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Existence of Nontrivial Solutions to Perturbed Schrödinger System

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Abstract. We are concerned with the multiplicity of semiclassical solutions of the following Schrödinger system involving critical nonlinearity and magnetic fields. Under proper conditions, we prove the existence and multiplicity of the nontrivial solutions to the perturbed system.

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1. Introduction

This paper is motivated by some works that have appeared in recent years concerning the nonlinear Schrödinger equation with electromagnetic fields and critical nonlinearity of the form

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}(\nabla+iA(x))^2\psi+W(x)\psi-K(x)|\psi|^{2^*-2}\psi-h\big(x,|\psi|^2\big)\psi, \tag{1.1}$$

where $\hbar$ is Planck's constant, i is the imaginary unit, 2^* is the critical exponent, $2^*=2N/(N-2)$, for $N\geq 3$, $A(x)=(A_1(x),A_2(x),\ldots,A_N(x)):\mathbb{R}^N\to\mathbb{R}^N$ is a real vector potential and $W(x)$ is a scalar electric potential. Knowledge of the solutions for the elliptic equation

$$-(\nabla+iA(x))^2u(x)+\lambda(W(x)-E)u(x)=\lambda K(x)|u|^{2^*-2}u+\lambda h\big(x,|u|^2\big)u \tag{1.2}$$

has a great importance in the study of standing-wave solutions of (1.1) i.e. the solutions of the type

$$\psi(x,t)=\exp\left(-\frac{iEt}{\hbar}\right)u(x),$$

where $\lambda^{-1}=\hbar^2/2m$. The transition from quantum mechanics to classical mechanics can be conducted by making $\hbar\to 0$. Therefore, the existence and multiplicity of solutions for $\hbar$ small has important physical interest.

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The problem in the case $A(x) \equiv 0$ has been explored by many authors including del Pino and Felmer [17, 18], Floer and WeinStein [22], Oh [24] and Wang [29]. For more results, we refer the reader to [1, 2, 4–6, 9, 10, 14–16, 25] and the reference therein.

As for as the Equation (1.2) in the case of $A(x) \neq 0$ is concerned, we recall Lions [21], Arioli and Szulkin [3], Cingolani [12] and the works of [7, 11, 13, 20, 26–28]. Among the works mentioned above, the corresponding authors have done a great deal of work and obtained many valuable results. Especially, many results have only been established in subcritical case by using various methods.

Motivated by the results just described, a natural question is whether the existence and multiplicity of results occur for the following perturbed Schrödinger system with critical nonlinearity and electromagnetic fields

$$\left\{ \begin{array}{ll} -(\varepsilon\nabla + iA(x))^2u + V(x)u = H_s\big(|u|^2, |v|^2\big)u + K(x)|u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ -(\varepsilon\nabla + iB(x))^2v + V(x)v = H_t\big(|u|^2, |v|^2\big)v + K(x)|v|^{2^*-2}v, & x \in \mathbb{R}^N. \end{array} \right. \tag{1.3}$$

To my knowledge, it seems there is a few work on the existence of solutions to (1.2), but to the system (1.3), there is almost no work on the existence and multiplicity of solutions. By using the similar idea or method of [19, 30] we will establish the two main results to (1.3).

Firstly, we make the following assumptions throughout the paper:

- (V_0) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(0) = \inf_{x\in\mathbb{R}^N} V(x) = 0$ (this is referred as critical frequency and first appeared in [9, 10]), and there is a constant $b > 0$ such that the set $\nu^b = \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure (the measure condition was first used in [4–6]);
- (A_0) $A(x), B(x) \in C(\mathbb{R}^N, \mathbb{R}^N), A(0) = B(0) = 0$;
- (K_0) $K(x) \in C(\mathbb{R}^N)$, $0 < \inf K \leq \sup K < \infty$;
- (H_1) $H \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, $H_s(s,t)$, $H_t(s,t) = o(1)$ as $|s| + |t| \to 0$;
- (H_2) there exist $2 < \alpha < 2^*$ and $C > 0$ such that

$$|H_s(s,t)|, |H_t(s,t)| \leq C\Big(1 + s^{\frac{\alpha-2}{2}} + t^{\frac{\alpha-2}{2}}\Big);$$

- (H_3) there exist $a_0 > 0$, $p, q > 2$, $\theta \in (2, 2^*)$ such that $H(s,t) \geq 2a_0(|s|^{p/2} + |t|^q/2)$ and

$$0 < \frac{\theta}{2} H(s,t) \leq sH_s(s,t) + tH_t(s,t) \quad \text{for all } s > 0, t > 0.$$

We can give the example of the nonlinearity H as follows:

$$H(s,t) = |s|^{\frac{\beta}{2}} + |t|^{\frac{\beta}{2}}, 4 < 2\beta < 6 = 2^* = \frac{2N}{N-2}, \quad \text{for } N = 3.$$

Next, we follow the two main results:

Theorem 1.1. *Assume that* (V_0), (A_0), (K_0) *and* (H_1)–(H_3) *hold. Then for any* $\sigma > 0$, *there exists* $\varepsilon_\sigma > 0$ *such that* $\varepsilon \leq \varepsilon_\sigma$, *the perturbed Schrödinger system* (1.3) *has one least energy solution* $(u_\varepsilon, v_\varepsilon)$ *satisfying*

$$\frac{\theta - 2}{2\theta} \int_{\mathbb{R}^N} \varepsilon^2\big(|\nabla|u_\varepsilon||^2 + |\nabla|v_\varepsilon||^2\big) + V(x)\big(|u_\varepsilon|^2 + |v_\varepsilon|^2\big) \leq \sigma\varepsilon^N. \tag{1.4}$$

Theorem 1.2. *Let* (V_0), (A_0), (K_0) *and* (H_1)-(H_3) *be satisfied. Moreover, assume that* $H(u,v)$ *is even in* (u,v), *then for any* $m \in N$ *and* $\sigma > 0$, *there is* $\varepsilon_{m\sigma} > 0$ *such that* $\varepsilon \leq \varepsilon_{m\sigma}$ *the system* (1.3) *has at least m pairs of solutions* $(u_\varepsilon, v_\varepsilon)$ *which satisfy the estimate* (1.4).

