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On the Exterior Degree of the Wreath Product of Finite Abelian Groups

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Abstract. The exterior degree $d^{\wedge}(G)$ of a finite group G has been recently introduced by Rezaei and Niroomand in order to study the probability that two given elements x and y of G commute in the nonabelian exterior square $G \wedge G$. This notion is related with the probability $d(G)$ that two elements of G commute in the usual sense. Motivated by a paper of Erovenko and Sury of 2008, we compute the exterior degree of a group which is the wreath product of two finite abelian p-groups (p prime). We find some numerical inequalities and study mostly abelian p-groups.

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1. Introduction

The present paper deals only with finite groups. A consistent body of scientific results is devoted to study the combinatorial conditions which influence the structure of finite groups in [1, 4, 5, 6, 17]. Denoting with $k(G)$ the *number of the G-conjugacy classes* $[x]_G = \{x^g \mid g \in G\}$ of a group G and with $C_G(x)$ the centralizer of x in G, it is shown in [1, 4, 5, 6, 17] that the *commutativity degree*

$$d(G) = \frac{|\{(x,y) \in G \times G \mid [x,y] = 1\}|}{|G|^2} = \frac{1}{|G|^2}\sum_{x\in G}|C_G(x)| = \frac{k(G)}{|G|}$$

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allows us to classify large classes of groups only looking at their numerical value of $d(G)$. The intriguing idea, which is behind most of the proofs of [1, 3, 4], is that $d(G)$ measures the distance of G from being abelian and so we may apply different techniques of combinatorial nature. We inform the reader that there are some recent contributions in [12, 19] which study the recognition of the structure of a group from inequalities of numerical nature. This approach might be useful to compare with our techniques of investigation.

Going back to illustrate our scopes, we mention that several authors call $d(G)$ the *probability of commuting pairs* of G. In fact, $\{(x,y) \in G \times G \mid [x,y] = 1\}$ can be regarded as a measurable subset of G^2 (with respect to the discrete measure over G^2) and $d(G)$ is defined exactly as a probability measure. Of course, $d(G) = 1$ if and only if G is abelian. As one may expect, $d(G)$ is an invariant, but it is not only invariant under isomorphisms of groups, but also under various generalizations, for instance the *isoclinisms* (see [5, 17]).

On the other hand, there is a recent interest in algebraic topology and in group theory in the study of the *nonabelian exterior square* $G \wedge G$ of G: we recall that $G \wedge G$ is the group generated by the symbols $g \wedge h$ and by the relations $gg' \wedge h = ((g')^g \wedge h^g)\ (g \wedge h)$, $g \wedge hh' = (g \wedge h)\ (g^h \wedge (h')^h)$ and $g \wedge g = 1$ for all $g, g', h, h' \in G$, where G acts on itself by conjugation via $(g')^g = g^{-1}g'g$.

A recent number of papers is in fact devoted to investigate a more specific invariant, which allows us to measure how far is G from being an abelian group of a prescribed type, for instance, elementary abelian of given rank. Niroomand and Rezaei [14] introduced the *exterior degree* of G

$$d^\wedge(G) = \frac{|\{(x,y) \in G \times G \mid x \wedge y = 1_{G\wedge G}\}|}{|G|^2} = \frac{1}{|G|}\sum_{i=1}^{k(G)} \frac{|C_G^\wedge(x_i)|}{|C_G(x_i)|},$$

where the last equality is precisely [14, Lemma 2.2]. The set

$$C_G^\wedge(x) = \{a \in G \mid a \wedge x = 1_{G\wedge G}\}$$

is called *exterior centralizer* of x in G and turns out to be a subgroup of G (see [13]) contained in $C_G(x)$. The *exterior center* of G is the set

$$Z^\wedge(G) = \{g \in G \mid 1_{G\wedge G} = g \wedge y \in G \wedge G, \forall y \in G\} = \bigcap_{x\in G} C_G^\wedge(x)$$

which is a subgroup of the center $Z(G)$ of G (see [13, 14, 15]). Originally, $C_G^\wedge(x)$ and $Z^\wedge(G)$ have been introduced for the study of properties of $G \wedge G$ and this justifies the use of these subgroups in our perspective of research.

$H_2(G,\mathbb{Z}) = M(G)$ denotes the second homology group of G with integral coefficients (also called *Schur multiplier* of G, see [11]) and plays a fundamental role in the study of the exterior degree, as noted in [14, 15, 16]. There is a classical result in [11], known as *Poincaré Duality*, which shows $H_2(G,\mathbb{Z}) \simeq H^2(G,\mathbb{C}^*)$. This means that the second homology group with coefficients in $\mathbb{Z}$ is isomorphic with the second cohomology group with coefficients in $\mathbb{C}^*$ and, in principle, we may use independently $H_2(G,\mathbb{Z})$ or $H^2(G,\mathbb{Z})$ for denoting the Schur multiplier. We prefer to use $H_2(G,\mathbb{Z}) = M(G)$, following [13, 14, 15, 16].

