a_1(X) = 1$. Balancing the degrees of $h(X)$ and $a_0(X)$, we conclude that $\deg(h(X)) = 1$ only. Assume that $h(X) = AX + B$, where $A \neq 0$, then

(3.11) $$a_0(X) = \frac{1}{2}AX^2 + (B+c)X + D.$$

Substituting $a_0(X)$, $a_1(X)$ and $h(X)$ in Equation (3.10) and setting all the coefficients of powers of X equal to zero, then we obtain a system of nonlinear algebraic equations such as

$$B^2 + Bc + AD = -\alpha, \quad \frac{3}{2}AB + Ac = -\beta, \quad \frac{A^2}{2} = -\gamma, \quad BD = 0,$$

and using Mathematica solving them, we obtain

(3.12) $\quad D = 0, B = \dfrac{i(-\beta + \sqrt{\beta^2 - 4\alpha\gamma})}{\sqrt{2\gamma}}, \quad A = -i\sqrt{2\gamma}, \quad c = -\dfrac{i(-\beta + 3\sqrt{\beta^2 - 4\alpha\gamma})}{2\sqrt{2\gamma}},$

(3.13) $\quad D = 0, B = \dfrac{i(\beta + \sqrt{\beta^2 - 4\alpha\gamma})}{\sqrt{2\gamma}}, \quad A = i\sqrt{2\gamma}, \quad c = -\dfrac{i(\beta + 3\sqrt{\beta^2 - 4\alpha\gamma})}{2\sqrt{2\gamma}},$

(3.14) $\quad D = 0, B = -\dfrac{i(\beta + \sqrt{\beta^2 - 4\alpha\gamma})}{\sqrt{2\gamma}}, \quad A = -i\sqrt{2\gamma}, \quad c = \dfrac{i(\beta + 3\sqrt{\beta^2 - 4\alpha\gamma})}{2\sqrt{2\gamma}},$

(3.15) $\quad D = 0, B = -\dfrac{i(-\beta + \sqrt{\beta^2 - 4\alpha\gamma})}{\sqrt{2\gamma}}, \quad A = i\sqrt{2\gamma}, \quad c = \dfrac{i(-\beta + 3\sqrt{\beta^2 - 4\alpha\gamma})}{2\sqrt{2\gamma}},$

(3.16) $\qquad\qquad D = -\dfrac{i\alpha}{\sqrt{2\gamma}}, \quad B = 0, \quad A = -i\sqrt{2\gamma}, \quad c = -\dfrac{i\beta}{\sqrt{2\gamma}},$

(3.17) $\qquad\qquad D = \dfrac{\alpha}{\sqrt{2\beta}}, \quad B = 0, \quad A = \sqrt{2\gamma}, \quad c = \dfrac{i\beta}{\sqrt{2\gamma}}.$

Using (3.12) into (3.11) and (3.5), we have

(3.18) $$Y = \frac{iX(\beta + 2X\gamma + \sqrt{\beta^2 - 4\alpha\gamma})}{2\sqrt{2\gamma}},$$

combining (3.18) with (3.3), we obtain the exact solution of KPP Equation (3.2) can be written as

$$u_1(\xi) = -\frac{e^{\frac{\left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\right)(i\sqrt{2}\xi + 4\sqrt{\gamma}\xi_0)}{4\sqrt{\gamma}}} \left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\right)}{-1 + 2e^{\frac{\left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\right)(i\sqrt{2}\xi + 4\sqrt{\gamma}\xi_0)}{4\sqrt{\gamma}}} \gamma},$$

where ξ_0 is an arbitrary constant. Thus the travelling wave solution of KPP Equation (3.1) can be written as

$$u_1(x,t) = -\frac{e^{\frac{\left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\right)\left(2i\sqrt{2}x\sqrt{\gamma} + t\left(\beta - 3\sqrt{\beta^2 - 4\alpha\gamma}\right) + 8\gamma\xi_0\right)}{8\gamma}} \left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\right)}{-1 + 2e^{\frac{\left(\beta + \sqrt{\beta^2 - 4\alpha\gamma}\right)\left(2i\sqrt{2}x\sqrt{\gamma} + t\left(\beta - 3\sqrt{\beta^2 - 4\alpha\gamma}\right) + 8\gamma\xi_0\right)}{8\gamma}} \gamma}.$$