These theorems extend the results in [19]. Observe that though the method used in our paper is similar to the one of [19], the procedure of the main results is not trivial. We must face our problem with complex-valued functions, at the same time, we need much more delicate estimates for the appearance of magnetic potentials $A(x)$ and $B(x)$.

This paper is organized as follows: in Section 2, we describe some preliminaries. Section 3 contains the behavior of (PS) sequences and technical Lemmas. Section 4 includes the proofs of the main results.

2. Preliminaries

Let $\lambda = \varepsilon^{-2}$. We think about the following equivalent problem

$$(2.1)\quad \begin{cases} -(\nabla + i\sqrt{\lambda}A(x))^2u + \lambda V(x)u = \lambda H_s\big(|u|^2,|v|^2\big)u + \lambda K(x)|u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ -(\nabla + i\sqrt{\lambda}B(x))^2v + \lambda V(x)v = \lambda H_t\big(|u|^2,|v|^2\big)v + \lambda K(x)|v|^{2^*-2}v, & x \in \mathbb{R}^N. \end{cases}$$

In order to prove Theorem 1.1 and Theorem 1.2, we need only prove the following result.

Theorem 2.1. *Assume that (V_0),(A_0),(K_0) and (H_1)–(H_3) hold. Then for $\sigma > 0$, there exists $\Lambda_\sigma > 0$ such that if $\lambda \geq \Lambda_\sigma$, the system* (2.1) *has at least one least energy solution (u_λ, v_λ) which satisfies*

$$\frac{\theta-2}{2\theta}\int_{\mathbb{R}^N}\big(|\nabla|u_\lambda||^2 + |\nabla|v_\lambda||^2 + \lambda V(x)\big(|u_\lambda|^2 + |v_\lambda|^2\big)\big) \leq \sigma\lambda^{1-\frac{N}{2}}.$$

Theorem 2.2. *Let (V_0),(A_0),(K_0) and (H_1)–(H_3) be satisfied. Moreover, assume that $H(u,v)$ is even in (u,v), then for any $m \in N$ and $\sigma > 0$, there is $\Lambda_{m\sigma} > 0$ such that $\lambda \geq \Lambda_{m\sigma}$ the system* (1.3) *has at least m pairs of solutions (u_λ, v_λ) which satisfy the estimate* (1.4).

For the convenience, we quote the following notations. Let $\nabla_A u = (\nabla + i\sqrt{\lambda}A)u$, $\nabla_B v = (\nabla + i\sqrt{\lambda}B)v$, $E_{\lambda,A}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N)\}$ and $E_{\lambda,B}(\mathbb{R}^N) = \{v \in L^2(\mathbb{R}^N) : \nabla_B v \in L^2(\mathbb{R}^N)\}$. It is obvious that $E_{\lambda,A}$ is the Hilbert subspace under the scalar product

$$(u,v)_{\lambda,A} = \mathrm{Re}\int_{\mathbb{R}^N}\big(\big(\nabla_A u, \overline{\nabla_A v}\big) + \lambda V(x)u\bar{v}\big),$$

the norm induced by the product $(\cdot,\cdot)$ is

$$\|u\|^2_{\lambda,A} = \int_{\mathbb{R}^N}\big(|\nabla_A u|^2 + \lambda V(x)|u|^2\big).$$

It is easily known that $E_{\lambda,A}$ is the closure of $C_0^\infty(\mathbb{R}^N,\mathbb{C})$. For $E_{\lambda,B}$,there exists the similar results to $E_{\lambda,A}$.

Remark 2.1. We have the following diamagnetic inequality(see [21]):

$$|\nabla_A u(x)| \geq |\nabla|u(x)|| \quad u \in E_{\lambda,A}(\mathbb{R}^N)$$

and

$$|\nabla_B v(x)| \geq |\nabla|v(x)|| \quad v \in E_{\lambda,B}(\mathbb{R}^N).$$

Indeed, since A, B is real-valued, we have

$$|\nabla|u(x)|| = \left|Re\left(\nabla u\frac{\bar{u}}{|u|}\right)\right| = \left|Re\big(\nabla u + i\sqrt{\lambda}Au\big)\frac{\bar{u}}{|u|}\right| \leq \big|\nabla u + i\sqrt{\lambda}Au\big| = |\nabla_A u(x)|$$

and

$$|\nabla|v(x)|| = \left|Re\left(\nabla v \frac{\bar{v}}{|v|}\right)\right| = \left|Re\big(\nabla v + i\sqrt{\lambda}Bv\big)\frac{\bar{v}}{|v|}\right| \le \big|\nabla v + i\sqrt{\lambda}Bv\big| = |\nabla_B v(x)|$$

(the bar denotes complex conjugation). These facts mean if $u \in E_{\lambda,A}(\mathbb{R}^N)$, $v \in E_{\lambda,B}(\mathbb{R}^N)$, then $|u|,|v| \in H^1(\mathbb{R}^N)$ and therefore $u,v \in L^p(\mathbb{R}^N)$ for any $p \in [2,2^*)$ i.e. if $u_n \rightharpoonup u$ in $E_{\lambda,A}$ ($v_n \rightharpoonup v$ in $E_{\lambda,B}$), then $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ for any $p \in [2,2^*)$ ($v_n \to v$ in $L^p_{loc}(\mathbb{R}^N)$) and $u_n \to u$ a.e. in $\mathbb{R}^N$($v_n \to v$ a.e. in $\mathbb{R}^N$).

