BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37***(1) (2014), 261–269*

The Comaximal Graph of a Lattice

[1]MOJGAN AFKHAMI AND [2]KAZEM KHASHYARMANESH
[1]Department of Mathematics, University of Neyshabur, P.O.Box 91136-899, Neyshabur, Iran
[2]Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O.Box 1159-91775, Mashhad, Iran
[1]mojgan.afkhami@yahoo.com, [2]khashyar@ipm.ir

Abstract. In this paper we introduce the comaximal graph of a finite bounded lattice L, denoted by $\Gamma(L)$. We also study the interplay of lattice-theoretic properties of L with graph-theoretic properties of $\Gamma(L)$.

2010 Mathematics Subject Classification: 05C10, 06A07, 06B10

Keywords and phrases: Lattice, comaximal graph, co-atom of a lattice, planarity.

1. Introduction

The investigation of graphs related to various algebraic structures is a very large and growing area of research. In particular, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see [11, 12, 15, 17, 20] for examples of recent results and further references. Several other classes of graphs associated with algebraic structures have been also actively investigated. For example, power graphs and divisibility graphs have been considered in [9, 10], zero-divisor graphs have been studied in [2–4, 6, 8], and cozero-divisor graphs have been introduced in [1].

Let R be a non-zero commutative ring with identity. In [16], Sharma and Bhatwadekar defined the comaximal graph on R, denoted by $\Gamma(R)$, with all elements of R being the vertices of $\Gamma(R)$, where two distinct vertices a and b are adjacent if and only if $Ra+Rb=R$. In [13] and [18], the authors considered a subgraph $\Gamma_2(R)$ of $\Gamma(R)$ consisting of non-unit elements of R, and studied several properties of the comaximal graph. Also the comaximal graph of a non-commutative ring was defined and studied in [19].

In this paper, we introduce and study the comaximal graph of a finite bounded lattice. In Section 2, we discuss algebraic properties of a lattice L and we characterize all maximal ideals of L. In Section 3, we define the comaximal graph of a lattice L, denoted by $\Gamma(L)$, and study some graph-theoretic properties of it. Moreover, among other things, we determine the clique number and chromatic number of this graph. Finally, in Section 4, we completely describe the planarity of the comaximal graph $\Gamma(L)$.

Now we recall some definitions and notation on graphs. We use the standard terminology of graphs following [5]. In a graph G, the *distance* between two distinct vertices a and b,

Communicated by Sanming Zhou.
Received: October 25, 2011; *Revised:* February 6, 2012.

denoted by $\mathrm{d}(a,b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a,b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a,b) : a$ and b are distinct vertices of $G\}$. The *girth* of G, denoted by $\mathrm{g}(G)$, is the length of the shortest cycle in G, if G contains a cycle; otherwise, we set $\mathrm{g}(G) := \infty$. Also, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. We say that G is *totally disconnected* if no two vertices of G are adjacent. Also, G is called an *empty graph* if its vertex-set is empty. A *clique* of a graph is a complete subgraph of it and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. For a positive integer r, an *r-partite graph* is one whose vertex-set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite graph* (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [5, p. 153]).

2. Basic definitions and properties

In this section, firstly we recall some definitions and notations on lattices.

A *lattice* is an algebra $L = (L, \wedge, \vee)$ satisfying the following conditions: for all $a, b, c \in L$,

1. $a \wedge a = a,\ a \vee a = a$,
2. $a \wedge b = b \wedge a,\ a \vee b = b \vee a$,
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c),\ a \vee (b \vee c) = (a \vee b) \vee c$, and
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

Note that in every lattice $a \wedge b = a$ always implies that $a \vee b = b$.

In the next theorem, we recall an equivalent definition of a lattice with respect to a partial order relation which will be used in this paper.

Theorem 2.1. [14, Theorem 2.1] *Let L be a lattice. One can define an order $\leqslant$ on L as follows:*

For any $a, b \in L$, we set $a \leqslant b$ if and only if $a \wedge b = a$. Then $(L, \leqslant)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) *and a least upper bound* (l.u.b.). *Conversely, let P be an ordered set such that, for every pair $a, b \in P$,* $\mathrm{g.l.b.}(a,b), \mathrm{l.u.b.}(a,b) \in P$. *For each a and b in P, we define $a \wedge b := \mathrm{g.l.b.}(a,b)$ and $a \vee b := \mathrm{l.u.b.}(a,b)$. Then $(P, \wedge, \vee)$ is a lattice.*

A lattice L is said to be *bounded* if there are elements 0 and 1 in L such that $0 \wedge a = 0$ and $a \vee 1 = 1$, for all $a \in L$.