Similarly, for the case of (3.13)-(3.17), the exact travelling wave solutions are

$$u_2(x,t) = -\frac{e^{\frac{6t\alpha\gamma + t\beta\sqrt{\beta^2 - 4\alpha\gamma} + i\sqrt{2}x\sqrt{\gamma}\sqrt{\beta^2 - 4\alpha\gamma} + 4\beta\gamma\xi_0}{4\gamma}} \left(-\beta + \sqrt{\beta^2 - 4\alpha\gamma}\right)}{e^{\frac{t\beta^2 + i\sqrt{2}x\beta\sqrt{\gamma} + 4\gamma\sqrt{\beta^2 - 4\alpha\gamma}\xi_0}{4\gamma}} + 2e^{\frac{6t\alpha\gamma + t\beta\sqrt{\beta^2 - 4\alpha\gamma} + i\sqrt{2}x\sqrt{\gamma}\sqrt{\beta^2 - 4\alpha\gamma} + 4\beta\gamma\xi_0}{4\gamma}} \gamma},$$

$$u_3(x,t) = \frac{e^{\frac{\beta\left(2i\sqrt{2}x\sqrt{\gamma}+t\left(\beta+3\sqrt{\beta^2-4\alpha\gamma}\right)+8\gamma\xi_0\right)}{8\gamma}}\left(-\beta+\sqrt{\beta^2-4\alpha\gamma}\right)}{\sqrt{\beta^2-4\alpha\gamma}\,e^{\frac{\left(2i\sqrt{2}x\sqrt{\gamma}+t\left(\beta+3\sqrt{\beta^2-4\alpha\gamma}\right)+8\gamma\xi_0\right)}{8\gamma}}+2e^{\beta\left(\frac{i\xi}{2\sqrt{2}\sqrt{\gamma}}+\xi_0\right)}\gamma},$$

$$u_4(x,t) = -\frac{e^{\left(\beta+\sqrt{\beta^2-4\alpha\gamma}\right)\xi_0}\left(\beta+\sqrt{\beta^2-4\alpha\gamma}\right)}{-e^{\frac{i\sqrt{2}x\sqrt{\gamma}\left(\beta+\sqrt{\beta^2-4\alpha\gamma}\right)+t\left(-6\alpha\gamma+\beta\left(\beta+\sqrt{\beta^2-4\alpha\gamma}\right)\right)}{4\gamma}}+2e^{\left(\beta+\sqrt{\beta^2-4\alpha\gamma}\right)\xi_0}\gamma},$$

$$u_5(x,t) = \frac{-\beta+\sqrt{-\beta^2+4\alpha\gamma}\tan\left[\frac{\sqrt{-\beta^2+4\alpha\gamma}\left(-t\beta+i\sqrt{2}x\sqrt{\gamma}+2\gamma\xi_0\right)}{4\gamma}\right]}{2\gamma},$$

$$u_6(x,t) = -\frac{\beta+\sqrt{-\beta^2+4\alpha\gamma}\tan\left[\frac{\sqrt{-\beta^2+4\alpha\gamma}\left(t\beta+i\sqrt{2}x\sqrt{\gamma}-2\gamma\xi_0\right)}{4\gamma}\right]}{2\gamma}.$$

The solutions are new exact solutions.

Case II: Assume the $m = 2$, by equating the coefficients of $Y^j(j = 0,1,2,3)$ on both sides of Equation (3.6), we have

$$(3.19) \qquad a_2'(X) = g(X)a_2(X),$$

$$(3.20) \qquad a_1'(X) - 2ca_2(X) = h(X)a_2(X) + g(X)a_1(X),$$

$$(3.21) \qquad a_0'(X) - ca_1(X) + 2a_2(X)(-\alpha X - \beta X^2 - \gamma X^3) = h(X)a_1(X) + g(X)a_0(X),$$

$$(3.22) \qquad a_1(X)(-\alpha X - \beta X^2 - \gamma X^3) = h(X)a_0(X).$$

Since $a_2(X)$ is a polynomial of X, then from (3.19) we deduce that $a_2(X)$ is constant and $g(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $h(X)$, $a_1(X)$ and $a_0(X)$, we conclude that either $\deg(h(X)) = 0$ or $\deg(h(X)) = 1$.

