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Multiple Solutions of Nonlinear Boundary Value Problems for Fractional Differential Equations

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Abstract. In this paper, we study nonlinear boundary value problems of fractional differential equations.

$$\begin{cases} \mathbf{D}^q_{0^+}x(t) = f(t,x(t)) & t \in J = [0,T] \\ g\big(x(0),x(T),x(\eta)\big) = 0 & \eta \in [0,T], \end{cases} \tag{0.1}$$

where $\mathbf{D}_{0^+}$ denotes the Caputo fractional derivative, $0 < q \leq 1$. Some new results on the multiple solutions are obtained by the use of the Amann theorem and the method of upper and lower solutions. An example is also given to illustrate our results.

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1. Introduction

Fractional derivatives provide an excellent tool for physics, mechanics, chemistry, engineering, etc, see [1, 6, 8, 13, 14, 15, 22, 28, 37]. There has been a significant development in fractional equations in recent years, see [2, 3, 5, 10, 11, 12, 21, 31, 34, 35], some papers deal with the existence of the solution of initial value problems [19, 20, 30, 32, 33] or linear boundary value problems for fractional differential equations by use of techniques of nonlinear analysis, see [4, 16, 23]. Recently, there is an increasing interest in the study of the existence on multiple solutions for the nonlinear fractional differential equations. Bai *et al.* [9] considered the existence and multiplicity of positive solutions of the nonlinear fractional

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differential equation boundary-value problem

$$\begin{cases} D^{\alpha}_{0+}u(t)+f(t,u(t))=0 \quad 0<t<1 \\ u(0)=u(1)=0, \end{cases} \tag{1.1}$$

where $1<\alpha\leq 2$, D^{α}_{0+} denotes the Riemann-Liouville fractional derivative, by means of Guo-Krasnoselskii fixed point theorem and Leggett-Williams fixed point theorem. In [17], Kaufmann and Mboumi considered the existence and multiplicity of positive solutions of the nonlinear fractional differential equation boundary-value problem

$$\begin{cases} D^{\alpha}_{0+}u(t)+a(t)f(u(t))=0 \quad 0<t<1 \\ u(0)=u'(1)=0, \end{cases} \tag{1.2}$$

where $1<\alpha\leq 2$, D^{α}_{0+} denotes the Riemann-Liouville fractional derivative, a is a positive and continuous function on $[0,1]$. Zhao *et al.* [36] studied the existence on multiple positive solutions for the nonlinear fractional differential equations

$$\begin{cases} D^{\alpha}_{0+}u(t)+f(t,u(t))=0 \qquad 0<t<1 \\ u(0)=u'(0)=u'(1)=0, \end{cases} \tag{1.3}$$

where D^{α}_{0+} denotes the Riemann-Liouville fractional derivative, by the properties of the Green function, the lower and upper solution method and fixed point theorem. Liu and Jia [23] studied the multiple solutions of the following nonlinear fractional two-point boundary value problem

$$\begin{cases} \mathbf{D}^{\alpha}_{0+}x(t)=f(t,x(t),x'(t)) \qquad\qquad t\in J=[0,1] \\ g_0(x(0),x'(0))=0, \\ g_1(x(1),x'(1))=0, \\ x''(0)=x'''(0)=\cdots=x^{n-1}(0)=0, \end{cases} \tag{1.4}$$

by using the Amann theorem and the method of upper and lower solutions,where $\mathbf{D}^{\alpha}_{0+}$ is the standard Caputo derivative, $n>2$ is an integer, $\alpha\in(n-1,n]$. But, it is not sufficient for us to define Caputo derivative $\mathbf{D}^{\alpha}_{0+}x(t)$, if $\alpha\in(n-1,n]$ and $x\in C^1[0,1]$ by the definition of Caputo derivative.

Motivated by the above, we focus on the multiple solutions for nonlinear fractional differential equations with nonlinear boundary value conditions:

$$\begin{cases} \mathbf{D}^{q}_{0+}x(t)=f(t,x(t)) \qquad t\in J=[0,T] \\ g(x(0),x(T),x(\eta))=0 \quad \eta\in[0,T], \end{cases} \tag{1.5}$$

where $\mathbf{D}_{0+}$ denotes the Caputo fractional derivative, $0<q\leq 1$, $f\in C(J\times R,R)$, $g\in C(R^3,R)$. Boundary value conditions in (1.5) include periodic boundary value, anti-periodic boundary value conditions. Therefore, we extend some previous results in many respects [6, 23, 25, 26, 34].