Definition 2.1. [7, Definition 39] *A non-empty subset I of a lattice L is called an* ideal *of L if and only if the following conditions are satisfied:*

(i) *For $a, b \in I$, $a \vee b \in I$.*

(ii) *For* $a \in I$ *and* $c \in L$, $a \wedge c \in I$.

An ideal I of L is *proper* if $I \neq L$.

Theorem 2.2. [7, Theorem 59] *For an ideal* I *of* L, *the following conditions are satisfied:*

(i) *If* $a \in I$ *and* $b \leqslant a$, *then* $b \in I$.
(ii) *If* $a \vee b \in I$, *then we have* $a, b \in I$.

Let I and J be ideals of a lattice L. Consider the set C of all elements c of L such that $c \leqslant a \vee b$, for some elements $a \in I$ and $b \in J$. Clearly, C is non-empty, because it obviously contains every element of I and of J. Also, by [7, Theorem 65], C is the least ideal (with respect to inclusion) containing I and J. We write $I \vee J$ for C. The ideal $I \vee J$ is said to be the ideal generated by the set-union $S = I \cup J$. If S consists of a single element a, then the ideal generated by the set $\{a\}$ is called the *principal ideal* generated by a; it consists of all $x \leqslant a$ and will be denoted by $[a]^\ell$ (see [7, Definition 41]). It is easy to see that, for each two principal ideals $[a]^\ell$ and $[b]^\ell$, we have the following equalities:

$$[a]^\ell \wedge [b]^\ell = [a \wedge b]^\ell, [a]^\ell \vee [b]^\ell = [a \vee b]^\ell.$$

A lattice L is said to be *distributive* if and only if, for all elements $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

In a distributive lattice L, for all $a, b, c \in L$, we have

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Proposition 2.1. *Let* L *be a distributive lattice and,* I *and* J *be ideals of* L. *Then*

$$I \vee J = \{a \vee b \mid a \in I,\ b \in J\}.$$

Proof. Set $S := \{a \vee b \mid a \in I,\ b \in J\}$. Since L is distributive, it is easy to check that S is an ideal of L which contains $I \cup J$. Now let K be an ideal of L which contains $I \cup J$. Then clearly K contains S. Since $I \vee J$ is the least ideal containing I and J, the result holds, and so we have $I \vee J = \{a \vee b \mid a \in I,\ b \in J\}$. ∎

Proposition 2.2. *Let* I *be an ideal of* L. *Then* $I = L$ *if and only if* $1 \in I$.

Proof. Clearly if $I = L$, then $1 \in I$. Now, suppose that $1 \in I$ and that $a \in L$. Then $a = a \wedge 1 \in I$. So $I = L$. ∎

In the following definition, we introduce a *unit* element in a lattice.

Definition 2.2. *An element* a *in* L *is said to be a* unit *if there exists an element* b *in* L *such that* $a \wedge b = 1$.

Note that 1 is the only unit element in every lattice, because if $a \wedge b = 1$, since $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$, we have $1 \leqslant a$ and $1 \leqslant b$ which implies that $a = 1 = b$.

Proposition 2.3. *Suppose that* $a, b \in L$. *Then* $[a]^\ell \vee [b]^\ell = L$ *if and only if* $a \vee b = 1$.

Proof. First let us assume that $[a]^\ell \vee [b]^\ell = L$. Then we have $[a \vee b]^\ell = L$. Thus $1 \leqslant a \vee b$, and also $a \vee b \leqslant 1$, which implies that $a \vee b = 1$.

Conversely, suppose that $a \vee b = 1$. Since, for any element $c \in L$, $c \leqslant 1$ we have that $c \in [a \vee b]^\ell$. Hence $[a \vee b]^\ell = L$, and so $[a]^\ell \vee [b]^\ell = L$. ∎

Definition 2.3. 1. *In a partially ordered set* $(P, \leqslant)$, *we say that* a *covers* b *or* b *is covered by* a, *in notation* $b \prec a$, *if and only if* $b < a$ *and there is no element* p *in* P *such that* $b < p < a$.

2. *An element* a *in* L *is called an atom if* $0 \prec a$. *Similarly,* a *is called a co-atom if* $a \prec 1$. *We denote the sets of all atoms and co-atoms in a lattice* L *by* $A(L)$ *and* $C(L)$, *respectively.*

A *maximal ideal* of L is a proper ideal which is maximal among all ideals of L. We denote the set of all maximal ideals of L by $\mathrm{Max}(L)$. Also, one can easily check that the set

$$J(L) := \bigcap_{\mathfrak{m} \in \mathrm{Max}(L)} \mathfrak{m}$$

is an ideal of L. We call it the *Jacobson radical* of L.