Very briefly, we mention that the interest for $C_G^\wedge(x)$ and $Z^\wedge(G)$ is due to the fact that they allow us to decide whether G is a *capable group* or not, that is, whether G is isomorphic to $E/Z(E)$ for some group E or not. Beyl and others [2] illustrate that capable groups are well known and subject to interesting classifications.

We noted that it is not available a precise computation of the exterior degree of wreath products of abelian groups as in [7], even if some general bounds are known by [14, 15, 16]. The present paper has been written to cover this aspect of the literature. Since the dihedral group D_8 of order 8 is isomorphic to the wreath product $C_2 \wr C_2$ of two copies of the cyclic group C_2 of order 2, we have precise values for $d^\wedge(D_8)$ already in [14, 15] and several other extraspecial p-groups (p any prime) can be constructed directly as wreath products of cyclic p-groups (see [10]). In fact we confirm not only the main results of [16], but provide new formulas for the exterior degree of wreath products of cyclic p-groups.

2. Preliminaries

Let L and H be groups and Ω a set with H acting on it. Let K be the direct product $K = \prod_{\omega\in\Omega} L_\omega$ of copies of $L_\omega = L$ indexed by the set Ω. The elements of K can be seen as arbitrary sequences (l_ω) of elements of L indexed by Ω with componentwise multiplication. Then the action of H on Ω extends in a natural way to an action of H on the group K by $h(l_\omega) = (l_{h^{-1}\omega})$. In this way, we have defined the group $L \wr_\Omega H$, wreath product of L by H with respect to Ω. The subgroup K of $L \wr_\Omega H$ is called a *basis*. Since H acts in a natural way on itself by left multiplication (notion of *left Cayley action*), we can choose $\Omega = H$. In this case, we write briefly $L \wr H$, omitting Ω, and the wreath product turns out to be the semidirect product $H \ltimes K$, that is, $L \wr H = H \ltimes K$. We will consider only this type of wreath product, also called *standard wreath product*. More specifically, we will focus on two abelian groups A and B and on $A \wr B$, considering the left Cayley action as just said. We will have

$$A \wr B = B \ltimes \underbrace{A \times A \times \cdots \times A}_{|B|-\text{times}} = B \ltimes A^{|B|},$$

that is, the semidirect product of B by $|B|$-copies of A (see [11, Chapter 6] or [10]). Several examples, which motivated our investigations, are listed below.

Example 2.1. The symmetric group

$$S_3 = \langle x, y \mid x^2 = y^3 = 1, x^{-1}yx = y^{-1}\rangle = \langle x\rangle \ltimes \langle y\rangle \simeq C_2 \ltimes A_3 \simeq C_2 \ltimes C_3$$

on 3 letters is isomorphic to the dihedral group D_6 of order 6, where $A_3 \simeq C_3$ denotes the alternating group on 3 elements. It is easy to check that $Z(S_3) = Z^\wedge(S_3) = 1$, $C_{S_3}(A_3) = A_3$ and $C_{S_3}(\langle x\rangle) = \langle x\rangle$. More generally, the dihedral group of order $2q$ is

$$D_{2q} = \langle x, y \mid x^2 = y^q = 1, x^{-1}yx = y^{-1}\rangle \simeq C_2 \ltimes C_q$$

(see [10]) and, in case $q \geq 3$ is an odd prime, it is possible to extend our considerations, up to isomorphisms, to all dihedral groups D_{2q}. We find again $C_{D_{2q}}(C_q) = C_q$, $C_{D_{2q}}(\langle x\rangle) = \langle x\rangle$ and $Z(D_{2q}) = Z^\wedge(D_{2q}) = 1$.

One of the key results in [14, 15] is the following bound, which restricts the values of the exterior degree by two functions depending on the size of the Schur multiplier.

Theorem 2.1. *(See* [14, Theorem 2.3]*) Let G be a group. Then*

$$\frac{d(G)}{|M(G)|} + \frac{|Z^\wedge(G)|}{|G|}\left(1 - \frac{1}{|M(G)|}\right) \leq d^\wedge(G) \leq d(G) - \left(\frac{p-1}{p}\right)\left(\frac{|Z(G)| - |Z^\wedge(G)|}{|G|}\right)$$

where p is the smallest prime number dividing the order of G.

Since capable groups are characterized to have trivial exterior center (see [2, 11]), the following consequences are clear.

