

L^∞ -Estimates of Rectangular Mixed Methods for Nonlinear Constrained Optimal Control Problem

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Abstract. In this paper, we investigate the rectangular mixed finite element methods for the quadratic convex optimal control problem governed by nonlinear elliptic equations with pointwise control constraints. The state and the co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. We derive L^∞ -error estimates for the rectangular mixed finite element approximation of nonlinear quadratic optimal control problems. Finally, we present some numerical examples which confirm our theoretical results.

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1. Introduction

Optimal control problems are playing increasingly important role in the design of modern life. They have various application backgrounds in the operation of physical, social, and economic processes. Efficient numerical methods are essential to successful applications of optimal control in practical problems. Finite element methods for state equations are widely used to solve optimal control problems. The finite element approximation of optimal control problem by piecewise constant functions is well investigated by Falk [13] and Geveci [14]. Arada *et al.* [1] discussed the discretization for semilinear elliptic optimal control problems. In [26], Malanowski discussed a constrained parabolic optimal control problems. Casas *et al.* presented the numerical results for elliptic boundary control problems in [7]. All of these papers are mainly focused on L^2 -estimates. Systematic introductions of the finite element method for optimal control problems can be found in [20, 21]. Error estimates in the L^∞ -norm can be obtained by other concepts in Hinze [16]. Moreover, Meyer and Rösch have studied the L^∞ -error estimates and superconvergence property for linear quadratic optimal control problem in [27].

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In many control problems, the objective functional contains gradient of the state variables. Thus the accuracy of gradient is important in numerical approximation of the state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods, see, for example, [2, 5, 6, 17]. However, there is only very limited research work on analyzing such elements for optimal control problems. Recently, we have done some preliminary work on error estimates and superconvergence of mixed finite element methods for optimal control problems in [8, 9, 10, 11, 12, 24, 25].

In this paper we discuss the L^∞ -error estimates of optimal order for nonlinear quadratic optimal control problem with pointwise control constraints using rectangular mixed finite element methods. We consider the following nonlinear quadratic optimal control problem:

$$(1.1) \quad \min_{u \in K \subset U} \left\{ \frac{1}{2} \|p - p_d\|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{\lambda}{2} \|u\|^2 \right\}$$

subject to the state equations

$$(1.2) \quad \operatorname{div} p + \phi(y) = f + u, \quad p = -A(x)\nabla y, \quad x \in \Omega,$$

with the boundary condition

$$(1.3) \quad y = 0, \quad x \in \partial\Omega,$$

where Ω is a bounded open set in \mathbb{R}^2 with Lipschitz continuous boundary $\partial\Omega$, p_d and y_d are the desirable objective functionals. We shall assume that $f \in H^1(\Omega)$, $U = L^\infty(\Omega)$, and $\lambda > 0$ is fixed. Here, K denotes the admissible set of the control variable, defined by

$$(1.4) \quad K = \{u \in L^\infty(\Omega) : \alpha(x) \leq u \leq \beta(x) \text{ a.e. in } \Omega\},$$

where $\alpha(x)$ and $\beta(x)$ are real functions in \mathbb{R} .

Let us state the assumptions on the functions $A(x)$ and $\phi(y)$.

- (A1) The coefficient matrix $A(x) = (a_{i,j}(x))_{2 \times 2} \in L^\infty(\Omega; \mathbb{R}^{2 \times 2})$ is a symmetric 2×2 -matrix and there are constants $c_0, c_1 > 0$ satisfying for any vector $\mathbf{X} \in \mathbb{R}^2$, $c_0 \|\mathbf{X}\|_{\mathbb{R}^2}^2 \leq \mathbf{X}^T A \mathbf{X} \leq c_1 \|\mathbf{X}\|_{\mathbb{R}^2}^2$.
- (A2) ϕ is of class C^2 with respect to the variable y , for any $R > 0$ the function $\phi(\cdot) \in W^{2,\infty}(-R, R)$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi'(y) \geq 0$.
- (A3) The two given functions satisfy the regularity $p_d \in (W^{2,s}(\Omega))^2$, $y_d \in W^{1,s}(\Omega)$, $s \geq 2$.

For $1 \leq s < \infty$ and m any nonnegative integer let $W^{m,s}(\Omega) = \{v \in L^s(\Omega); D^\alpha v \in L^s(\Omega) \text{ if } |\alpha| \leq m\}$ denote the Sobolev spaces endowed with the norm $\|v\|_{m,s}^s = \sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^s(\Omega)}^s$, and the semi-norm $|v|_{m,s}^s = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^s(\Omega)}^s$. We set $W_0^{m,s}(\Omega) = \{v \in W^{m,s}(\Omega) : v|_{\partial\Omega} = 0\}$. For $s = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. Let $\|\cdot\|_{0,\infty}$ denote the maximum norm.