In the following theorem, we characterize all maximal ideals of L in terms of the co-atoms of L.

Theorem 2.3. *In a lattice* L, *we have*

$$\mathrm{Max}(L) = \{[\mathfrak{m}]^{\ell} \mid \mathfrak{m} \in C(L)\},$$

and so the number of maximal ideals in L *is equal to the number of co-atoms of* L; *in other words, we have* $|\mathrm{Max}(L)| = |C(L)|$.

Proof. Let I be a maximal ideal of L. Then we have the following cases:

Case 1. There exists a co-atom $\mathfrak{m} \in C(L)$ such that $\mathfrak{m} \in I$. Then clearly $I \supseteq [\mathfrak{m}]^{\ell}$. Now, if there is an element $a \in I \setminus [\mathfrak{m}]^{\ell}$, then we have that $a \vee \mathfrak{m} \in I$. Since $\mathfrak{m} \leqslant a \vee \mathfrak{m}$, one can conclude that $a \vee \mathfrak{m} = 1$ or $a \vee \mathfrak{m} = \mathfrak{m}$. If $a \vee \mathfrak{m} = 1$, then, by Proposition 2.2, $I = L$ which is impossible. Otherwise, $a \vee \mathfrak{m} = \mathfrak{m}$. In this situation, $a \leqslant \mathfrak{m}$, and so $a \in [\mathfrak{m}]^{\ell}$ which is again impossible. Thus $I = [\mathfrak{m}]^{\ell}$.

Case 2. Assume that I doesn't contain any co-atom. Let S be the set of all maximal elements in I. If $|S| > 1$, then assume that a and b are two distinct elements in S. Since I is an ideal, we have $a \vee b \in I$. Also $a \leqslant a \vee b$ and $b \leqslant a \vee b$. Since a and b are maximal elements in I, we have $a = a \vee b = b$, which is impossible. Therefore, S is singleton. Let a be the unique maximal element in I. Then $a \leqslant \mathfrak{m}$, for some $\mathfrak{m} \in C(L)$. Thus $I \subset [\mathfrak{m}]^{\ell}$ which is impossible.

Also, it is easy to check that, for each $\mathfrak{m} \in C(L)$, $[\mathfrak{m}]^{\ell}$ is a maximal ideal in L. Hence the results follow. ∎

3. Comaximal graph of a lattice

In the rest of the paper, we assume that L is a finite bounded lattice. We define the comaximal graph of a lattice L, denoted by $\Gamma(L)$, as an undirected graph with all elements of L being the vertices, and two distinct vertices a and b are adjacent if and only if $[a]^{\ell} \vee [b]^{\ell} = L$ or equivalently $a \vee b = 1$.

In the comaximal graph $\Gamma(L)$, the vertex 1, which is the only unit element in L, is adjacent to all other vertices, and so $\Gamma(L)$ is a refinement of a star graph with center 1. Thus we consider all non-unit elements $L \setminus \{1\}$ as vertex-set and denote this set by $W(L)$. By Theorem 2.3, one can easily see that $W(L)$ is the set $\bigcup_{\mathfrak{m} \in \mathrm{Max}(L)} \mathfrak{m}$.

We begin this section with the following proposition.

Proposition 3.1. *An induced subgraph of* $\Gamma(L)$ *with vertex-set* $J(L)$ *is totally disconnected and it is disjoint from an induced subgraph with vertices in* $W(L) \setminus J(L)$.

Proof. Suppose that a and b are arbitrary elements in $J(L)$. If a and b are adjacent, then we have $a \vee b = 1$. Also, since $a, b \in J(L)$, there exists $\mathfrak{m} \in C(L)$ such that $a \leqslant \mathfrak{m}$ and $b \leqslant \mathfrak{m}$. This implies that $1 = a \vee b \leqslant \mathfrak{m}$ which is impossible. So a is not adjacent to b.

Now, suppose that $a \in J(L)$ and $b \in W(L) \setminus J(L)$. Since there exists $\mathfrak{m} \in C(L)$ such that $a \leqslant \mathfrak{m}$ and $b \leqslant \mathfrak{m}$, again one can conclude that a is not adjacent to b. ∎

In Proposition 3.1, we showed that all vertices in $J(L)$ are isolated vertices. Therefore we ignore these isolated vertices and consider an induced subgraph of $\Gamma(L)$ with vertex-set $W(L) \setminus J(L)$, which will be denoted by $\Gamma_2(L)$. Note that $\Gamma(L)$ is totally disconnected if and only if $|C(L)| = 1$ and, in this situation, by Theorem 2.3, we have $W(L) = J(L)$. So in the rest of this section, for studying some basic properties of $\Gamma_2(L)$, we assume that $|C(L)| \geqslant 2$.