Corollary 2.1. *(See* [14, Corollary 2.5]*) Let G be a group. Then $d^{\wedge}(G) \leq d(G)$. Moreover, if G is capable, then $\frac{1}{|G|} \leq d^{\wedge}(G) \leq d(G)$.*

There are a series of information which can be found in [11] about $M(A \wr B)$ that we list in the next lines. Given an arbitrary abelian group A,

$$A \sharp A = \frac{A \otimes A}{U(A)}, \text{ where } U(A) = \langle a \otimes b + b \otimes a \mid a, b \in A \rangle$$

and

$$\mathrm{Inv}(A) = \{a \in A \mid a^2 = 1\}.$$

The structure of $A \sharp A$ is described by the following result.

Theorem 2.2. *(See* [11, Lemma 6.3.4]*) Let $A = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_t}$ be a decomposition of an abelian group A for $n_1, n_2, \ldots, n_t \geq 1$ and s the number of even n_i for $1 \leq i \leq t$. Then*

$$A \sharp A = \bigoplus_{1 \leq i \leq j}^{t} C_{(n_i, n_j)} \oplus C_2^s.$$

Two classic results of Blackburn show that we may compute $M(A \wr B)$ once we know $A \sharp A$ and $\mathrm{Inv}(A)$. The first is very general.

Theorem 2.3. *(See* [11, Theorem 6.3.3]*) Let A and B be two abelian groups. Then*

$$M(A \wr B) = M(A) \oplus M(B) \oplus (B \otimes B)^{\frac{1}{2}(|A| - |\mathrm{Inv}(A)| - 1)} \oplus (B \sharp B)^{|\mathrm{Inv}(A)|}.$$

The second is an application and deals with $M(P_n)$, where P_n is a Sylow p-subgroup of the symmetric group S_{p^n}. It is well known by a result of Kaloujnine (see [11, Section 6]) that P_n has order p^k with $k = 1 + p + p^2 + \cdots + p^{n-1}$ and that $P_1 \simeq C_p$, $P_2 \simeq C_p \wr C_p$, $P_3 = C_p \wr (C_p \wr C_p)$ and so on until $P_n = P_1 \wr P_{n-1}$. Moreover P_{n-1}/P'_{n-1} is an elementary abelian p-group of order p^{n-1} for all n. The following result is very important after we note that any p-group can be embedded in a p–group whose Schur multiplier is elementary abelian [11, Corollary 6.3.6]. Therefore most of the groups which have been studied in [1, 4, 5, 6, 13, 14, 15, 17] turns out to have the Schur multipliers equal to $M(P_n)$.

Theorem 2.4. *(See* [11, Theorem 6.3.5]*) If P_n is a Sylow p-subgroup of the symmetric group S_{p^n}, then $M(P_n) = C_p^s$, where $s = \frac{1}{12}(p-1)(n-1)n(2n-1)$ if $p \neq 2$ and $s = \frac{1}{6}n(n^2-1)$ if $p = 2$.*

We may be more specific on $|\mathrm{Inv}(A)|$ when A is a cyclic group in Theorem 2.3. Before to proceed, the following observation is fundamental and motivates us to concentrate on p-groups.

Remark 2.1. An abelian group can be always written as direct sum of its Sylow p-subgroups by a well known result of decomposition (see [10]). On the other hand, we know that the exterior degree is a multiplicative function, that is, the exterior degree of a direct product (of finitely many groups) equals the product of the values of the exterior degree of each factor (see [14]). Therefore it is reasonable to reduce the study of the exterior degree of abelian groups only to the case of abelian p-groups. Therefore we will concentrare mostly on p-groups from now on.

We know in fact that each finite cyclic group C_n can be written as a direct sum

$$C_n \simeq C_{p_1^{m_1}} \oplus C_{p_2^{m_2}} \oplus \cdots \oplus C_{p_r^{m_r}}$$

of cyclic groups $C_{p_i^{m_i}}$, where $p_i \geq 2$ are primes such that $n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$.

There is a good description of $|\mathrm{Inv}(C_n)|$ in [8, 9] by the function

$$\xi : n \in \mathbb{N} \mapsto \xi(n) = \begin{cases} 1, \text{ if } 8|n, \\ -1, \text{ if } 2|n \text{ and } 4 \not| n, \in \{-1,0,1\} \\ 0, \text{ otherwise.} \end{cases}$$

Theorem 2.5. *(See* [8, Lemma 2, Theorem 2]*) Let $n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$ be a prime decomposition of n with $p_i < p_{i+1}$ and $m_i > 0$ for all $1 \leq 1 \leq r-1$. Then*

$$|\mathrm{Inv}(C_n)| = 2^{r+\xi(n)}.$$

In particular, if $r = 1$, then $n = p^m$ and

$$|\mathrm{Inv}(C_{p^m})| = 2^{1+\xi(p^m)}.$$

The wreath product of cyclic p-groups is described below.