Case II (i): Taking $\deg(h(X)) = 0$, and suppose that $h(X) = A$, then we obtain $a_1(X)$ and $a_0(X)$ as

$$(3.23) \qquad a_1(X) = B + (A + 2c)X,$$

$$(3.24) \qquad a_0(X) = D + (AB + Bc)X + \left(\frac{A^2}{2} + \frac{3}{2}Ac + c^2 + \alpha\right)X^2 + \frac{2X^3\beta}{3} + \frac{X^4\gamma}{2},$$

where B and D are constants. Substituting $h(X)$, $a_0(X)$ and $a_1(X)$ in (3.22) and setting all the coefficients of powers of X equal to zero, then we have a system of nonlinear algebraic equations and using Mathematica solving them with respect to D, B, A, α, β, we obtain

$$(3.25) \qquad D = 0, \quad B = -\frac{2ic^2\sqrt{2}}{9\sqrt{\gamma}}, \quad A = -\frac{4c}{3}, \quad \alpha = -\frac{4c^2}{9}, \quad \beta = -ic\sqrt{2\gamma},$$

$$(3.26) \qquad D = 0, \quad B = \frac{2ic^2\sqrt{2}}{9\sqrt{\gamma}}, \quad A = -\frac{4c}{3}, \quad \alpha = -\frac{4c^2}{9}, \quad \beta = ic\sqrt{2\gamma},$$

$$(3.27) \qquad D = 0, \quad B = 0, \quad A = -\frac{4c}{3}, \quad \alpha = -\frac{2c^2}{9}, \quad \beta = 0,$$

Using (3.25) into (3.23), (3.24) and (3.5), we obtain

$$(3.28) \qquad\qquad Y = \frac{c}{3}X + \frac{i\sqrt{2\gamma}}{2}X^2,$$

$$(3.29) \qquad\qquad Y = \frac{2i\sqrt{2}c^2}{9\sqrt{\gamma}} - cX - \frac{i\sqrt{\gamma}}{\sqrt{2}}X^2.$$

Combining (3.28) with (3.3), we obtain the two exact solutions of (3.2) and then the two exact solutions of KPP equation (3.1) can be written as

$$u_7(x,t) = \frac{\sqrt{2}ce^{\frac{1}{3}c(x-ct+6\xi_0)}}{-1 - 3ie^{\frac{1}{3}c(x-ct+6\xi_0)}\sqrt{\gamma}},$$

$$u_8(x,t) = \frac{\sqrt{2}c}{e^{-\frac{1}{3}c(x-ct+6\xi_0)} - 3i\sqrt{\gamma}},$$

and also for (3.29), we obtain the exact solution of KPP Equation (3.1) as the following

$$u_9(x,t) = -\frac{ic\left(-3 + \tanh\left[\frac{1}{6}c\left(x-ct+18i\sqrt{\gamma}\xi_0\right)\right]\right)}{3\sqrt{2}\sqrt{\gamma}}.$$

Similarly, the exact travelling wave solutions of (3.26) are

$$u_{10}(x,t) = \frac{\sqrt{2}ce^{\frac{1}{3}c(x-ct+6\xi_0)}}{-1 + 3ie^{\frac{1}{3}c(x-ct+6\xi_0)}\sqrt{\gamma}},$$

$$u_{11}(x,t) = \frac{\sqrt{2}c}{e^{-\frac{1}{3}c(x-ct+6\xi_0)} + 3i\sqrt{\gamma}},$$

$$u_{12}(x,t) = \frac{ic\left(-3 + \tanh\left[\frac{1}{6}c\left(x-ct-18i\sqrt{\gamma}\xi_0\right)\right]\right)}{3\sqrt{2}\sqrt{\gamma}},$$

and for (3.27), using (3.23), (3.24) and (3.5), we obtain

$$(3.30) \qquad\qquad Y = -\frac{1}{3}cX - \frac{1}{2}i\sqrt{2\gamma}X^2,$$

$$(3.31) \qquad\qquad Y = -\frac{1}{3}cX + \frac{1}{2}i\sqrt{2\gamma}X^2,$$

where from (3.30) and (3.31), we have the exact travelling wave solutions given in the following pages.