In the next theorem, we study the connectedness and diameter of $\Gamma_2(L)$.

Theorem 3.1. *The graph* $\Gamma_2(L)$ *is connected and* $\operatorname{diam}(\Gamma_2(L)) \leqslant 3$.

Proof. Let x and y be two distinct vertices in $W(L) \setminus J(L)$. Since $x, y \notin J(L)$, there exist maximal ideals $[\mathfrak{m}]^\ell$ and $[\mathfrak{m}']^\ell$ such that $x \notin [\mathfrak{m}]^\ell$ and $y \notin [\mathfrak{m}']^\ell$. Thus $x \vee \mathfrak{m} = 1$ and $y \vee \mathfrak{m}' = 1$. This means that x is adjacent to $\mathfrak{m}$ and y is adjacent to $\mathfrak{m}'$. Now, if $\mathfrak{m} = \mathfrak{m}'$, then we have the path $x - \mathfrak{m} - y$. Otherwise, $\mathfrak{m} \neq \mathfrak{m}'$. Therefore $\mathfrak{m} \leqslant \mathfrak{m} \vee \mathfrak{m}'$ and since $\mathfrak{m}$ is a co-atom, $\mathfrak{m} \vee \mathfrak{m}' = \mathfrak{m}$ or $\mathfrak{m} \vee \mathfrak{m}' = 1$. Now, if $\mathfrak{m} \vee \mathfrak{m}' = \mathfrak{m}$, then we have that $\mathfrak{m}' \leqslant \mathfrak{m}$ which is impossible. So $\mathfrak{m} \vee \mathfrak{m}' = 1$ and one can find the path $x - \mathfrak{m} - \mathfrak{m}' - y$ between x and y. Therefore $\Gamma_2(L)$ is connected, and with the above discussion, we have $\operatorname{diam}(\Gamma_2(L)) \leqslant 3$. ∎

Proposition 3.2. *The graph* $\Gamma_2(L)$ *is complete if and only if* $W(L) \setminus J(L) = C(L)$.

Proof. First suppose that $\Gamma_2(L)$ is complete. Suppose to the contrary that there exists an element $x \in (W(L) \setminus J(L)) \setminus C(L)$. So there is a co-atom $\mathfrak{m}$ in $C(L)$ such that $x \leqslant \mathfrak{m}$. Thus $x \vee \mathfrak{m} \neq 1$. This means that x and $\mathfrak{m}$ are not adjacent which is a contradiction.

Conversely, suppose that $W(L) \setminus J(L) = C(L)$. Since, for each two distinct elements $\mathfrak{m}$ and $\mathfrak{m}'$ in $C(L)$, $\mathfrak{m} \vee \mathfrak{m}' = 1$, we have $\mathfrak{m}$ is adjacent to $\mathfrak{m}'$. Thus the induced subgraph of $\Gamma_2(L)$ with vertex-set $C(L)$ is complete, and hence the result holds. ∎

In the following theorem, we study complete n-partite comaximal graphs.

Theorem 3.2. *For a positive integer n, the graph* $\Gamma_2(L)$ *is a complete n-partite graph if and only if* $|C(L)| = n$ *and, for each two distinct maximal ideals* $[\mathfrak{m}]^\ell$ *and* $[\mathfrak{m}']^\ell$, $[\mathfrak{m}]^\ell \cap [\mathfrak{m}']^\ell = J(L)$.