The article is organized as follow. In Section 2, we prepare some material need to prove our results. In Section 3, it is devoted to the multiple solutions for Equation (1.5) by means of the Amann theorem and the method of upper and lower solutions. In Section 4, we give an example that illustrates our results.

2. Background material and preliminaries

For the convenience of the readers, we first present some useful definitions and fundamental facts of fractional calculus theory, which can be found in [18, 27, 29].

Definition 2.1. [18, 27, 29] *Caputo's derivative for a function $f \in C^n[0,\infty)$ can been written as*

$$\mathbf{D}^s_{0+} f(x) = \frac{1}{\Gamma(n-s)} \int_0^x \frac{f^{(n)}(t)dt}{(x-t)^{s+1-n}}, \quad n = [s]+1 \tag{2.1}$$

where $[s]$ denotes the integer part of real number $s > 0$.

Definition 2.2. [18, 27, 29] *For $s > 0$, the integral*

$$I^s_{0+} f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}} dt, \tag{2.2}$$

is called Riemann-Liouville fractional integral of order s.

Lemma 2.1. *(cf.* [18, p. 93–96]*) Let $u \in C^m[0,1]$ and $q \in (m-1,m)$, $m \in N$ and $v \in C[0,1]$. Then, for $t \in [0,1]$,*

(1) $\mathbf{D}^q_{0+} I^q v(t) = v(t)$;

(2) $I^q \mathbf{D}^q_{0+} u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0)$;

(3) $\lim_{t\to 0^+} \mathbf{D}^q_{0+} u(t) = \lim_{t\to 0^+} I^q u(t) = 0$.

Let E be a Banach space, $P \subset E$ be a cone. A cone P is called solid if it contains interior points, i.e. $\mathring{P} \neq \emptyset$. Every cone P in E defines a partial ordering in E given by $x \preceq y$ if and only if $y - x \in P$. If $x \preceq y$ and $x \neq y$, we write $x \prec y$, if a cone P is solid and $y - x \in \mathring{P}$, we write $x \prec\prec y$. A cone P is said to be normal if there exists a constant $N > 0$ such that $0 \preceq x \preceq y$ implies $\| x \| \leq N \| y \|$. If P is normal, then every ordered interval $[x,y] = \{z \in E \mid x \preceq z \preceq y\}$ is bounded. In this paper, the partial ordering " $\preceq$ " is always given by P.

Lemma 2.2. [7] *Let E be a Banach space, and $P \subset E$ be a normal solid cone. Suppose that there exist $\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2 \in E$ with $\alpha_1 \prec \beta_1 \prec \alpha_2 \prec \beta_2$ and $A : [\alpha_1,\ \beta_2] \longrightarrow E$ is a completely continuous strongly increasing operator such that*

$$\alpha_1 \preceq A\alpha_1,\ A\beta_1 \prec \beta_1,\ \alpha_2 \prec A\alpha_2,\ A\beta_2 \preceq \beta_2.$$

Then the operator A has at least three fixed points $x_1,\ x_2,\ x_3$ such that

$$\alpha_1 \preceq x_1 \prec\prec \beta_1,\ \alpha_2 \prec\prec x_2 \preceq \beta_2,\ \alpha_2 \not\preceq x_3 \not\preceq \beta_1.$$

3. Existence results

Let $E = \{x(t) \mid x \in C(J)\}$ be a Banach space endowed with the norm $\| x \|_E = \max_{t\in J} \mid x(t) \mid$. And define the cone $P \subset E$ by

$$P = \{x \in E \mid x(t) \geq 0, t \in [0,T]\}.$$

Obviously, P is a normal solid cone in E, and $x \preceq y \in E$ if and only if $x(t) \leq y(t)$ for $t \in [0,T]$.