Lemma 2.1. *Let $A = C_{p^m}$ and $B = C_{p^n}$ where p is an odd prime and $m, n \geq 1$ integers. Then*

$$p^{\lfloor \frac{1}{2} n(p^m-3) \rfloor} \leq |M(A \wr B)| \leq p^{\lfloor \frac{1}{2} n(p^m+1) \rfloor}.$$

Moreover, the lower bound is achieved when $U(A) = B \otimes B$ and the upper bound when $U(B) = 0$.

Proof. The Künneth Formula [11, Theorem 2.2.10] shows that

$$M(C_{p^m} \oplus C_{p^n}) = M(C_{p^m}) \oplus M(C_{p^n}) \oplus (C_{p^m} \otimes C_{p^n}) = C_{p^m} \otimes C_{p^n} = C_{p^{(m,n)}}$$

We apply Theorem 2.3 and find

$$\begin{aligned} M(A \wr B) &= M(C_{p^m} \wr C_{p^n}) \\ &= M(C_{p^m}) \oplus M(C_{p^n}) \oplus (C_{p^n} \otimes C_{p^n})^{\frac{1}{2}(p^m - |\mathrm{Inv}(C_{p^m})| - 1)} \oplus (C_{p^n} \sharp C_{p^n})^{|\mathrm{Inv}(C_{p^m})|} \\ &= (C_{p^n} \otimes C_{p^n})^{\frac{1}{2}(p^m - |\mathrm{Inv}(C_{p^m})| - 1)} \oplus (C_{p^n} \sharp C_{p^n})^{|\mathrm{Inv}(C_{p^m})|} \end{aligned}$$

but p is odd, then $\xi(p) = \xi(p^m) = 0$ and $|\mathrm{Inv}(C_{p^m})| = 2$ by Theorem 2.5, and

$$= (C_{p^n} \otimes C_{p^n})^{\frac{1}{2}(p^m-3)} \oplus (C_{p^n} \sharp C_{p^n})^2 = C_{p^n}^{\frac{1}{2}(p^m-3)} \oplus (C_{p^n} \sharp C_{p^n})^2.$$

If $U(B) = B \otimes B$, then $B \sharp B = 0$ and

$$M(A \wr B) = C_{p^n}^{\frac{1}{2}(p^m-3)}.$$

If $U(B) = 0$, then $B \sharp B = B \otimes B$ and

$$M(A \wr B) = C_{p^n}^{\frac{1}{2}(p^m-3)} \oplus C_{p^n}^2 = C_{p^n}^{\frac{1}{2}(p^m+1)}.$$

If $U(B)$ is a nontrivial proper subgroup of $B \otimes B$, then $0 \leq |B \sharp B| \leq |B \otimes B|$ and

$$|C_{p^n}^{\frac{1}{2}(p^m-3)}| \leq |M(A \wr B)| \leq |C_{p^n}^{\frac{1}{2}(p^m+1)}|,$$

as claimed. ∎

Lemma 2.2. *Let $A = C_{2^m}$ and $B = C_{2^n}$ and $m, n \geq 1$ integers.*

(i) *If* $m = 1$, *then* $|M(A \wr B)| \leq 2^{\lfloor \frac{1}{2}n \rfloor}$.

(ii) *If* $m = 2$, *then* $2^{\lfloor \frac{1}{2}n \rfloor} \leq |M(A \wr B)| \leq 2^{\lfloor \frac{5}{2}n \rfloor}$.

(iii) *If* $m \geq 3$, *then* $2^{\lfloor \frac{1}{2}n(2^m-5) \rfloor} \leq |M(A \wr B)| \leq 2^{\lfloor \frac{1}{2}n(2^m+5) \rfloor}$.

Moreover, the lower bounds are achieved when $U(B) = B \otimes B$ *and the upper bounds when* $U(B) = 0$.

Proof. By Theorem 2.5, we should distinguish three cases in order to apply the same argument of Lemma 2.1. If $m = 1$, then $\xi(2) = -1$ and $|\mathrm{Inv}(C_2)| = 1$. In this case we get

$$2^{\frac{1}{2}n(2^1-2)} \leq |M(A \wr B)| \leq 2^{\frac{1}{2}n(2^1-1)}.$$

If $m = 2$, then $\xi(4) = 0$ and $|\mathrm{Inv}(C_4)| = 2$. In this case, we get

$$2^{\frac{1}{2}n(2^2-3)} \leq |M(A \wr B)| \leq p^{\frac{1}{2}n(2^2+1)}.$$

If $m \geq 3$, then $\xi(2^m) = 1$ and $|\mathrm{Inv}(C_{2^m})| = 4$. In this case, we get