Remark 2.2. In general, $E_{\lambda,A}(\mathbb{R}^N) \not\subseteq H^1(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) \not\subseteq E_{\lambda,A}(\mathbb{R}^N)$. However, it was proved by Szulkin [3] that if Ω is a bounded domain with regular boundary, then $E_{\lambda,A}(\Omega)$ and $H^1(\Omega)$ are equivalent, where $E_{\lambda,A}(\Omega) = \{u \in L^2(\Omega) : \nabla_A u \in L^2(\Omega)\}$ with the norm

$$\|u\|^2_{E_{\lambda,A}(\Omega)} = \int_\Omega \big(|\nabla_A u|^2 + |u|^2\big).$$

From Remark 2.1, for each $p \in [2,2^*)$, there is $c_p > 0$ (independent of λ) such that, if $\lambda > 1$, we have

$$\left(\int_{\mathbb{R}^N} |u|^p\right)^{\frac{1}{p}} \le c_p\left(\int_{\mathbb{R}^N} |\nabla|u||^2\right)^{\frac{1}{2}} \le c_p\left(\int_{\mathbb{R}^N} |\nabla_A u|^2\right)^{\frac{1}{2}} \le c_p\|u\|_{E_{\lambda,A}}.$$

Set $E_\lambda = E_{\lambda,A} \times E_{\lambda,B}$ and $\|(u,v)\|^2_\lambda = \|u\|^2_{\lambda,A} + \|v\|^2_{\lambda,B}$ for $(u,v) \in E_\lambda$. The energy functional associated with (2.1) is defined by

$$\begin{aligned} J_\lambda(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u + i\sqrt{\lambda}Au|^2 + |\nabla v + i\sqrt{\lambda}Bv|^2 + \lambda V(x)\big(|u|^2+|v|^2\big)\big) - \lambda\int_{\mathbb{R}^N} G(x,u,v) \\ &= \frac{1}{2}\|(u,v)\|^2_\lambda - \lambda\int_{\mathbb{R}^N} G(x,u,v) \quad \text{for } (u,v) \in E_\lambda, \end{aligned}$$

where $G(x,u,v) = (K(x))/2^*(|u|^{2^*} + |v|^{2^*}) + 1/2H(|u|^2,|v|^2)$.

Under the assumptions of Theorem 2.1, standard arguments [30] indicate that $J_\lambda \in C^1(E_\lambda,\mathbb{R})$ and the critical points of J_λ are weak solutions of (2.1).

3. Technical lemmas

Similar to the proof of Lemma 3.1 in [19], the following result can be obtained.

Lemma 3.1. *Assume that the assumptions of Theorem 2.1 hold and $\{(u_n,v_n)\}$ is a $(PS)_c$ sequence for J_λ. Then $c \ge 0$ and $\{(u_n,v_n)\}$ is bounded in E_λ.*

Proof. By (H_3), we have

$$\begin{aligned} &J_\lambda(u_n,v_n) - \frac{1}{\theta}J'_\lambda(u_n,v_n)(u_n,v_n) \\ &= \left(\frac{1}{2}-\frac{1}{\theta}\right)\|(u_n,v_n)\|^2_\lambda + \left(\frac{1}{\theta}-\frac{1}{2^*}\right)\lambda\int_{\mathbb{R}^N} K(x)\big(|u_n|^{2^*} + |v_n|^{2^*}\big) \\ &\quad + \lambda\int_{\mathbb{R}^N} \frac{1}{\theta}\big(|u_n|^2H_s\big(|u_n|^2,|v_n|^2\big) + |v_n|^2H_t\big(|u_n|^2,|v_n|^2\big)\big) - \frac{1}{2}H\big(|u_n|^2,|v_n|^2\big) \\ &\ge \left(\frac{1}{2}-\frac{1}{\theta}\right)\|(u_n,v_n)\|^2_\lambda \ge 0. \end{aligned}$$

Together with $J_\lambda(u_n,v_n)\to c$ and $J'_\lambda(u_n,v_n)\to 0$ in E_λ^{-1}, we have $\{(u_n,v_n)\}$ is bounded in E_λ and $c\geq 0$. The proof is completed. ∎

By Lemma 3.1, $(PS)_c$ sequence $\{(u_n,v_n)\}$ is bounded in E_λ. So we can assume $(u_n,v_n)\rightharpoonup(u,v)$ in E_λ. By Remark 2.1, passing to a subsequence, $u_n\to u$ and $v_n\to v$ in $L^p_{loc}(\mathbb{R}^N)$ for any $p\in[2,2^*)$ and $u_n\to u, v_n\to v$ a.e. in $\mathbb{R}^N$. It is standard that (u,v) is a critical point of J_λ, namely a weak solution of (2.1).

Lemma 3.2. *Let $s\in[2,2^*)$. There is a subsequence $\{(u_{n_j},v_{n_j})\}$ such that for any $\varepsilon>0$, there exists $r_\varepsilon>0$ with*

$$\limsup_{i\to\infty}\int_{B_i\setminus B_r}|u_{n_i}|^s+|v_{n_i}|^s\leq\varepsilon,$$

for all $r\geq r_\varepsilon$, where $B_r:=\{x\in\mathbb{R}^N:|x|\leq r\}$.

Proof. The proof of Lemma 3.2 is similar to the one of Lemma 3.4 [23]. ∎

Let $\eta\in C^\infty(\mathbb{R}^+)$, satisfying $0\leq\eta(t)\leq 1, t\geq 0$, $\eta(t)=1$, if $t\leq 1$, and $\eta(t)=0$, if $t\geq 2$. Define $\widetilde{u}_j(x)=\eta(2|x|/j)u(x)$, $\widetilde{v}_j(x)=\eta(2|x|/j)v(x)$, then $\widetilde{u}_j\to u$ in $E_{\lambda,A}$ and $\widetilde{v}_j\to v$ in $E_{\lambda,A}$.