Theorem 3.1. *Let $h \in C(J)$, $0 < q \le 1$, $\lambda \ne \mu+\gamma$ and $d,\lambda,\mu,\gamma \in R$. Then the solution of the boundary problem*

$$\begin{cases} \mathbf{D}^q_{0^+}x(t) = h(t) & t \in J = [0,T] \\ \lambda x(0) - \mu x(T) - \gamma x(\eta) = d, & \eta \in [0,T], \end{cases} \tag{3.1}$$

can be represented by

$$\begin{aligned} x(t) = {}& \frac{\mu}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds + \frac{\gamma}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\times \\ & \int_0^\eta (\eta-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds + \frac{d}{\lambda-(\mu+\gamma)}. \end{aligned} \tag{3.2}$$

Proof. Assume x satisfies (3.1), then Lemma 2.1 implies

$$x(t) = I^q_{0^+}h(t) + c_0 = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds + c_0.$$

By the boundary condition, we can obtain that

$$\begin{aligned} c_0 = {}& \frac{\mu}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds \\ & + \frac{\gamma}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}h(s)ds + \frac{d}{\lambda-(\mu+\gamma)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} x(t) = {}& \frac{\mu}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds + \frac{\gamma}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)} \\ & \int_0^\eta (\eta-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds + \frac{d}{\lambda-(\mu+\gamma)}. \end{aligned}$$ ∎

Definition 3.1. *We say that $x(t)$ is a generalized solution of the fractional differential equation (3.1) if $x \in C(J,E)$ and satisfies (3.2). Similarly, we may give the definition of generalized solutions of (1.5).*

Remark 3.1. Obviously, if $x(t) \in C^1(J,E)$ is a solution of (3.1), it is easily to get that $x(t) \in C(J,E)$ is a generalized solution of (3.1) in virtue of Theorem 3.1. However, by the following simple example, a generalized solution of (3.1) is not a solutions of (3.1) in general.

Example 3.1. Let $h(t) = a$ (a is a constant), $q = 1/2$. According to (3.2), we get

$$\begin{aligned} x(t) &= \frac{\mu}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds + \frac{\gamma}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}h(s)ds \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds + \frac{d}{\lambda-(\mu+\gamma)} \\ &= \frac{2a}{\Gamma(1/2)}t^{1/2} + \frac{\mu}{\lambda-(\mu+\gamma)}\frac{2a}{\Gamma(1/2)}T^{1/2} + \frac{\gamma}{\lambda-(\mu+\gamma)}\frac{2a}{\Gamma(1/2)}\eta^{1/2} + \frac{d}{\lambda-(\mu+\gamma)}, \end{aligned}$$

which implies that $x(t) \notin C^1([0,1],E)$. By the definition of Caputo derivative, we could not define Caputo derivative $\mathbf{D}^q_{0^+}x(t)$.

Theorem 3.2. *If* $0 < q \le 1$, $x \in C^1(J)$, $\lambda > \mu + \gamma$, $\mu,\ \gamma \ge 0$ *and*

$$\begin{cases} \mathbf{D}_{0+}^q x(t) \ge 0 & t \in J = [0,T] \\ \lambda x(0) - \mu x(T) - \gamma x(\eta) \ge 0, & \eta \in [0,T], \end{cases}$$

then $x(t) \ge 0$.

Proof. For any $h \in C(J)$, $h(t) \ge 0$, $d \ge 0$. Consider the following equation

$$\begin{cases} \mathbf{D}_{0+}^q x(t) = h(t) & t \in J = [0,T] \\ \lambda x(0) - \mu x(T) - \gamma x(\eta) = d, \end{cases}$$

by Theorem 3.1 and Lemma 2.1, we get

$$x(t) = \frac{\mu}{\lambda - (\mu+\gamma)} \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} h(s) ds + \frac{\gamma}{\lambda - (\mu+\gamma)} \frac{1}{\Gamma(q)} \int_0^\eta (\eta - s)^{q-1} h(s) ds$$
$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds + \frac{d}{\lambda - (\mu+\gamma)}.$$

It is easy to see that $x(t) \ge 0$. We complete the proof. ∎

Definition 3.2. *Let* $\alpha, \beta \in C^1(J)$. α *is called a lower solution of boundary value problem (1.5) if it satisfies*

$$\begin{cases} \mathbf{D}_{0+}^q \alpha(t) \le f(t, \alpha(t)) & t \in J = [0,T] \\ g(\alpha(0), \alpha(T), \alpha(\eta)) \ge 0, & \eta \in [0,T]. \end{cases}$$