$$2^{\frac{1}{2}n(2^m-5)} \leq |M(A \wr B)| \leq 2^{\frac{1}{2}n(2^m+5)}.$$

∎

Remark 2.2. Lemma 2.1 shows that

$$|M(A \wr B)| \in \{p^{\lfloor \frac{1}{2}n(p^m-3) \rfloor}, p^{\lfloor \frac{1}{2}n(p^m-2) \rfloor}, p^{\lfloor \frac{1}{2}n(p^m-1) \rfloor}, p^{\lfloor \frac{1}{2}np^m \rfloor}, p^{\lfloor \frac{1}{2}n(p^m+1) \rfloor}\},$$

that is, we have just five choices for $|M(A \wr B)|$ and of the above type, for all $m, n \geq 1$. A similar situation happens in Lemma 2.2 (iii), where we find only eleven possible values of $|M(A \wr B)|$ between $2^{\lfloor \frac{1}{2}n(2^m-5) \rfloor}$ and $2^{\lfloor \frac{1}{2}n(2^m+5) \rfloor}$.

The following example is done for convenience of the reader.

Example 2.2. The Schur multipliers of metacyclic p-groups have been computed by Austin, Beyl and Ng independently, see [11, Theorem 2.11.3, Proposition 2.11.4] or [2]. It is well known that $C_2 \wr C_2 \simeq D_8$, which is a metacyclic 2-group, has $M(D_8) \simeq C_2$. We find exactly this result if $m = n = 1$ in Lemma 2.2 (i). On the other hand, P_2 is a Sylow 2-subgroup of S_4 of order 8 and is well known that $P_2 \simeq C_2 \wr C_2 \simeq D_8$. From Theorem 2.4, $s = 1$ and again $M(P_2) \simeq C_2$ is confirmed.

Erovenko and Sury [7] showed that if B is an abelian group of order n and A is an arbitrary abelian group, then the commutativity degree of the wreath product $A \wr B$ tends to $\frac{1}{n^2}$ as the order of A tends to infinity. By the way, Sury has recently investigated some combinatorial properties of wreath products in [18].

Theorem 2.6. *(See* [7, Theorem 1.1]*) Let* A *and* $B = \{b_1, b_2, ..., b_n\}$ *be two abelian groups. Then*

$$d(A \wr B) = \frac{1}{n^2|A|^n} \sum_{s,t=1}^{n} |A|^{\alpha(s,t)},$$

where $\alpha(s,t) = |B : \langle b_s, b_t \rangle|$.

Immediately, we may draw the following conclusion.

Corollary 2.2. *Let* A *and* $B = \{b_1, b_2, ..., b_n\}$ *be two abelian groups. If* $A \wr B$ *is capable, then*

$$\frac{1}{n^2\ |A|^n} \leq d^{\wedge}(A \wr B) \leq \frac{1}{n^2|A|^n} \sum_{s,t=1}^{n} |A|^{\alpha(s,t)}$$

Proof. The upper bound $d^{\wedge}(A \wr B) \leq d(A \wr B)$ is always true by Theorems 2.1 and 2.6. The lower bound follows by Corollary 2.1 because $A \wr B$ is capable. ∎

3. Main theorems

The p-group $E_1 = \langle a,b,c \mid a^p = b^p = c^p = 1, [a,c] = [b,c] = 1, [a,b] = c\rangle$ is extraspecial of order p^3 and exponent p and has $|M(E_1)| = p^2$. It was investigated recently in [16] under our perspective. [16, Theorem 2.2 (i)] shows that

$$d^{\wedge}(E_1) = \sum_{g \in E_1} |C^{\wedge}_{E_1}(g)| = \frac{p^3+p^2-1}{p^5}, \tag{3.1}$$

where the first equality is clear from the definitions but the second depends on the fact that $|C^{\wedge}_{E_1}(g)| = p$ for all $g \in E_1$. Moreover, Niroomand [16] proved a series of results for $d^{\wedge}(P)$ in which the presence of a bound of the form (3.1) for an arbitrary p-group P implies that $P/Z^{\wedge}(P)$ is elementary abelian (see [16, Theorems 2.4 and 2.6]). Similar conditions were studied already in [1, 4, 5, 17] for the commutativity degree and have motivated us to look for a specific type of inequalities in our investigations, which has the formal aspect of (3.1).

We need to recall from [13] that the map

$$\varphi : g \in C_G(x) \mapsto x \wedge g \in M(G) \tag{3.2}$$

is a monomorphism of groups such that $\ker \varphi = C^{\wedge}_G(x)$ and $C_G(x)/C^{\wedge}_G(x)$ is isomorphic to a subgroup of $M(G)$ for all $x \in G$. Consequently,

$$|C_G(x) : C^{\wedge}_G(x)| \leq |M(G)| \tag{3.3}$$

and, in case φ is surjective, we find

$$|C_G(x) : C^{\wedge}_G(x)| = |M(G)|. \tag{3.4}$$

The following example is instructive.

Example 3.1.