Proof. At first suppose that $\Gamma_2(L)$ is a complete n-partite graph. If $|C(L)| > n$, then there exists a part with at least two co-atoms, say $\mathfrak{m}$ and $\mathfrak{m}'$. But we have $\mathfrak{m} \vee \mathfrak{m}' = 1$, and so $\mathfrak{m}$ is adjacent to $\mathfrak{m}'$ which is impossible. Hence $|C(L)| \leqslant n$. If $|C(L)| < n$, then there exists a part which doesn't contain any co-atom. Let x belong to this part. Then $x \leqslant \mathfrak{m}$, for some $\mathfrak{m} \in C(L)$. Clearly, $x \vee \mathfrak{m} \neq 1$, and so x is not adjacent to $\mathfrak{m}$ which is impossible, since by our assumption $\Gamma_2(L)$ is a complete n-partite graph. Thus we have $|C(L)| = n$. Now, assume to the contrary that there exist two distinct maximal ideals $[\mathfrak{m}]^\ell$ and $[\mathfrak{m}']^\ell$ with $[\mathfrak{m}]^\ell \cap [\mathfrak{m}']^\ell \neq J(L)$. Let V and V' be two parts that contain $\mathfrak{m}$ and $\mathfrak{m}'$, respectively. Also, suppose that $x \in ([\mathfrak{m}]^\ell \cap [\mathfrak{m}']^\ell) \setminus J(L)$. Then we have $x \vee \mathfrak{m} \neq 1$ and $x \vee \mathfrak{m}' \neq 1$. So x is not adjacent to both $\mathfrak{m}$ and $\mathfrak{m}'$. Since $\Gamma_2(L)$ is a complete n-partite graph, we have $x \in V \cap V'$, which is a contradiction.

For the converse statement, set $V_i := [\mathfrak{m}_i]^\ell \setminus J(L)$, where $\mathfrak{m}_i \in C(L)$. Then one can easily check that $\Gamma_2(L)$ is a complete n-partite graph with parts V_i, for $i = 1, \ldots, n$. ∎

Corollary 3.1. (i) *If $\Gamma_2(L)$ is n-partite, then $|C(L)| \leqslant n$.*
(ii) *The graph $\Gamma_2(L)$ is a complete bipartite graph if and only if $|C(L)| = 2$.*

In the following result, we investigate the girth of $\Gamma_2(L)$.

Theorem 3.3. *In the graph $\Gamma_2(L)$, we have* $\mathrm{g}(\Gamma_2(L)) \in \{3,4,\infty\}$.

Proof. If $|C(L)| \geqslant 3$, then, by choosing distinct elements $\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{m}_3$ in $C(L)$, we have the cycle $\mathfrak{m}_1-\mathfrak{m}_2-\mathfrak{m}_3-\mathfrak{m}_1$, and so $\mathrm{g}(\Gamma_2(L)) = 3$. If $|C(L)| = 2$, then $\mathrm{Max}(L) = \{[\mathfrak{m}]^\ell,[\mathfrak{m}']^\ell\}$ and $\Gamma_2(L)$ is a complete bipartite graph. In this situation, if $|[\mathfrak{m}]^\ell \setminus [\mathfrak{m}']^\ell|, |[\mathfrak{m}']^\ell \setminus [\mathfrak{m}]^\ell| \geqslant 2$, then clearly $\mathrm{g}(\Gamma_2(L)) = 4$. Otherwise, we have $\mathrm{g}(\Gamma_2(L)) = \infty$. ∎

In the next result, we determine the clique number and chromatic number of $\Gamma_2(L)$.

Theorem 3.4. *In the graph $\Gamma_2(L)$, we have*

$$\chi(\Gamma_2(L)) = \omega(\Gamma_2(L)) = |C(L)|.$$

Proof. Assume that t is the number of co-atoms in L and $\mathrm{Max}(L) = \{[\mathfrak{m}_1]^\ell,\ldots,[\mathfrak{m}_t]^\ell\}$. Put $S_1 := [\mathfrak{m}_1]^\ell$ and $S_i := [\mathfrak{m}_i]^\ell \setminus \bigcup_{j=1}^{i-1}[\mathfrak{m}_j]^\ell$, for $i = 2,\ldots,t$. Clearly there is no adjacency between vertices in S_i and $S_i \cap S_j = \emptyset$, for all $i \neq j$. Also $W(L) = \bigcup_{i=1}^{t} S_i$. Thus we have $\chi(\Gamma_2(L)) \leqslant t$. Now, since the set $\{\mathfrak{m}_1,\ldots,\mathfrak{m}_t\}$ forms a clique for $\Gamma_2(L)$, one can conclude that $t \leqslant \omega(\Gamma_2(L))$. On the other hand, it is clear that $\omega(\Gamma_2(L)) \leqslant \chi(\Gamma_2(L))$. So the result holds. ∎

Since 1 is the only unit element in L which is adjacent to all other vertices, by Theorem 3.4, we have the following result.

Corollary 3.2. *The following equalities hold.*

$$\chi(\Gamma(L)) = \omega(\Gamma(L)) = |C(L)|+1$$

We end this section with the following proposition.