β *is called a upper solution of boundary value problem (1.5) if it satisfies*

$$\begin{cases} \mathbf{D}_{0+}^q \beta(t) \ge f(t, \beta(t)) & t \in J = [0,T] \\ g(\beta(0), \beta(T), \beta(\eta)) \le 0, & \eta \in [0,T]. \end{cases}$$

In the sequel, we need the following hypotheses:

(H1) $f : J \times R \to R$ is strictly increasing with respect to the second variable.
(H2) $g(u_2, v_2, w_2) - g(u_1, v_1, w_1) \ge -\lambda(u_2 - u_1) + \mu(v_2 - v_1) + \gamma(w_2 - w_1)$,
where $u_1 \le u_2$, $v_1 \le v_2$, $w_1 \le w_2$, $\lambda > \mu + \gamma$, $\mu > 0, \gamma \ge 0$.

Theorem 3.3. *Assume that (H1) and (H2) hold. And there exist two lower solutions* α_1, α_2 *and two upper solutions* β_1, β_2 *of boundary value problem (1.5) such that* α_2, β_1 *are not the solutions of the boundary value problem (1.5) with* $\alpha_1 \prec \beta_1 \prec \alpha_2 \prec \beta_2$. *Then the boundary value problem (1.5) has at least three distinct generalized solutions* x_1, x_2, x_3 *and satisfies*

$$\alpha_1 \preceq x_1 \prec\prec \beta_1,\ \alpha_2 \prec\prec x_2 \preceq \beta_2,\ \alpha_2 \npreceq x_3 \npreceq \beta_1.$$

Proof. We will prove the theorem in view of Lemma 2.2. For any $u \in [\alpha_1,\ \beta_2]$, consider the following problem

$$\text{(3.3)} \quad \begin{cases} \mathbf{D}_{0+}^q x(t) = f(t, u(t)) & t \in J = [0,T] \\ \lambda x(0) - \mu x(T) - \gamma x(\eta) = g(u(0), u(T), u(\eta)) + \lambda u(0) - \mu u(T) - \gamma u(\eta), \end{cases}$$

by Theorem 3.1, we have

$$
\begin{aligned}
x(t) = {} & \frac{\mu}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1} f(s,u(s))ds \\
& + \frac{\gamma}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1} f(s,u(s))ds \\
& + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s))ds \\
& + \frac{1}{\lambda-(\mu+\gamma)}[g(u(0),u(T),u(\eta))+\lambda u(0)-\mu u(T)-\gamma u(\eta)] =: (Au)(t).
\end{aligned}
$$

It is easy to see that x is the generalized solution of the boundary value problem (1.5) if and only if x is the fixed point of A. We show that $A : [\alpha_1, \beta_2] \longrightarrow E$ is completely continuous.

First, we prove that A is continuous. For $u_1,\ u_2 \in [\alpha_1,\ \beta_2]$,

$$
\begin{aligned}
& \mid Au_2 - Au_1 \mid \\
& = \Bigg| \frac{\mu}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\big[f(s,u_2(s))-f(s,u_1(s))\big]ds \\
& \quad + \frac{\gamma}{\lambda-(\mu+\gamma)}\frac{1}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}\big[f(s,u_2(s))-f(s,u_1(s))\big]ds \\
& \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[f(s,u_2(s))-f(s,u_1(s))\big]ds \\
& \quad + \frac{1}{\lambda-(\mu+\gamma)}\big[g(u_2(0),u_2(T),u_2(\eta))-g(u_1(0),u_1(T),u_1(\eta)) \\
& \quad + \lambda(u_2(0)-u_1(0))-\mu(u_2(T)-u_1(T))-\gamma(u_2(\eta)-u_1(\eta))\big]\Bigg|, \\
& \le \left(\frac{\mu}{\lambda-(\mu+\gamma)}\frac{T^q}{\Gamma(q+1)}+\frac{\gamma}{\lambda-(\mu+\gamma)}\frac{\eta^q}{\Gamma(q+1)}+\frac{t^q}{\Gamma(q+1)}\right) \\
& \quad \max_{s\in J} \mid f(s,u_2(s))-f(s,u_1(s)) \mid \\
& \quad + \Bigg|\frac{1}{\lambda-(\mu+\gamma)}\big[g(u_2(0),u_2(T),u_2(\eta))-g(u_1(0),u_1(T),u_1(\eta)) \\
& \quad + \lambda(u_2(0)-u_1(0))-\mu(u_2(T)-u_1(T))-\gamma(u_2(\eta)-u_1(\eta))\big]\Bigg|,
\end{aligned}
$$

in view of the continuity of $f,\ g$, we have A is continuous.