Lemma 3.3.

$$\lim_{j\to\infty}Re\int_{\mathbb{R}^N}\Big(H_s\big(|u_{n_j}|^2,|v_{n_j}|^2\big)u_{n_j} -H_s\big(|u_{n_j}-\widetilde{u}_j|^2,|v_{n_j}-\widetilde{v}_j|^2\big)(u_{n_j}-\widetilde{u}_j)-H_s\big(|\widetilde{u}_j|^2,|\widetilde{v}_j|^2\big)\widetilde{u}_j\Big)\bar{\varphi}=0$$

and

$$\lim_{j\to\infty}Re\int_{\mathbb{R}^N}\Big(H_t\big(|u_{n_j}|^2,|v_{n_j}|^2\big)v_{n_j} -H_t\big(|u_{n_j}-\widetilde{u}_j|^2,|v_{n_j}-\widetilde{v}_j|^2\big)(v_{n_j}-\widetilde{v}_j)-H_t\big(|\widetilde{u}_j|^2,|\widetilde{v}_j|^2\big)\widetilde{v}_j\Big)\bar{\psi}=0,$$

uniformly in $(\varphi,\psi)\in E_\lambda$ with $\|(\varphi,\psi)\|_{E_\lambda}\leq 1$.

Proof. Similar to the proof of Lemma 3.6 [23], so we omit it. ∎

Lemma 3.4. *One has along a subsequence*

$$J_\lambda(u_n-\widetilde{u}_n,v_n-\widetilde{v}_n)\to c-J_\lambda(u,v)$$

and

$$J'_\lambda(u_n-\widetilde{u}_n,v_n-\widetilde{v}_n)\to 0 \text{ in } E_\lambda^{-1}.$$

Proof. Since $\widetilde{u}_j\to u$ in $E_{\lambda A}$, $\widetilde{v}_j\to v$ in $E_{\lambda,B}$ and $(u_j,v_j)\rightharpoonup(u,v)$ in E_λ, one has

$$\begin{aligned}&J_\lambda(u_n-\widetilde{u}_n,v_n-\widetilde{v}_n)\\ =\;& J_\lambda(u_n,v_n)-J_\lambda(\widetilde{u}_n,\widetilde{v}_n)+\frac{\lambda}{2^*}\int_{\mathbb{R}^N}K(x)\big(|u_n|^{2^*}-|u_n-\widetilde{u}_n|^{2^*}-|\widetilde{u}_n|^{2^*}\big)\\ &+\frac{\lambda}{2^*}\int_{\mathbb{R}^N}K(x)\big(|v_n|^{2^*}-|v_n-\widetilde{v}_n|^{2^*}-|\widetilde{v}_n|^{2^*}\big)\\ &+\frac{\lambda}{2}\int_{\mathbb{R}^N}H\big(|u_n|^2,|v_n|^2\big)-H\big(|u_n-v_n|^2,|\widetilde{u}_n-\widetilde{v}_n|^2\big)-H\big(|\widetilde{u}_n|^2,|\widetilde{v}_n|^2\big)+o(1).\end{aligned}$$

Along the lines in proving the Brezis-Lieb lemma, it is easy to check that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\big(|u_n|^{2^*}-|u_n-\widetilde{u}_n|^{2^*}-|\widetilde{u}_n|^{2^*}\big)=0,$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\big(|v_n|^{2^*}-|v_n-\widetilde{u}_n|^{2^*}-|\widetilde{v}_n|^{2^*}\big)=0$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} H\big(|u_n|^2,|v_n|^2\big)-H\big(|u_n-v_n|^2,|\widetilde{u}_n-\widetilde{v}_n|^2\big)-H\big(|\widetilde{u}_n|^2,|\widetilde{v}_n|^2\big)=0.$$

Note that $J_\lambda(u_n,v_n)\to c$ and $J_\lambda(\widetilde{u}_n,\widetilde{v}_n)\to J_\lambda(u,v)$, we have that

$$J_\lambda(u_n-\widetilde{u}_n,v_n-\widetilde{v}_n)\to c-J_\lambda(u,v).$$

For any $(\varphi,\psi)\in E_\lambda$, we have

$$\begin{aligned}
& J'_\lambda(u_n-\widetilde{u}_n,v_n-\widetilde{v}_n)(\varphi,\psi)\\
= \;& J'_\lambda(u_n,v_n)(\varphi,\psi)-J'_\lambda(\widetilde{u}_n,\widetilde{v}_n)(\varphi,\psi)\\
& +\lambda\mathrm{Re}\int_{\mathbb{R}^N} K(x)\big(|u_n|^{2^*-2}u_n-|u_n-\widetilde{u}_n|^{2^*-2}(u_n-\widetilde{u}_n)-|\widetilde{u}_n|^{2^*-2}\widetilde{u}_n\big)\bar{\varphi}\\
& +\lambda\mathrm{Re}\int_{\mathbb{R}^N} K(x)\big(|v_n|^{2^*-2}v_n-|v_n-\widetilde{v}_n|^{2^*-2}(v_n-\widetilde{v}_n)-|\widetilde{v}_n|^{2^*-2}\widetilde{v}_n\big)\bar{\psi}\\
& +\lambda\mathrm{Re}\int_{\mathbb{R}^N} \big(H_s\big(|u_n|^2,|v_n|^2\big)u_n-H_s\big(|u_n-\widetilde{u}_n|^2,|v_n-\widetilde{v}_n|^2\big)(u_n-\widetilde{u}_n)\\
& \quad -H_s\big(|\widetilde{u}_n|^2,|\widetilde{v}_n|^2\big)\widetilde{u}_n\big)\bar{\varphi}\\
& +\lambda\mathrm{Re}\int_{\mathbb{R}^N} \big(H_t(|u_n|^2,|v_n|^2)v_n-H_t\big(|u_n-\widetilde{u}_n|^2,|v_n-\widetilde{v}_n|^2\big)(v_n-\widetilde{v}_n)\\
& \quad -H_t\big(|\widetilde{u}_n|^2,|\widetilde{v}_n|^2\big)\widetilde{v}_n\big)\bar{\psi}.
\end{aligned}$$

It is standard to check that

$$\lim_{n\to\infty}\mathrm{Re}\int_{\mathbb{R}^N} K(x)\big(|u_n|^{2^*-2}u_n-|u_n-\widetilde{u}_n|^{2^*-2}(u_n-\widetilde{u}_n)-|\widetilde{u}_n|^{2^*-2}\widetilde{u}_n\big)\bar{\varphi}=0$$

and

$$\lim_{n\to\infty}\mathrm{Re}\int_{\mathbb{R}^N} K(x)\big(|v_n|^{2^*-2}v_n-|v_n-\widetilde{v}_n|^{2^*-2}(v_n-\widetilde{v}_n)-|\widetilde{v}_n|^{2^*-2}\widetilde{v}_n\big)\bar{\psi}=0$$

uniformly in $(\varphi,\psi)\in E_\lambda$ with $\|(\varphi,\psi)\|_\lambda\le 1$. Therefore, the conclusion required holds by Lemma 3.3. The proof is completed. ∎