$$u_{13}(x,t) = \frac{2^{1/6}}{\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)}\left(92^{1/3}c\gamma - \frac{\left(3i2^{2/3}c^2 e^{c(x-ct+6\xi_0 i)}\sqrt{\gamma}\right)}{\left(\sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)^2 - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)}\right)^{1/3}}\right.$$
$$\left. + \left(\sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)^2 - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)}\right)^{1/3}\right),$$

$$u_{14}(x,t) = \frac{1}{\left(2\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)\right)}\left(18\sqrt{2}c\gamma + \frac{\left(6(-2)^{5/6}c^2 e^{c(x-ct+6\xi_0 i)}\sqrt{\gamma}\right)}{\left(\sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)^2 - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)}\right)^{1/3}}\right.$$
$$\left. + i2^{1/6}(i+\sqrt{3})\left(\sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)^2 - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)}\right)^{1/3}\right),$$

$$u_{15}(x,t) = \frac{1}{\left(2\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)\right)}\left(18\sqrt{2}c\gamma + \frac{\left(32^{5/6}(i+\sqrt{3})c^2 e^{c(x-ct+6\xi_0 i)}\sqrt{\gamma}\right)}{\left(\sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)^2 - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)}\right)^{1/3}}\right.$$
$$\left. + 2^{1/6}(-1-i\sqrt{3})\left(\sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)^2 - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)}\right)^{1/3}\right),$$

$$u_{16}(x,t) = \frac{2^{1/6}}{\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)}\left(-92^{1/3}c\gamma - \frac{\left(3i2^{2/3}c^2 e^{c(x-ct+6\xi_0 i)}\sqrt{\gamma}\right)}{\left(-c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)^2 - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right) + \sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)^2}}\right)^{1/3}}\right.$$
$$\left. - \left(\sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)^2} - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)\right)^{1/3}\right),$$

$$u_{17}(x,t) = \frac{2^{1/6}}{\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)}\left(-92^{1/3}c\gamma + \frac{\left(3(-1)^{5/6}2^{2/3}c^2 e^{c(x-ct+6\xi_0 i)}\sqrt{\gamma}\right)}{\left(-c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)^2 - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right) + \sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}}}\right)^{1/3}}\right.$$
$$\left. - (-1)^{2/3}\left(\sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)^2} - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)\right)^{1/3}\right),$$

$$u_{18}(x,t) = \frac{2^{1/6}}{\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)}\left(-92^{1/3}c\gamma + \frac{\left(3(-1)^{1/6}2^{2/3}c^2 e^{c(x-ct+6\xi_0 i)}\sqrt{\gamma}\right)}{\left(-c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)^2 - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right) + \sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}}}\right)^{1/3}}\right.$$
$$\left. + (-1)^{1/3}\left(\sqrt{c^6 e^{2c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} + 27i\gamma^{3/2}\right)^2} - c^3 e^{c(x-ct+6\xi_0 i)}\left(e^{c(x-ct+6\xi_0 i)} - 27i\gamma^{3/2}\right)\right)^{1/3}\right),$$

$$u_{19}(x,t) = \frac{2^{1/6}}{(-e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2})}\left(92^{1/3}c\gamma - \frac{(3i2^{2/3}c^2 e^{c(x-ct-6\xi_0 i)}\sqrt{\gamma})}{(-c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)})^2 - c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2}))^{1/3}}\right.$$
$$\left.+\left(\sqrt{c^6 e^{2c(x-ct-12\xi_0 i)}(e^{cx-c^2 t}-27ie^{6\xi_0 ci}\gamma^{3/2})^2}-27ie^{6\xi_0 ci}\gamma^{3/2}(e^{cx-c^2 t}-27ie^{6\xi_0 ci}\gamma^{3/2}))^{1/3}\right)\right),$$