Next, we claim that A is a compact operator. For $0 \le t_1 \le t_2 \le T$,

$$
\begin{aligned}
& \mid Au(t_2) - Au(t_1) \mid \\
& = \frac{1}{\Gamma(q)}\left|\int_0^{t_2}(t_2-s)^{q-1} f(s,u(s))ds - \int_0^{t_1}(t_1-s)^{q-1} f(s,u(s))dsds\right| \\
& \le \frac{\varpi}{\Gamma(q+1)}\left|\int_0^{t_1}((t_2-s)^{q-1}-(t_1-s)^{q-1})ds\right| + \frac{\varpi}{\Gamma(q+1)}\left|\int_{t_1}^{t_2}(t_2-s)^{q-1}ds\right| \\
& \le \frac{2\varpi}{\Gamma(q+1)}(t_2-t_1)^q,
\end{aligned}
$$

where $\varpi = \max_{s\in J} \mid f(s,u(s)) \mid$, which show that A is equicontinuous. It is obvious that A is uniformly bounded for all $u \in [\alpha_1, \beta_2]$. Therefore, A is compact operator by Ascoli-Arzela theorem.

We show A is strongly increasing operator. For any $u_1,\ u_2 \in [\alpha_1, \beta_2]$, with $u_1 \prec u_2$ i.e. $u_1(t) \le u_2(t)$ and $u_1(t) \not\equiv u_2(t)$. In view of (H1), we have for $\forall\, t \in J$

$$f(t,u_2(t)) - f(t,u_1(t)) \ge 0.$$

There exists an interval $[a,b] \subset [0,T]$ such that $u_1(t) < u_2(t)$ for $t \in [a,b]$ through the fact $u_1(t) \not\equiv u_2(t)$. Hence, by (H1) again

$$f(t,u_2(t)) - f(t,u_1(t)) > 0 \quad t \in [a,b]. \tag{3.4}$$

By (3.4), we have for $\forall\, t \in J$,

$$\begin{aligned}
&(Au_2)(t) - (Au_1)(t) \\
&= \frac{\mu}{\lambda-(\mu+\gamma)} \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1}[f(s,u_2(s)) - f(s,u_1(s))]ds \\
&\quad + \frac{\gamma}{\lambda-(\mu+\gamma)} \frac{1}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1}[f(s,u_2(s)) - f(s,u_1(s))]ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[f(s,u_2(s)) - f(s,u_1(s))]ds \\
&\quad + \frac{1}{\lambda-(\mu+\gamma)} \big[g(u_2(0),u_2(T),u_2(\eta)) - g(u_1(0),u_1(T),u_1(\eta)) \\
&\quad + \lambda(u_2(0)-u_1(0)) - \mu(u_2(T)-u_1(T)) - \gamma(u_2(\eta)-u_1(\eta))\big] \\
&> \frac{\mu}{\lambda-(\mu+\gamma)} \frac{1}{\Gamma(q)} \int_a^b (T-s)^{q-1}[f(s,u_2(s)) - f(s,u_1(s))]ds > 0.
\end{aligned}$$

Thus, $Au_1(t) \prec Au_2(t)$, for $t \in J$, and we get A is strongly increasing operator.