Let $u_n^1=u_n-\widetilde{u}_n, v_n^1=v_n-\widetilde{v}_n$, then $u_n-u=u_n^1+(\widetilde{u}_n-u), v_n-v=v_n^1+(\widetilde{v}_n-v)$. So $(u_n,v_n)\to(u,v)$ in E_λ if and only if $(u_n^1,v_n^1)\to(0,0)$ in E_λ. Observe that

$$J_\lambda\big(u_n^1,v_n^1\big)-\frac{1}{2}J'_\lambda\big(u_n^1,v_n^1\big)\big(u_n^1,v_n^1\big)\ge\frac{\lambda}{N}K_{\min}\int_{\mathbb{R}^N}\big(|u_n^1|^{2^*}+|v_n^1|^{2^*}\big),$$

where $K_{\min}=\inf_{x\in\mathbb{R}^N}K(x)>0$. Hence by Lemma 3.4, we get

$$|u_n^1|_{2^*}^{2^*}+|v_n^1|_{2^*}^{2^*}\le\frac{N(c-J_\lambda(u,v))}{\lambda K_{\min}}+o(1). \tag{3.1}$$

Now, we determine the energy level of the functional J_λ below which the $(PS)_c$ condition holds.

Let $V_b(x) = \max\{V(x), b\}$, where b is the positive constant in the assumption (V_0). Since the set ν^b has finite measure and $u_n^1, v_n^1 \to 0$ in $L^2_{loc}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} V(x)\big(|u_n^1|^2 + |v_n^1|^2\big) = \int_{\mathbb{R}^N} V_b(x)\big(|u_n^1|^2 + |v_n^1|^2\big) + o(1).$$

By (H_2) and (H_3), there exists $C_b > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} K(x)\big(|u|^{2^*} + |v|^{2^*}\big) + |u|^2 H_s\big(|u|^2, |v|^2\big) + |v|^2 H_t\big(|u|^2, |v|^2\big) \\ \leq\ & b\big(\|u\|_2^2 + \|v\|_2^2\big) + C_b\big(\|u\|_{2^*}^{2^*} + \|v\|_{2^*}^{2^*}\big). \end{aligned}$$

Let S be the best Sobolev constant

$$S\|u\|_{2^*}^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2,$$

for all $u \in H^1(\mathbb{R}^N)$.

Lemma 3.5. *Under the assumptions of Theorem 2.1, there is a constant $\alpha_0 > 0$ independent of λ such that, for any $(PS)_c$ sequence $\{(u_n, v_n)\} \subset E_\lambda$ for J_λ with $(u_n, v_n) \rightharpoonup (u, v)$, either $(u_n, v_n) \to (u, v)$ or $c - J_\lambda(u, v) \geq \alpha_0 \lambda^{1-N/2}$.*

Proof. Assume that $(u_n, v_n) \not\to (u, v)$, then

$$\liminf_{n\to\infty} \big\|\big(u_n^1, v_n^1\big)\big\|_\lambda > 0 \quad \text{and} \quad c - J_\lambda(u, v) > 0.$$

By the Sobolev inequality and the diamagnetic inequality, we have

$$S\big(\|u_n^1\|_{2^*}^2 + \|v_n^1\|_{2^*}^2\big) \leq \lambda C_b\big(\|u_n^1\|_{2^*}^{2^*} + \|v_n^1\|_{2^*}^{2^*}\big) + o(1).$$

It is easy to show that $\liminf_{n\to\infty}(\|u_n^1\|_{2^*}^{2^*} + \|v_n^1\|_{2^*}^{2^*}) > 0$. Thus, by (3.1), we get

$$\begin{aligned} S &\leq \lambda C_b\big(\|u_n^1\|_{2^*}^{2^*} + \|v_n^1\|_{2^*}^{2^*}\big)^{\frac{2^*-2}{2^*}} + o(1) \leq \lambda C_b\left(\frac{N(c - J_\lambda(u, v))}{\lambda K_{\min}}\right)^{\frac{2}{N}} + o(1) \\ &= \lambda^{1-\frac{2}{N}} C_b\left(\frac{N}{K_{\min}}\right)^{\frac{2}{N}} (c - J_\lambda(u, v))^{\frac{2}{N}} + o(1). \end{aligned}$$

Therefore, we have $\alpha_0 \lambda^{1-N/2} \leq c - J_\lambda(u, v) + o(1)$, where $\alpha_0 = S^{N/2} C_b^{-N/2} N^{-1} K_{\min}$. The proof is completed. ∎

Lemma 3.6. *Under the assumptions of Theorem 2.1, there is a constant $\alpha_0 > 0$ independent of λ such that, if a sequence $\{(u_n, v_n)\} \subset E_\lambda$ satisfies*

$$J_\lambda(u_n, v_n) \to c < \alpha_0 \lambda^{1-N/2}, J_\lambda'(u_n, v_n) \to 0 \text{ in } E_\lambda^{-1},$$

then $\{(u_n, v_n)\}$ is relatively compact in E_λ.