(i) The group E_1 satisfies (3.3) properly, because $|C_{E_1}(x) : C^{\wedge}_{E_1}(x)| = p$ for all $x \in E_1$ and $|M(E_1)| = p^2$.

(ii) The extraspecial p-group of order p^3 and exponent p^2 with $p \neq 2$ is $E_2 = \langle a,b,c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [a,c] = [b,c] = 1, [a,b] = c\rangle$ and it satisfies (3.4), because $|C_{E_2}(x) : C^{\wedge}_{E_2}(x)| = |M(E_2)| = 1$ for all $x \in E_2$.

(iii) A cyclic group C_n has $M(C_n) = 1$ (see [11]) and satisfies (3.4), because $|C_{C_n}(x) : C^{\wedge}_{C_n}(x)| = |M(C_n)| = 1$ for all $x \in C_n$.

If $G = P$ is a p-group, then it is not hard to see that $M(P)$ is also a p-group (see [11]) and it is meaningful to introduce

$$u_x = \log_p \frac{|M(P)|}{|C_P(x) : C^{\wedge}_P(x)|} \tag{3.5}$$

in order to measure the gap among (3.3) and (3.4).

Of course, u_x depends on x and $|C_P(x) : C^{\wedge}_P(x)| \cdot p^{u_x} = |M(P)|$ is a bound depending on x. In particular, $u_x = 0$ if and only if $|C_P(x) : C^{\wedge}_P(x)| = |M(P)|$, which is exactly (3.4). Immediately, we observe that all groups with trivial Schur multiplier must satisfy (3.4) and then they have $u_x = 0$. Example 3.1 (ii) and (iii) belong to this case and so they are indicative of a more general fact.

Theorem 3.1. *Let $A = C_{p^m}$, $B = C_{p^n}$, p odd prime, $\alpha(s,t) = |B : \langle b_s, b_t\rangle|$ for $b_s, b_t \in B$ and $m,n,s,t \geq 1$. Then*

$$\frac{1}{p^{\lfloor \frac{1}{2}(2mp^n+n(p^m+5))\rfloor}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)} \leq d^{\wedge}(A \wr B).$$

Moreover, there exist elements $x_1, x_2, \ldots, x_{k(A\wr B)} \in A \wr B$ such that $u = u_{x_1} + u_{x_2} + \cdots + u_{x_{k(A\wr B)}}$ and

$$d^{\wedge}(A \wr B) \leq \frac{1}{p^{m(p^n-1)+n}} + \frac{u}{p^{\lfloor \frac{1}{2}(2mp^n+n(p^m+1))\rfloor}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}.$$

Proof. First of all,

$$|A \wr B| = |B| \cdot |A|^{|B|} = p^n \cdot (p^m)^{p^n} = p^n \cdot p^{mp^n} = p^{n+mp^n}. \tag{3.6}$$

Notice that $Z(A \wr B) = \{(a,a,\ldots,a) \mid a \in A\}$ is the set of elements of $A^{|B|}$ in which the components are equal, that is, the diagonal subgroup of $A^{|B|}$ and so $|Z(A \wr B)| = |A| \geq |Z^{\wedge}(A \wr B)|$. We will prove before the upper bound and then the lower bound.

Since for all $i = 1, 2, \ldots, k(A \wr B)$

$$\left|\frac{C^{\wedge}_{A\wr B}(x_i)}{C_{A\wr B}(x_i)}\right| = \frac{u_{x_i}}{|M(A \wr B)|},$$

we get

$$d^{\wedge}(A \wr B) = \frac{1}{|A \wr B|} \sum_{i=1}^{k(A\wr B)} \left|\frac{C^{\wedge}_{A\wr B}(x_i)}{C_{A\wr B}(x_i)}\right|$$
$$= \frac{1}{|A \wr B|}\left(|Z^{\wedge}(A \wr B)| + \frac{k(A \wr B) - |Z^{\wedge}(A \wr B)|}{|M(A \wr B)|}\right)$$

and, if $u = u_{x_1} + u_{x_2} + \cdots + u_{k(A\wr B)}$, then the above quantity becomes

$$= \frac{u\, k(A \wr B)}{|A \wr B|\, |M(A \wr B)|} + \frac{|Z^{\wedge}(A \wr B)|}{|A \wr B|}\left(1 - \frac{u}{|M(A \wr B)|}\right)$$
$$= u\, \frac{d(A \wr B)}{|M(A \wr B)|} + \frac{|Z^{\wedge}(A \wr B)|}{|A \wr B|}\left(1 - \frac{u}{|M(A \wr B)|}\right)$$
$$\leq u\, \frac{d(A \wr B)}{|M(A \wr B)|} + \frac{|A|}{|B| \cdot |A|^{|B|}}\left(1 - \frac{u}{|M(A \wr B)|}\right)$$
$$= u\, \frac{d(A \wr B)}{|M(A \wr B)|} + \frac{1}{|B| \cdot |A|^{|B|-1}}\left(1 - \frac{u}{|M(A \wr B)|}\right). \tag{3.7}$$