Now, we prove $\alpha_1 \preceq A\alpha_1$. Consider the following problem

$$\begin{cases}
\mathbf{D}_{0+}^q A\alpha_1(t) = f(t,\alpha_1(t)) \quad t \in J = [0,T] \\
\lambda A\alpha_1(0) - \mu A\alpha_1(T) - \gamma A\alpha_1(\eta) = g(\alpha_1(0),\alpha_1(T),\alpha_1(\eta)) \\
\qquad\qquad\qquad\qquad\qquad\qquad + \lambda\alpha_1(0) - \mu\alpha_1(T) - \gamma\alpha_1(\eta).
\end{cases}$$

Set $\alpha(t) = A\alpha_1(t) - \alpha_1(t)$. In view of α_1 a lower solution of Equation (1.1), we get

$$\mathbf{D}_{0+}^q \alpha(t) = \mathbf{D}_{0+}^q A\alpha_1(t) - \mathbf{D}_{0+}^q \alpha_1(t) = f(t,\alpha_1(t)) - \mathbf{D}_{0+}^q \alpha_1(t) \ge 0,$$

$$\begin{aligned}
\lambda\alpha(0) - \mu\alpha(T) - \gamma\alpha(\eta) &= \lambda A\alpha_1(0) - \mu A\alpha_1(T) - \gamma A\alpha_1(\eta) \\
&\quad - (\lambda\alpha_1(0) - \mu\alpha_1(T) - \gamma\alpha_1(\eta)) \\
&= g(\alpha_1(0),\alpha_1(T),\alpha_1(\eta)) \ge 0.
\end{aligned}$$

By Theorem 3.2, we know that $\alpha(t) \ge 0$ and $\alpha_1 \preceq A\alpha_1$.

Similarly, we have $\alpha_2 \preceq A\alpha_2$. We know $\alpha_2 \ne A\alpha_2$, since α_2 is a lower solution of Equation (1.5), and not is a solution of Equation (1.5). Thus $\alpha_2 \prec A\alpha_2$. According to the same way, we can get $A\beta_1 \prec \beta_1$, $A\beta_2 \preceq \beta_2$.

In view of Lemma 2.2, we know that A has at least three fixed points $x_1,\ x_2,\ x_3 \in [\alpha_1, \beta_2]$, moreover

$$\alpha_1 \preceq x_1 \prec\prec \beta_1,\ \alpha_2 \prec\prec x_2 \preceq \beta_2,\ \alpha_2 \not\preceq x_3 \not\preceq \beta_1.$$ ∎

Remark 3.2. (1) In a similar way, we can deal with multiple solutions for problem (1.5) with more general nonlinear boundary conditions

$$g(x(t_0),\ x(t_1),\dots,\ x(t_m))=0,$$

where $0=t_0<t_1<t_2<\cdots<t_m=T$ under some conditions.

(2) We may discuss the extension to fractional order between 1 and 2, and even higher order in the same way.

(3) We have to consider the extension to Riemann-Liouville fractional derivatives by different methods, because of the fact (cf. [27, p.70])

$${}_aD_t^{-p}\big({}_aD_t^{p}x(t)\big)=x(t)-\sum_{j=1}^{n}\big[{}_aD_t^{p-j}x(t)\big]_{t=a}\frac{(t-a)^{p-j}}{\Gamma(p-j+1)},\quad n-1\le p\le n.$$

See [27] for the definition of the Riemann-Liouville fractional integral ${}_aD_t^{-p}$.

4. Example

Consider the following problems

$$\begin{cases}\mathbf{D}_{0^+}^{1/2}x(t)=4t^{1/2}/\pi\arctan(e^{x(t)}) & t\in J=[0,1]\\ x(0)-1/4x(1)=1/2x(\eta),\end{cases}\tag{4.1}$$

where $f(t,x(t))=4t^{1/2}/\pi\arctan(e^{x(t)})$, $g(x(0),\ x(1),\ x(\eta))=x(0)-1/4x(1)-1/2x(\eta)$, $\eta=1/2$, $\lambda=1$, $\mu=1/4$, $\gamma=1/2$. We can easily verify that (H1) and (H2) hold. It is easy to see that $\alpha_1=t+2$, $\alpha_2=2t+7$ are the lower solutions, $\beta_1=4t^2+3,\beta_2=16t^2/3+8$ are the upper solutions of Equation (4.1).Thus, all the condition of Theorem 3.3 are satisfied and the problem (4.1) has at least three fixed points $x_1,\ x_2,\ x_3\in[\alpha_1,\beta_2]$, moreover

$$\alpha_1\preceq x_1\prec\prec\beta_1,\ \alpha_2\prec\prec x_2\preceq\beta_2,\ \alpha_2\not\preceq x_3\not\preceq\beta_1.$$

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