$$u_{20}(x,t) = \frac{-1}{(2(e^{c(x-ct-6\xi_0 i)}-27i\gamma^{3/2}))}\left(18\sqrt{2}c\gamma + \frac{(6(-2)^{5/6}c^2 e^{c(x-ct-6\xi_0 i)}\sqrt{\gamma})}{(-c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2})^2 + \sqrt{c^6 e^{2c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2}))^{1/3}}}\right.$$
$$\left.+ i2^{1/6}(i+\sqrt{3})\left(\sqrt{c^6 e^{2c(x-ct-12\xi_0 i)}(e^{cx-c^2 t}-27ie^{6\xi_0 ci}\gamma^{3/2})^2 - c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2}))^{1/3}}\right.\right.$$

$$u_{21}(x,t) = \frac{-1}{(2(e^{c(x-ct-6\xi_0 i)}-27i\gamma^{3/2}))}\left(18\sqrt{2}c\gamma + \frac{(32^{5/6}(i+\sqrt{3})c^2 e^{c(x-ct-6\xi_0 i)}\sqrt{\gamma})}{(-c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2})^2 + \sqrt{c^6 e^{2c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2}))}}\right.$$
$$\left.+ 2^{1/6}(-1-i\sqrt{3})\left(\sqrt{c^6 e^{2c(x-ct-12\xi_0 i)}(e^{cx-c^2 t}-27ie^{6\xi_0 ci}\gamma^{3/2})^2} - c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2}))^{1/3}\right)\right),$$

$$u_{22}(x,t) = \frac{2^{1/6}}{(-ie^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2})}\left(-9i2^{1/3}c\gamma + \frac{(32^{2/3}c^2 e^{c(x-ct-6\xi_0 i)}\sqrt{\gamma})}{(-c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2})^2 + \sqrt{c^6 e^{2c(x-ct-12\xi_0 i)}(e^{cx-c^2 t}+27ie^{6\xi_0 ci}\gamma^{3/2})^2}}\right.$$
$$\left.- i\left(\sqrt{c^6 e^{2c(x-ct-12\xi_0 i)}(e^{c\xi}+27ie^{6\xi_0 ci}\gamma^{3/2})^2} - c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}-27i\gamma^{3/2}))^{1/3}\right)\right),$$

$$u_{23}(x,t) = \frac{1}{(2(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2}))}\left(18\sqrt{2}c\gamma + \frac{(32^{5/6}(-i+\sqrt{3})c^2 e^{c(x-ct-6\xi_0 i)}\sqrt{\gamma})}{(-c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}-27i\gamma^{3/2})^2 + \sqrt{c^6 e^{2c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}-27i\gamma^{3/2}))}}\right.$$
$$\left.+ i2^{1/6}(i+\sqrt{3})\left(\sqrt{c^6 e^{2c(x-ct-12\xi_0 i)}(e^{cx-c^2 t}+27ie^{6\xi_0 ci}\gamma^{3/2})^2} - c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}-27i\gamma^{3/2}))^{1/3}\right)\right),$$

$$u_{24}(x,t) = \frac{-1}{(2(e^{c(x-ct-6\xi_0 i)}+27i\gamma^{3/2}))}\left(-18\sqrt{2}c\gamma + \frac{(32^{5/6}(i+\sqrt{3})c^2 e^{c(x-ct-6\xi_0 i)}\sqrt{\gamma})}{(-c^3 e^{c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}-27i\gamma^{3/2})^2 + \sqrt{c^6 e^{2c(x-ct-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}-27i\gamma^{3/2})}}\right.$$
$$\left.+ 2^{1/6}(1+i\sqrt{3})\left(\left(\sqrt{c^6 e^{2c(x-ct-12\xi_0 i)}(e^{cx-c^2 t}+27ie^{6\xi_0 ci}\gamma^{3/2})^2}\right)^{1/3} - c^3 e^{c(\xi-6\xi_0 i)}(e^{c(x-ct-6\xi_0 i)}-27i\gamma^{3/2}))\right)\right).$$