Now Theorem 2.6 implies

$$d(A \wr B) = \frac{1}{p^{2n} p^{mp^n}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)} = \frac{1}{p^{2n+mp^n}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)} \tag{3.8}$$

and, if we replace (3.8) in (3.7) and use (3.6), then we get

$$= \frac{u}{|M(A \wr B)|}\left(\frac{1}{p^{2n+mp^n}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right) + \frac{1}{p^{n+mp^n-m}}\left(1 - \frac{u}{|M(A \wr B)|}\right)$$

$$\leq \frac{u}{|M(A\wr B)|}\left(\frac{1}{p^{2n+mp^n}}\sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right)+\frac{1}{p^{n+mp^n-m}}.$$

But the lower bound in Lemma 2.1 implies $\frac{1}{|M(A\wr B)|} \leq \frac{1}{p^{\lfloor \frac{1}{2}n(p^m-3)\rfloor}}$ and so we may upper bound with

$$\leq \frac{u}{p^{\lfloor \frac{1}{2}n(p^m-3)\rfloor}}\left(\frac{1}{p^{2n+mp^n}}\sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right)+\frac{1}{p^{n+mp^n-m}}$$

$$=\frac{u}{p^{\lfloor \frac{1}{2}(n(p^m+1)+2mp^n)\rfloor}}\sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}+\frac{1}{p^{n+m(p^n-1)}},$$

as claimed.

On the other hand,

$$d^{\wedge}(A\wr B)=\frac{d(A\wr B)}{|M(A\wr B)|}+\frac{|Z^{\wedge}(A\wr B)|}{|A\wr B|}\left(1-\frac{1}{|M(A\wr B)|}\right)\geq \frac{d(A\wr B)}{|M(A\wr B)|}$$

and by Theorem 2.6 and the upper bound of Lemma 2.1 we get

$$=\frac{1}{|M(A\wr B)|}\left(\frac{1}{p^{2n+mp^n}}\sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right)\geq \frac{1}{p^{\lfloor \frac{1}{2}n(p^m+1)\rfloor}}\left(\frac{1}{p^{2n+mp^n}}\sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right)$$

$$=\frac{1}{p^{\lfloor \frac{1}{2}(n(p^m+5)+2mp^n)\rfloor}}\sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}$$

as claimed. ∎

The even case is described below.

Theorem 3.2. *Let $A=C_{2^m}$, $B=C_{2^n}$, $\alpha(s,t)=|B:\langle b_s,b_t\rangle|$ for $b_s,b_t\in B$, $m,n,s,t\geq 1$ and suitable $x_1,x_2,\ldots,x_{k(A\wr B)}\in A\wr B$ such that $u=u_{x_1}+u_{x_2}+\cdots+u_{x_{k(A\wr B)}}$.*

(i) *If $m=1$, then*

$$\frac{1}{2^{\lfloor \frac{1}{2}(m2^{n+1}+5n)\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}\leq d^{\wedge}(A\wr B)\leq \frac{1}{2^{n+m2^n-m}}+\frac{u}{2^{2n+m2^n}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}$$

(ii) *If $m=2$, then*

$$\frac{1}{2^{\lfloor \frac{5}{2}(m2^{n+1}+5n)\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}\leq d^{\wedge}(A\wr B)\leq \frac{1}{2^{n+m2^n-m}}+\frac{u}{2^{\lfloor \frac{1}{2}(m2^{n+1}+5n)\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}$$

(iii) *If $m\geq 3$, then*

$$\frac{1}{2^{\lfloor \frac{1}{2}(m2^{n+1}+n(2^m+9))\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}\leq d^{\wedge}(A\wr B)\leq \frac{1}{2^{n+m2^n-m}}$$

$$+\frac{u}{2^{\lfloor \frac{1}{2}(m2^{n+1}+n(2^m-1))\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}.$$

Proof. We follow the argument of the proof of Theorem 3.1. From Theorem 2.6,

$$d^{\wedge}(A \wr B) \leq \frac{u}{|M(A \wr B)|} \left(\frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \right) + \frac{1}{2^{n+m2^n-m}}$$

and we should distinguish three cases in view of Lemma 2.2. If $m = 1$, then

$$d^{\wedge}(A \wr B) \leq \frac{u}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} + \frac{1}{2^{n+m2^n-m}}.$$

If $m = 2$, then

$$d^{\wedge}(A \wr B) \leq \frac{u}{2^{\lfloor \frac{1}{2}n \rfloor}} \left(\frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \right) + \frac{1}{2^{n+m2^n-m}}.$$