Proof. From $J_\lambda(u_n, v_n) \to c$ and $J_\lambda'(u_n, v_n) \to 0$, we get $\{(u_n, v_n)\} \subset E_\lambda$ is a $(PS)_c$ sequence for J_λ. By $c < \alpha_0 \lambda^{1-N/2}$, we have $c - J_\lambda(u, v) < \alpha_0 \lambda^{1-N/2} - J_\lambda(u, v)$. Together with $J_\lambda(u, v) \geq 0$ and Lemma 3.5, we get the required conclusion. ∎

Lemma 3.7. *Under the assumptions of Theorem 2.1, there exist $\alpha_\lambda, \rho_\lambda > 0$ such that*

$$J_\lambda(u, v) > 0,\ \ 0 < \|(u, v)\|_\lambda < \rho_\lambda;\ J_\lambda(u, v) \geq \alpha_\lambda,\ \text{if } \|(u, v)\|_\lambda = \rho_\lambda.$$

Proof. By $(H_1)-(H_3)$, for $\delta \le (4\lambda C_2)^{-1}$, there exists C_δ such that

$$\int_{\mathbb{R}^N} G(x,u,v) \le \delta\big(\|u\|_2^2+\|v\|_2^2\big)+C_\delta\big(\|u\|_{2^*}^{2^*}+\|v\|_{2^*}^{2^*}\big).$$

Thus

$$J_\lambda(u,v) \ge \frac{1}{2}\|(u,v)\|_\lambda^2 - \lambda\delta\big(\|u\|_2^2+\|v\|_2^2\big)-\lambda C_\delta\big(\|u\|_{2^*}^{2^*}+\|v\|_{2^*}^{2^*}\big).$$

Observe that $\|u\|_2^2+\|v\|_2^2 \le C_2\|(u,v)\|_\lambda^2$, we have

$$J_\lambda(u,v) \ge \frac{1}{4}\|(u,v)\|_\lambda^2 - \lambda C_\delta\big(\|u\|_{2^*}^{2^*}+\|v\|_{2^*}^{2^*}\big),$$

which implies that the conclusions required hold. The proof is completed. ∎

Lemma 3.8. *Under the assumptions of Theorem 2.1, for any finite dimensional subspace $F \subset E_\lambda$, one has $J_\lambda(u,v) \to -\infty$ as $(u,v)\in F$, $\|(u,v)\|_\lambda \to \infty$.*

Proof. By the assumptions of Theorem 2.1

$$J_\lambda(u,v) \le \frac{1}{2}\|(u,v)\|_\lambda^2 - \lambda a_0\big(|u|_p^p+|v|_q^q\big)$$

for all $(u,v)\in E_\lambda$. Since all norms in a finite dimensional space are equivalent and $(p,q>2)$, we easily obtain the desired conclusion. ∎

Define the functional

$$\Phi_\lambda(u,v)=\frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla u+i\sqrt{\lambda}Au|^2+|\nabla v+i\sqrt{\lambda}Bv|^2+\lambda V(x)\big(|u|^2+|v|^2\big)\big)$$
$$-a_0\lambda\int_{\mathbb{R}^N}(|u|^p+|v|^q).$$

It is obvious that $\Phi_\lambda \in C^1(E_\lambda)$ and $J_\lambda(u,v)\le\Phi_\lambda(u,v)$ for any $(u,v)\in E_\lambda$.

Note that

$$\inf\left\{\int_{\mathbb{R}^N}|\nabla\phi|^2 : \phi\in C_0^\infty(\mathbb{R}^N,\mathbb{R}), |\phi|_p=1\right\}=0$$

and

$$\inf\left\{\int_{\mathbb{R}^N}|\nabla\psi|^2 : \psi\in C_0^\infty(\mathbb{R}^N,\mathbb{R}), |\psi|_q=1\right\}=0.$$

For any $\delta>0$, there exist $\phi_\delta,\psi_\delta\in C_0^\infty(\mathbb{R}^N,\mathbb{R})$, with $|\phi_\delta|_p=|\psi_\delta|_q=1$ and $\operatorname{supp}\phi_\delta, \operatorname{supp}\psi_\delta \subset B_{r_\delta}(0)$ such that $|\nabla\phi_\delta|_2^2, |\nabla\psi_\delta|_2^2<\delta$.

Let $e_\lambda(x)=(\phi_\delta(\sqrt{\lambda}x),\psi_\delta(\sqrt{\lambda}x))$, then $\operatorname{supp} e_\lambda \subset B_{\lambda^{-1/2}r_\delta}(0)$. For $t\ge 0$, we have

$$\Phi_\lambda(te_\lambda)=\frac{t^2}{2}\|e_\lambda\|_\lambda^2-a_0\lambda t^p\int_{\mathbb{R}^N}|\phi_\delta(\sqrt{\lambda}x)|^p-a_0\lambda t^q\int_{\mathbb{R}^N}|\psi_\delta(\sqrt{\lambda}x)|^q$$
$$=\lambda^{1-\frac{N}{2}}I_\lambda(t\phi_\delta,t\psi_\delta),$$

where

$$I_\lambda(u,v)=\frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla u|^2+|\nabla v|^2+A\big(\lambda^{-\frac{1}{2}}x\big)|u|^2+B(\lambda^{-\frac{1}{2}}x)|v|^2+V\big(\lambda^{-\frac{1}{2}}x\big)\big(|u|^2+|v|^2\big)\big)$$
$$-a_0\int_{\mathbb{R}^N}(|u|^p+|v|^q).$$

It is obvious that

$$\max_{t\geq 0} I_\lambda(t\phi_\delta, t\psi_\delta) = \frac{p-2}{2p(pa_0)^{\frac{2}{p-2}}}\left\{\int_{\mathbb{R}^N}|\nabla\phi_\delta|^2 + A\big(\lambda^{-\frac{1}{2}}x\big)|\phi_\delta|^2 + V\big(\lambda^{-\frac{1}{2}}x\big)|\phi_\delta|^2\right\}^{\frac{p}{p-2}}$$
$$+\frac{q-2}{2q(qa_0)^{\frac{2}{q-2}}}\left\{\int_{\mathbb{R}^N}|\nabla\psi_\delta|^2 + B\big(\lambda^{-\frac{1}{2}}x\big)|\psi_\delta|^2 + V\big(\lambda^{-\frac{1}{2}}x\big)|\psi_\delta|^2\right\}^{\frac{q}{q-2}}.$$