Proposition 3.3. *Let L' be a finite bounded lattice such that $\Gamma_2(L) \cong \Gamma_2(L')$. Then we have $|C(L)| = |C(L')|$ and $|L \setminus J(L)| = |L' \setminus J(L')|$.*

Proof. Since $\Gamma_2(L) \cong \Gamma_2(L')$, we have $\omega(\Gamma_2(L)) = \omega(\Gamma_2(L'))$. Therefore, by Theorem 3.4, we conclude that $|C(L)| = |C(L')|$. Also, clearly $|L \setminus J(L)| = |L' \setminus J(L')|$. ∎

4. Planar comaximal graph of a lattice

In this section, we investigate the planarity of the graph $\Gamma(L)$. Since the vertices in the set $J(L)$ are isolated vertices, one can easily see that $\Gamma(L)$ is planar if and only if $\Gamma_2(L)$ is planar. Therefore, we ignore the isolated vertices and study the planarity of $\Gamma_2(L)$.

We begin this section with the following lemma.

Lemma 4.1. *If $\Gamma_2(L)$ is planar, then $|C(L)| \leqslant 4$.*

Proof. Assume to the contrary that $|C(L)| \geqslant 5$. Let $\mathfrak{m}_1,\ldots,\mathfrak{m}_5$ be distinct elements in $C(L)$. Clearly, for each i,j with $1 \leqslant i \neq j \leqslant 5$, we have $\mathfrak{m}_i \vee \mathfrak{m}_j = 1$, and so $\mathfrak{m}_i$ is adjacent to $\mathfrak{m}_j$. Thus K_5 is a subgraph of $\Gamma_2(L)$, and hence, by Kuratowski's Theorem, it is not planar which is a contradiction. Hence $|C(L)| \leqslant 4$. ∎

If $|C(L)| = 1$, then $\Gamma_2(L)$ is an empty graph. Now, suppose that $|C(L)| = 2$. By part (ii) of Corollary 3.1, we have that $\Gamma_2(L)$ is a complete bipartite graph. So $\Gamma_2(L)$ is planar if and only if $|[\mathfrak{m}_1]^\ell \setminus [\mathfrak{m}_2]^\ell| \leqslant 2$ or $|[\mathfrak{m}_2]^\ell \setminus [\mathfrak{m}_1]^\ell| \leqslant 2$, where $C(L) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$.

Hence the only remaining cases to consider are $|C(L)| = 3$ and $|C(L)| = 4$.

Notations 4.1. To simplify notation, we denote the maximal ideal $[\mathfrak{m}]^\ell$, where $\mathfrak{m} \in C(L)$, by $\mathfrak{m}$. Suppose that $\mathrm{Max}(L) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_t\}$, where $t > 1$. We set $S_j := \mathfrak{m}_j \setminus \bigcup_{\mathfrak{m}_i \neq \mathfrak{m}_j} \mathfrak{m}_i$, $S_{j_1 j_2} := (\mathfrak{m}_{j_1} \cap \mathfrak{m}_{j_2}) \setminus \bigcup_{i \notin \{j_1, j_2\}} \mathfrak{m}_i$ and $S_{j_1 j_2 j_3} := (\mathfrak{m}_{j_1} \cap \mathfrak{m}_{j_2} \cap \mathfrak{m}_{j_3}) \setminus \bigcup_{i \notin \{j_1, j_2, j_3\}} \mathfrak{m}_i$, where $1 \leqslant j_1 < j_2 < j_3 \leqslant t$.

Note that each element in S_i is adjacent to all elements of S_j, for $i \neq j$, and also it is adjacent to all elements in $S_{j_1 j_2}$ and $S_{j_1 j_2 j_3}$, where $j_1, j_2, j_3 \notin \{i\}$.

Now consider the case where $|C(L)| = 3$. Put $\mathrm{Max}(L) := \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$. If there exist distinct i and j with $1 \leqslant i, j \leqslant 3$ such that $|S_i|, |S_j| \geqslant 3$, then $K_{3,3}$ is a subgraph of $\Gamma_2(L)$, and so it is not planar.

Assume that there exists a unique S_i, say S_1, such that $|S_1| \geqslant 3$. In this situation $\Gamma_2(L)$ is planar if and only if $|S_2| = |S_3| = 1$ and $S_{23} = \emptyset$. Now, suppose that, for all $1 \leqslant i \leqslant 3$, $|S_i| \leqslant 2$. At first assume that, for all i, $|S_i| = 2$. If there exists a non-empty S_{ij}, then without loss of generality, we may assume that $S_{12} \neq \emptyset$ and in this situation, we have the subdivision of $K_{3,3}$ in $\Gamma_2(L)$ as it is shown in Figure 1, where $a_1, a_1' \in S_1$, $a_2, a_2' \in S_2$, $a_3, a_3' \in S_3$ and $b \in S_{12}$. Thus $\Gamma_2(L)$ is not planar.