The outline of this paper is as follows. In next section, we construct the rectangular mixed finite element discretization for optimal control problems governed by nonlinear elliptic equations with pointwise control constraints. In Section 3, we derive a L^∞ -error estimates of optimal order for the lowest order Raviart-Thomas mixed finite element approximation for the optimal control problem. Numerical examples are presented in Section 4.

2. Mixed methods of optimal control problem

In this section, we shall describe the mixed finite element discretization of nonlinear convex optimal control problem (1.1)–(1.3). First, we introduce the co-state elliptic equation

$$(2.1) \quad -\operatorname{div}(A(x)(\nabla z + p - p_d)) + \phi'(y)z = y - y_d, \quad x \in \Omega,$$

with the boundary condition

$$(2.2) \quad z = 0, \quad x \in \partial\Omega.$$

It is assumed that both the elliptic Equations (1.2) and (2.1) have sufficient regularity.

Then, we recall the following existed results from Proposition 6.2 and Proposition 6.3 in [1] which is very useful for our work.

Lemma 2.1. *Let u_1, u_2 be in $L^\infty(\Omega)$, y_1, y_2 be the corresponding solutions of (1.2), and let z_1, z_2 be the corresponding solutions of (2.1). Then $y_1 - y_2, z_1 - z_2$ satisfy the estimate*

$$(2.3) \quad \|y_1 - y_2\|_2 \leq C\|u_1 - u_2\|_0,$$

$$(2.4) \quad \|z_1 - z_2\|_2 \leq C\|y_1 - y_2\|_0,$$

where $C > 0$ does not depend on u_1, u_2 and y_1, y_2 .

Let

$$V = H(\operatorname{div}; \Omega) = \{v \in (L^2(\Omega))^2, \operatorname{div}v \in L^2(\Omega)\}, \quad W = L^2(\Omega),$$

endowed with the norm given by

$$\|v\|_{\operatorname{div}} = \|v\|_{H(\operatorname{div}; \Omega)} = (\|v\|_{0, \Omega}^2 + \|\operatorname{div}v\|_{0, \Omega}^2)^{1/2}.$$

We recast (1.1)–(1.3) as the following weak forms: find $(p, y, u) \in V \times W \times U$ such that

$$(2.5) \quad \min_{u \in K \subset U} \left\{ \frac{1}{2} \|p - p_d\|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{\lambda}{2} \|u\|^2 \right\}$$

$$(2.6) \quad (A^{-1}p, v) - (y, \operatorname{div}v) = 0, \quad \forall v \in V,$$

$$(2.7) \quad (\operatorname{div}p, w) + (\phi(y), w) = (f + u, w), \quad \forall w \in W.$$

It is well known (see e.g., [19]) that the optimal control problem (2.5)–(2.7) has a solution (p, y, u) , and that a triplet (p, y, u) is the solution of (2.5)–(2.7) if and only if there is a co-state $(q, z) \in V \times W$ such that (p, y, q, z, u) satisfies the following optimality conditions:

$$(2.8) \quad (A^{-1}p, v) - (y, \operatorname{div}v) = 0, \quad \forall v \in V,$$

$$(2.9) \quad (\operatorname{div}p, w) + (\phi(y), w) = (f + u, w), \quad \forall w \in W,$$

$$(2.10) \quad (A^{-1}q, v) - (z, \operatorname{div}v) = -(p - p_d, v), \quad \forall v \in V,$$

$$(2.11) \quad (\operatorname{div}q, w) + (\phi'(y)z, w) = (y - y_d, w), \quad \forall w \in W,$$

$$(2.12) \quad (z + \lambda u, \tilde{u} - u)_U \geq 0, \quad \forall \tilde{u} \in K,$$

where $(\cdot, \cdot)_U$ is the inner product of U . In the rest of the paper, we shall simply write the product as (\cdot, \cdot) whenever no confusion should be caused.

Introducing the following projection [3]:

$$(2.13) \quad \Pi_{[\alpha, \beta]}(g(x)) = \max(\alpha(x), \min(g(x), \beta(x))), \quad \text{a.e. } x \in \Omega,$$

we can directly express the control from above optimality condition:

$$(2.14) \quad u(x) = \Pi_{[\alpha, \beta]} \left(-\frac{1}{\lambda} z(x) \right).$$

Let \mathcal{T}_h be regular rectangulation of Ω , with boundary elements only allowed to have one curved side. They are assumed to satisfy the angle condition which means that there is a positive constant C such that for all $T \in \mathcal{T}_h$, $C^{-1}h_T^2 \leq |T| \leq Ch_T^2$, where $|T|$ is the area of T and h_T is the diameter of T . Let $h = \max h_T$. In addition C or c denotes a general positive constant independent of h .