Recall that $A(0)=0,\ B(0)=0,\ V(0)=0$ and $\operatorname{supp}\phi_\delta, \operatorname{supp}\psi_\delta \subset B_{r_\delta}(0)$. Therefore, there exists $\Lambda_\delta > 0$ such that for all $\lambda \geq \Lambda_\sigma$, we get

$$\max_{t\geq 0} J_\lambda(t\phi_\delta, t\psi_\delta) \leq \left(\frac{p-2}{2p(pa_0)^{\frac{2}{p-2}}}(5\delta)^{\frac{p}{p-2}} + \frac{q-2}{2q(qa_0)^{\frac{2}{q-2}}}(5\delta)^{\frac{q}{q-2}}\right)\lambda^{1-\frac{N}{2}}. \tag{3.2}$$

It follows from (3.2) that

Lemma 3.9. *Under the assumptions of Theorem 2.1, for any $\sigma > 0$ there exists $\Lambda_\sigma > 0$ such that for each $\lambda \geq \Lambda_\sigma$, there is $\bar{e}_\lambda \in E_\lambda$ with $\|\bar{e}_\lambda\|_\lambda > \rho_\lambda$, $J_\lambda(\bar{e}_\lambda) \leq 0$ and*

$$\max_{t\geq 0} J_\lambda(t\bar{e}_\lambda) \leq \sigma\lambda^{1-\frac{N}{2}},$$

where ρ_λ is defined from Lemma 3.7.

Proof. We can choose $\delta < 0$ so small that

$$\left(\frac{p-2}{2p(pa_0)^{\frac{2}{p-2}}}(5\delta)^{\frac{p}{p-2}} + \frac{q-2}{2q(qa_0)^{\frac{2}{q-2}}}(5\delta)^{\frac{q}{q-2}}\right)\lambda^{1-\frac{N}{2}} \leq \sigma.$$

We take $e_\lambda(x) = (\phi_\delta(\sqrt{\lambda}x), \psi_\delta(\sqrt{\lambda}x))$ and $\Lambda_\sigma = \Lambda_\delta$. Let $\bar{t}_\lambda > 0$ be such that $\bar{t}_\lambda\|e_\lambda\|_\lambda > \rho_\lambda$ and $J_\lambda(te_\lambda) \leq 0$ for all $t \geq \bar{t}_\lambda$. So we take $\bar{e}_\lambda = \bar{t}_\lambda e_\lambda$. The required conclusion holds. ∎

For any $m \in N$, we can choose m functions $\phi_\delta^i \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi_\delta^i \cap \operatorname{supp}\phi_\delta^j = \emptyset,\ i \neq j$, $|\phi_\delta^i|_p = 1$ and $|\nabla\phi_\delta^i|_2^2 < \delta$. Similarly, one can also get m functions $\psi_\delta^i \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp}\psi_\delta^i \cap \operatorname{supp}\psi_\delta^j = \emptyset,\ \ i \neq j$, $|\psi_\delta^i|_q = 1$ and $|\nabla\psi_\delta^i|_2^2 < \delta$. Let $r_\delta^m > 0$ be such that $\operatorname{supp}(\phi_\delta^i, \psi_\delta^i) \subset B_{r_\delta^m}^i(0)$ for $i = 1, 2, \ldots, m$.

Set $e_\lambda^i(x) = \big(\phi_\delta^i(\sqrt{\lambda}x), \psi_\delta^i(\sqrt{\lambda}x)\big) = \big(f_\lambda^i, g_\lambda^i\big)$, $i = 1, 2, \ldots, m$, then $\operatorname{supp} e_\lambda^i(x) \subset B_{\lambda^{-1/2}r_\delta^m}(0)$. Let $F_{\lambda\delta}^m = \operatorname{span}\{e_\lambda^1, e_\lambda^2, \ldots, e_\lambda^m\}$. For each

$$(u,v) = \sum_{i=1}^m k_i e_\lambda^i \in F_{\lambda\delta}^m,$$

we get

$$\int_{\mathbb{R}^N}\big(|\nabla_A u|^2 + |\nabla_B v|^2\big) = \sum_{i=1}^m |k_i|^2\left(\int_{\mathbb{R}^N}|\nabla_A f_\lambda^i|^2 + \int_{\mathbb{R}^N}|\nabla_B g_\lambda^i|^2\right),$$
$$\int_{\mathbb{R}^N} V(x)\big(|u|^2 + |v|^2\big) = \sum_{i=1}^m |k_i|^2\left(\int_{\mathbb{R}^N}V(x)|f_\lambda^i|^2 + \int_{\mathbb{R}^N}V(x)|g_\lambda^i|^2\right),$$
$$\frac{1}{2^*}\int_{\mathbb{R}^N}K(x)\big(|u|^{2^*} + |v|^{2^*}\big) = \frac{1}{2^*}\sum_{i=1}^m |k_i|^{2^*}\left(\int_{\mathbb{R}^N}K(x)|f_\lambda^i|^{2^*} + \int_{\mathbb{R}^N}K(x)|g_\lambda^i|^{2^*}\right)$$

and

$$\int_{\mathbb{R}^N} H(u,v) = \sum_{i=1}^{m} \int_{\mathbb{R}^N} H\big(k_i f_\lambda^i, k_i g_\lambda^i\big).$$

Therefore

$$J_\lambda(u,v) = \sum_{i=1}^{m} J_\lambda\left(k_i e_\lambda^i\right)$$

and

$$J_\lambda(k_i e_\lambda^i) \le \phi_\lambda\left(k_i e_\lambda^i\right).$$

Set $\beta_\delta := max\{|(\phi_\delta^i, \psi_\delta^i)|_2^2 : \ i = 1,2,\ldots,m\}$ and choose some $\Lambda_{m\delta} > 0$ so that

$$V(\lambda^{\frac{1}{2}} x) \le \frac{\delta}{\beta_\delta} \quad \text{for all} \quad |x| \le r_\delta^m \quad \text{and} \quad \lambda \ge \Lambda_{m\delta}$$

Similar to the proof mentioned above, we can obtain the following inequality

$$\max_{(u,v)\in F_{\lambda\delta}^m} J_\lambda(u,v) \le \left(\frac{m(p-2)}{2p(pa_0)^{\frac{2}{p-2}}} (5\delta)^{\frac{p}{p-2}} + \frac{m(q-2)}{2q(qa_0)^{\frac{2}{q-2}}} (5\delta)^{\frac{q}{q-2}} \right) \lambda^{\frac{2-N}{2}}. \tag{3.3}$$

By using the estimate, we can get the following.