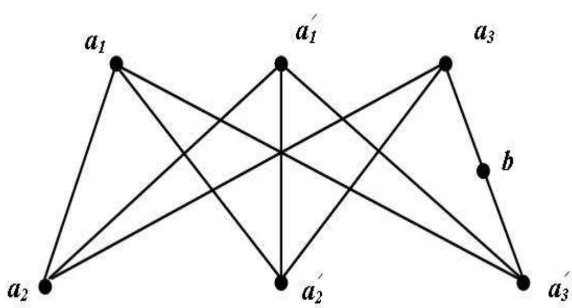

Figure 1. A subdivision of $K_{3,3}$

Suppose that there are distinct S_i and S_j, without loss of generality, S_2 and S_3 with $|S_2| = |S_3| = 2$. In this situation, if $S_{1,3}, S_{1,2} \neq \emptyset$, then $\Gamma_2(L)$ has the following subdivision of K_5, where $a_1 \in S_1$, $a_2, a_2' \in S_2$, $a_3, a_3' \in S_3$, $b \in S_{13}$ and $c \in S_{12}$. Hence $\Gamma_2(L)$ is not planar.

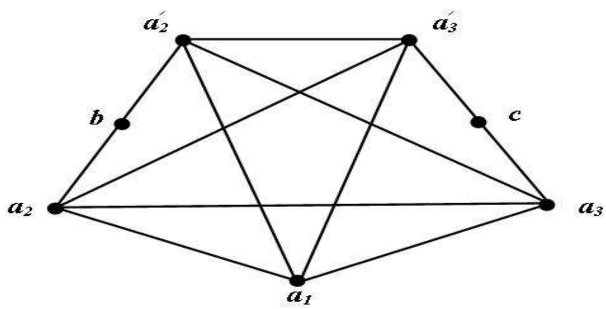

Figure 2. A subdivision of K_5

If there exists only one S_i with $|S_i| = 2$ or, for all i, $|S_i| = 1$, then one can easily check that $\Gamma_2(L)$ is planar.

By the above discussions, we have the following theorem.

Theorem 4.2. *Suppose that $|C(L)| = 3$. Then $\Gamma_2(L)$ is planar if and only if one of the following conditions hold.*

(i) $|S_i| \geqslant 3$ *and* $|S_j| = 1$*, for all* $j \neq i$*, and* $S_{jk} = \emptyset$*, for* $j \neq i \neq k$*, where* $1 \leqslant i, j, k \leqslant 3$.

(ii) $|S_i| = 2$*, for all* $1 \leqslant i \leqslant 3$*, and* $S_{ij} = \emptyset$*, for all* $1 \leqslant i < j \leqslant 3$.

(iii) $|S_i| = |S_j| = 2$*, for some* i *and* j *with* $1 \leqslant i < j \leqslant 3$*, and* $S_{ki} = \emptyset$ *or* $S_{kj} = \emptyset$*, where* $k \notin \{i, j\}$ *and* $1 \leqslant k \leqslant 3$.

(iv) *There is a unique* S_i *with* $|S_i| = 2$*, and* $|S_j| = 1$*, for all* $j \neq i$*, where* $1 \leqslant i, j \leqslant 3$.

(v) *For all* $1 \leqslant i \leqslant 3$, $|S_i| = 1$.

Now, to complete the study of planarity of $\Gamma_2(L)$, we only need to consider the case where $|C(L)| = 4$.

Theorem 4.3. *Assume that $|C(L)| = 4$. Then $\Gamma_2(L)$ is planar if and only if one of the following conditions hold.*

(i) *For all* $1 \leqslant i \leqslant 4$, $|S_i| = 1$.

(ii) *There exists only one* S_i *with* $|S_i| = 2$ *and, in this situation,* $S_{jk} = \emptyset$ *for all* $j, k \notin \{i\}$*, and* $S_{jkl} = \emptyset$ *for* $j, k, l \notin \{i\}$.

Proof. If one of the conditions (i) or (ii) holds true, then one can easily check that $\Gamma_2(L)$ is planar.

Conversely, suppose that $\Gamma_2(L)$ is planar and assume to the contrary that neither (i) nor (ii) is satisfied. So there is some i with $1 \leqslant i \leqslant 4$, say $i = 1$, such that $|S_1| \geqslant 2$. Now we have the following cases:

Case 1. There exists some j with $2 \leqslant j \leqslant 4$, say $j = 2$, such that $|S_2| \geqslant 2$. Then the vertices of the set $\{a_1, a_2, c\} \cup \{b_1, b_2, d\}$ form the graph $K_{3,3}$, where $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$ and $d \in S_4$. Thus $\Gamma_2(L)$ is not planar which is the required contradiction.