Now, if we solve system of nonlinear algebraic equations with respect to D, B, c, α, γ, we obtain other solutions

$$(3.32) \qquad D = 0, \quad \gamma = 0, \quad B = \frac{A^3}{9\beta}, \quad \alpha = -\frac{A^2}{6}, \quad c = -\frac{5A}{6},$$

$$(3.33) \qquad D = 0, \quad \gamma = 0, \quad B = 0, \quad \alpha = \frac{A^2}{6}, \quad c = -\frac{5A}{6},$$

where the exact travelling wave solutions are

$$u_{25}(x,t) = \frac{A^2 \left(1 - 2e^{\frac{A(x-ct)}{6} + 3A\beta\xi_0}\right)}{6 \left(-1 + e^{\frac{A(x-ct)}{6} + 3A\beta\xi_0}\right)^2 \beta}, \qquad u_{26}(x,t) = \frac{A^2 \left(1 + 2e^{\frac{A(x-ct)}{6} + 3A\beta\xi_0}\right)}{6 \left(1 + e^{\frac{A(x-ct)}{6} + 3A\beta\xi_0}\right)^2 \beta},$$

$$u_{27}(x,t) = \frac{A^2 e^{3A\beta\xi_0} \left(-2e^{\frac{A(x-ct)}{6}} + e^{3A\beta\xi_0}\right)}{6 \left(e^{\frac{A(x-ct)}{6}} - e^{3A\beta\xi_0}\right)^2 \beta}, \qquad u_{28}(x,t) = \frac{A^2 e^{3A\beta\xi_0} \left(2e^{\frac{A(x-ct)}{6}} + e^{3A\beta\xi_0}\right)}{6 \left(e^{\frac{A(x-ct)}{6}} + e^{3A\beta\xi_0}\right)^2 \beta},$$

and

$$u_{29}(x,t) = \frac{A^2 e^{\frac{1}{3}A(x-ct+3\xi_0)} \left(1 - 2i\sqrt{6}e^{\frac{1}{6}A(x-ct+3\xi_0)}\sqrt{\beta} - 6e^{\frac{A(x-ct)}{3} + A\xi_0}\beta\right)}{\left(1 + 6e^{\frac{A(x-ct)}{3} + A\xi_0}\beta\right)^2},$$

$$u_{30}(x,t) = \frac{A^2 e^{\frac{1}{3}A(x-ct+3\xi_0)} \left(1 + 2i\sqrt{6}e^{\frac{1}{6}A(x-ct+3\xi_0)}\sqrt{\beta} - 6e^{\frac{A(x-ct)}{3} + A\xi_0}\beta\right)}{\left(1 + 6e^{\frac{A(x-ct)}{3} + A\xi_0}\beta\right)^2},$$

respectively.

Case II (ii): Taking $\deg(h(X)) = 1$, and suppose that $h(X) = Ax + B$ and $A \neq 0$, then we obtain $a_1(X)$ and $a_0(X)$ as

$$a_1(X) = D + \left(B + \frac{2c}{-1+c^2}\right)X + \frac{A}{2}X^2,$$

$$a_0(X) = R + \left(BD + \frac{cD}{-1+c^2}\right)X + \left(\frac{c\left(-B + 2c + Bc^2\right)}{2\left(-1+c^2\right)^2} + \frac{\left(-B^2 + 2Bc + B^2c^2 - AD + Ac^2D\right)}{2\left(-1+c^2\right)}\right)$$

$$+ \frac{\alpha}{-1+c^2}\Big)X^2 + \left(\frac{Ac}{6\left(-1+c^2\right)} + \frac{A\left(-3B + 4c + 3Bc^2\right)}{6\left(-1+c^2\right)}\right)X^3 + \left(\frac{A^2}{8} + \frac{\beta}{2\left(-1+c^2\right)}\right)X^4,$$

where D and R are constants. Substituting $h(X)$, $a_0(X)$, $a_1(X)$ and $a_2(X)$ in (3.22) setting all the coefficients of powers of X equal to zero, then we obtain a system of nonlinear algebraic equations another and using Mathematica solving them, we can obtain several new exact solution other. We emphasize that our results can be found to have potentially useful applications in mathematical physics and applied mathematics including numerical simulation.

4. Conclusion

The first integral method described herein is not only efficient but also has the merit of being widely applicable. we described this method for finding some new exact solutions for the KPP equation, that it contains the various equations. So, the proposed method can be

extended to solve the nonlinear problems which arise in the soliton theory and other areas.

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