Lemma 3.10. *Under the assumptions of Lemma 3.7, for any $m \in N$ and $\sigma > 0$, there exists $\Lambda_{m\sigma} > 0$ such that for each $\lambda \ge \Lambda_{m\delta}$, we can take a m-dimensional subspace F satisfying*

$$\max_{(u,v)\in F} J_\lambda(u,v) \le \sigma \lambda^{\frac{2-N}{2}}.$$

Proof. choose $\delta > 0$ so small that

$$\left(\frac{m(p-2)}{2p(pa_0)^{\frac{2}{p-2}}} (5\delta)^{\frac{p}{p-2}} + \frac{m(q-2)}{2q(qa_0)^{\frac{2}{q-2}}} (5\delta)^{\frac{q}{q-2}} \right) \le \sigma$$

and take $F = F_{\lambda\delta}^m$. By (3.3), we get the conclusion as required. ∎

4. Proof of the main results

Firstly, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. By Lemma 3.9, for any $0 < \sigma < \alpha_0$, there exists $\Lambda_\sigma > 0$ such that for each $\lambda \ge \Lambda_\sigma$, we get $c_\lambda \le \sigma \lambda^{1-N/2}$, where

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} J_\lambda(\gamma(t)),$$
$$\Gamma_\lambda = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, \gamma(1) = \bar{e}_\lambda\}.$$

In virtue of Lemma 3.5, J_λ satisfies the $(PS)_{c_\lambda}$ condition. Hence, by the mountain pass theorem, there exists $(u_\lambda, v_\lambda) \in E_\lambda$ satisfying $J_\lambda'(u_\lambda, v_\lambda) = 0$ and $J_\lambda(u_\lambda, v_\lambda) = c_\lambda$. Therefore, (u_λ, v_λ) is a weak solution of (2.1).

Moreover, it is well known that (u_λ, v_λ) is one least energy solution of (2.1).

Note that $J_\lambda(u_\lambda, v_\lambda) \le \sigma \lambda^{1-N/2}$ and $J_\lambda'(u_\lambda, v_\lambda) = 0$, we have

$$\begin{aligned} J_\lambda(u_\lambda, v_\lambda) &= J_\lambda(u_\lambda, v_\lambda) - \frac{1}{\theta} J_\lambda'(u_\lambda, v_\lambda)(u_\lambda, v_\lambda) \\ &= (\frac{1}{2} - \frac{1}{\theta}) \|(u_\lambda, v_\lambda)\|_\lambda^2 + (\frac{1}{\theta} - \frac{1}{2^*}) \lambda \int_{\mathbb{R}^N} K(x)(|u_\lambda|^{2^*} + |v_\lambda|^{2^*}) \end{aligned}$$

$$+\lambda\int_{\mathbb{R}^N}\frac{1}{\theta}(|u_\lambda|^2H_s(|u_\lambda|^2,|v_\lambda|^2)+|v_\lambda|^2H_t(|u_\lambda|^2,|v_\lambda|^2))-\frac{1}{2}H(|u_\lambda|^2,|v_\lambda|^2)$$
$$\geq(\frac{1}{2}-\frac{1}{\theta})\|(u_\lambda,v_\lambda)\|_\lambda^2.$$

So the diamagnetic inequality implies that

$$\frac{\theta-2}{2\theta}\int_{\mathbb{R}^N}(|\nabla|u_\lambda||^2+|\nabla|v_\lambda||^2+\lambda V(x)(|u_\lambda|^2+|v_\lambda|^2))\leq\sigma\lambda^{1-\frac{N}{2}}.$$

The proof is completed. ∎

Secondly, we give the proof of Theorem 2.2.

Proof of Theorem 2.2. By Lemma 3.10, for any $m\in N$ and $\sigma\in(0,\alpha_0)$, there exists $\Lambda_{m\sigma}$ such that for $\lambda\geq\Lambda_{m\sigma}$, we can choose a m-dimensional subspace F with $\max J_\lambda(F)\leq\sigma\lambda^{1-N/2}$. By Lemma 3.8, there is $R>0$ (depending on λ and m) such that $J_\lambda(u)\leq 0$ for all $u\in F|B_R$.

Denote the set of all symmetric (in the sense that $-\Omega=\Omega$) and closed subsets of E_λ by Σ. For each $\Omega\in\Sigma$, let gen(Ω) be the Krasnoselski genus and let

$$i(A):=\min_{h\in\Gamma_m} gen\ h(\Omega)\cap\partial B_{\rho_\lambda}$$

where Γ_m is the set of all odd homeomorphisms $h\in C(E_\lambda,E_\lambda)$ and ρ_λ is the number of Lemma 3.7. Then i is a version of Benci's pseudoindex [8]. Let

$$c_{\lambda_j}=\inf_{i(\Omega)\geq j}\sup_{u\in\Omega}J_\lambda(u),\quad 1\leq j\leq m.$$

Since $J_\lambda(u)\geq\alpha_\lambda$ for all $u\in\partial B_{\rho_\lambda}$ (see Lemma 3.7) and $i(F)=dimF=m$,

$$\alpha_\lambda\leq c_{\lambda_1}\leq c_{\lambda_2}\leq\cdots\leq c_{\lambda_m}\leq\sup_{(u,v)\in F^m_{\lambda\sigma}}J_\lambda(u,v)\leq\sigma\lambda^{1-\frac{N}{2}}.$$

In connection with Lemma 3.6, we know that J_λ satisfies the $(PS)_{c_{\lambda_j}}$ condition at all levels c_{λ_j}. By the critical point theory, all c_{λ_j} are critical levels and J_λ has at least m pairs of nontrivial critical points. Finally, as in the proof of Theorem 2.1, we easily get these solutions are the least energy solutions. ∎

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