Case 2. For all $2 \leqslant j \leqslant 4$, $|S_j| = 1$ and we have $S_{jk} \neq \emptyset$, for some $1 < j < k$, say $S_{23} \neq \emptyset$ or $S_{234} \neq \emptyset$. If $S_{23} \neq \emptyset$, then the vertices of the set $\{a_1, a_2, d\} \cup \{b, c, e\}$ form the graph $K_{3,3}$, where $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$ and $e \in S_{23}$. Thus $\Gamma_2(L)$ is not planar, a contradiction. Now if $S_{234} \neq \emptyset$, then the vertices of the set $\{a_1, a_1', a_2, a_3, a_4, b\}$ form a subdivision of K_5, where $a_1, a_1' \in S_1$, $a_2 \in S_2$, $a_3 \in S_3$, $a_4 \in S_4$ and $b \in S_{234}$. Therefore $\Gamma_2(L)$ is not planar which is again a contradiction.

Therefore, if $\Gamma_2(L)$ is planar, then one of the conditions (i) or (ii) holds. ∎

Acknowledgement. The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

References

[1] M. Afkhami and K. Khashyarmanesh, The cozero-divisor graph of a commutative ring, *Southeast Asian Bull. Math.* **35** (2011), no. 5, 753–762.

[2] D. F. Anderson, M. C. Axtell and J. A. Stickles, Jr., Zero-divisor graphs in commutative rings, in *Commutative Algebra—Noetherian and Non-Noetherian Perspectives*, 23–45, Springer, New York, 2011.

[3] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), no. 2, 434–447.

[4] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), no. 1, 208–226.

[5] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[6] F. DeMeyer and L. DeMeyer, Zero divisor graphs of semigroups, *J. Algebra* **283** (2005), no. 1, 190–198.

[7] T. Donnellan, *Lattice Theory*, Pergamon Press, Oxford, 1968.

[8] E. Estaji and K. Khashyarmanesh, The zero-divisor graph of a lattice, *Results Math.* **61** (2012), no. 1-2, 1–11.

[9] A. V. Kelarev and S. J. Quinn, A combinatorial property and power graphs of groups, in *Contributions to General Algebra, 12 (Vienna, 1999)*, 229–235, Heyn, Klagenfurt.

[10] A. V. Kelarev and S. J. Quinn, Directed graphs and combinatorial properties of semigroups, *J. Algebra* **251** (2002), no. 1, 16–26.

[11] A. Kelarev, J. Ryan and J. Yearwood, Cayley graphs as classifiers for data mining: the influence of asymmetries, *Discrete Math.* **309** (2009), no. 17, 5360–5369.

[12] C. H. Li and C. E. Praeger, On the isomorphism problem for finite Cayley graphs of bounded valency, *European J. Combin.* **20** (1999), no. 4, 279–292.

[13] H. R. Maimani, M. Salimi, A. Sattari and S. Yassemi, Comaximal graph of commutative rings, *J. Algebra* **319** (2008), no. 4, 1801–1808.

[14] J. B. Nation, *Notes on Lattice Theory,* Cambridge studies in advanced mathematics, Vol. 60, Cambridge University Press, Cambridge, 1998.

[15] C. E. Praeger, C. H. Li and A. C. Niemeyer, Finite transitive permutation groups and finite vertex-transitive graphs, in *Graph Symmetry (Montreal, PQ, 1996)*, 277–318, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 497 Kluwer Acad. Publ., Dordrecht.

[16] P. K. Sharma and S. M. Bhatwadekar, A note on graphical representation of rings, *J. Algebra* **176** (1995), no. 1, 124–127.

[17] A. Thomson and S. Zhou, Gossiping and routing in undirected triple-loop networks, *Networks* **55** (2010), no. 4, 341–349.

[18] H. -J. Wang, Graphs associated to co-maximal ideals of commutative rings, *J. Algebra* **320** (2008), no. 7, 2917–2933.

[19] H. -J. Wang, Co-maximal graph of non-commutative rings, *Linear Algebra Appl.* **430** (2009), no. 2-3, 633–641.

[20] S. Zhou, A class of arc-transitive Cayley graphs as models for interconnection networks, *SIAM J. Discrete Math.* **23** (2009), no. 2, 694–714.