If $m \geq 3$, then

$$d^{\wedge}(A \wr B) \leq \frac{u}{2^{\lfloor \frac{1}{2}n(2^m-5) \rfloor}} \left(\frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \right) + \frac{1}{2^{n+m2^n-m}}.$$

On the other hand,

$$d^{\wedge}(A \wr B) \geq \frac{d(A \wr B)}{|M(A \wr B)|}$$

and the following cases should be considered by Lemma 2.2 and Theorem 2.6. If $m = 1$, then we may lower bound with

$$\geq \frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \geq \frac{1}{2^{\lfloor \frac{1}{2}n \rfloor}} \frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}.$$

If $m = 2$, then we have analogously

$$\geq \frac{1}{2^{\lfloor \frac{5}{2}n \rfloor}} \left(\frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \right).$$

If $m \geq 3$, then we have analogously

$$\geq \frac{1}{2^{\lfloor \frac{1}{2}n(2^m+5) \rfloor}} \left(\frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \right).$$

∎

We end with an application to the Sylow p-subgroups P_n of the symmetric group S_{p^n}, described in Theorem 2.4.

Theorem 3.3. *Let P_n be a capable Sylow p-subgroup of S_{p^n} and $u = u_{x_1} + \cdots + u_{x_{k(P_n)}}$ for suitable $x_1, \ldots, x_{k(P_n)} \in P_n$.*

(i) *If $p \neq 2$, then*

$$d^{\wedge}(P_n) = \frac{u\, d(P_n)}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} + \frac{1}{p^{\frac{1-p^n}{1-p}}} \left(1 - \frac{u}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} \right).$$

(ii) *If $p = 2$, then*

$$d^{\wedge}(P_n) = \frac{u\, d(P_n)}{p^{\frac{1}{6}n(n^2-1)}} + \frac{1}{p^{\frac{1-p^n}{1-p}}} \left(1 - \frac{u}{p^{\frac{1}{6}n(n^2-1)}} \right).$$

Proof. (i). We know from Theorem 2.4 that $P_n = P_1 \wr P_{n-1}$,

$$|P_n| = 1 + p + p^2 + \cdots + p^{n-1} = \frac{1-p^n}{1-p}$$

and $M(P_n) = C_p^s$, where $s = \frac{1}{12}(p-1)(n-1)n(2n-1)$ if $p \neq 2$. Moreover, P_n is capable, then $Z^\wedge(P_n) = 1$. We may repeat the proof of Theorem 3.1 and get

$$\begin{aligned}
d^\wedge(P_n) &= \frac{1}{|P_n|} \sum_{i=1}^{k(P_n)} \left| \frac{C^\wedge_{P_n}(x_i)}{C_{P_n}(x_i)} \right| = \frac{1}{|P_n|} \left(|Z^\wedge(P_n)| + \frac{k(P_n) - |Z^\wedge(P_n)|}{|M(P_n)|} \right) \\
&= \frac{u\,k(P_n)}{|P_n|\,|M(P_n)|} + \frac{|Z^\wedge(P_n)|}{|P_n|} \left(1 - \frac{u}{|M(P_n)|} \right) = u\,\frac{d(P_n)}{|M(P_n)|} + \frac{|Z^\wedge(P_n)|}{|P_n|} \left(1 - \frac{u}{|M(P_n)|} \right) \\
&= u\,\frac{d(P_n)}{|M(P_n)|} + \frac{1}{|P_n|} \left(1 - \frac{u}{|M(P_n)|} \right) = \frac{u}{|M(P_n)|} \left(d(P_n) - \frac{1}{|P_n|} \right) + \frac{1}{|P_n|} \\
&= \frac{u}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} \left(d(P_n) - \frac{1}{p^{1+p+p^2+\ldots+p^{n-1}}} \right) + \frac{1}{p^{1+p+p^2+\ldots+p^{n-1}}} \\
&= \frac{u}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} \left(d(P_n) - \frac{1}{p^{\frac{1-p^n}{1-p}}} \right) + \frac{1}{p^{\frac{1-p^n}{1-p}}} \\
&= \frac{u\,d(P_n)}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} + \frac{1}{p^{\frac{1-p^n}{1-p}}} \left(1 - \frac{u}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} \right).
\end{aligned}$$

(ii). In case $p = 2$, it is enough to replace the term $\frac{1}{12}(p-1)(n-1)n(2n-1)$ with $\frac{1}{6}n(n^2-1)$ by Theorem 2.4. ∎

The importance of Theorem 3.3 is due to the fact that it provides a relation among $d^\wedge(P_n)$ and $d(P_n)$. Since there are several results on the commutativity degree in [1, 4, 5, 6], the term $d(P_n)$ is well known and then Theorem 3.3 is significant.

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