Let $V_h \times W_h \subset V \times W$ denote the Raviart-Thomas space [30] of the lowest order associated with the rectangulation \mathcal{T}_h of Ω . P_k denotes the space of polynomials of total degree at most k , $Q_{m,n}$ indicates the space of polynomials of degree no more than m and n in x and y , respectively. Let $V(T) = \{v \in Q_{1,0}(T) \times Q_{0,1}(T)\}$. We define

$$\begin{aligned} V_h &:= \{v_h \in V : \forall T \in \mathcal{T}_h, v_h|_T \in V(T)\}, \\ W_h &:= \{w_h \in W : \forall T \in \mathcal{T}_h, w_h|_T \in P_0(T)\}, \\ K_h &:= \{\tilde{u}_h \in K : \forall T \in \mathcal{T}_h, \tilde{u}_h|_T \in P_0(T)\}. \end{aligned}$$

By the definition of finite element subspace, the mixed finite element discretization of (2.5)–(2.7) is as follows: compute $(p_h, y_h, u_h) \in V_h \times W_h \times K_h$ such that

$$(2.15) \quad \min_{u_h \in K_h} \left\{ \frac{1}{2} \|p_h - p_d\|^2 + \frac{1}{2} \|y_h - y_d\|^2 + \frac{\lambda}{2} \|u_h\|^2 \right\}$$

$$(2.16) \quad (A^{-1}p_h, v_h) - (y_h, \operatorname{div} v_h) = 0, \quad \forall v_h \in V_h,$$

$$(2.17) \quad (\operatorname{div} p_h, w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \quad \forall w_h \in W_h.$$

It is well known that the optimal control problem (2.15)–(2.17) again has a solution (p_h, y_h, u_h) , and that a triplet (p_h, y_h, u_h) is the solution of (2.15)–(2.17) if and only if there is a co-state $(q_h, z_h) \in V_h \times W_h$ such that $(p_h, y_h, q_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(2.18) \quad (A^{-1}p_h, v_h) - (y_h, \operatorname{div} v_h) = 0, \quad \forall v_h \in V_h,$$

$$(2.19) \quad (\operatorname{div} p_h, w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \quad \forall w_h \in W_h,$$

$$(2.20) \quad (A^{-1}q_h, v_h) - (z_h, \operatorname{div} v_h) = -(p_h - p_d, v_h) \quad \forall v_h \in V_h,$$

$$(2.21) \quad (\operatorname{div} q_h, w_h) + (\phi'(y_h)z_h, w_h) = (y_h - y_d, w_h), \quad \forall w_h \in W_h,$$

$$(2.22) \quad (z_h + \lambda u_h, \tilde{u}_h - u_h) \geq 0, \quad \forall \tilde{u}_h \in K_h.$$

By using (2.13) and (2.22), we can easily obtain the following results:

Lemma 2.2. *Let $(p_h, y_h, q_h, z_h, u_h)$ is the optimal solution of (2.18)–(2.22), then u_h is given by*

$$(2.23) \quad u_h = \Pi_{[\alpha, \beta]} \left(-\frac{1}{\lambda} z_h \right).$$

Let $P_h : W \rightarrow W_h$ be the orthogonal L^2 -projection into W_h define by [2]:

$$(2.24) \quad (P_h w - w, \chi) = 0, \quad w \in W, \quad \chi \in W_h,$$

which satisfies

$$(2.25) \quad \|P_h w - w\|_{0,q} \leq C \|w\|_{t,q} h^t, \quad 0 \leq t \leq 1, \text{ if } w \in W \cap W^{t,q}(\Omega),$$

$$(2.26) \quad \|P_h w - w\|_{-r} \leq C \|w\|_r h^{r+t}, \quad 0 \leq r, t \leq 1, \text{ if } w \in H^t(\Omega),$$

$$(2.27) \quad (\operatorname{div} v, w - P_h w) = 0, \quad w \in W, \quad v \in V_h.$$

Let $\pi_h : V \rightarrow V_h$ be the Raviart-Thomas projection [28], which satisfies

$$(2.28) \quad (\operatorname{div}(\pi_h v - v), w) = 0, \quad v \in V, \quad w \in W_h,$$

$$(2.29) \quad \|\pi_h v - v\|_{0,q} \leq C \|v\|_{t,q} h^t, \quad 1/q < t \leq 1, \text{ if } v \in V \cap W^{t,q}(\Omega)^2,$$

$$(2.30) \quad \|\operatorname{div}(\pi_h v - v)\|_{0,\infty} \leq C \|\operatorname{div} v\|_t h^t, \quad 0 \leq t \leq 1, \text{ if } v \in V \cap H^t(\operatorname{div}; \Omega).$$

We have the commuting diagram property

$$(2.31) \quad \operatorname{div} \circ \pi_h = P_h \circ \operatorname{div} : V \rightarrow W_h \quad \text{and} \quad \operatorname{div}(I - \pi_h)V \perp W_h.$$

Furthermore, we also define the standard L^2 -orthogonal projection $Q_h : K \rightarrow K_h$, which satisfies: for any $\tilde{u} \in K$

$$(2.32) \quad (\tilde{u} - Q_h \tilde{u}, \tilde{u}_h)_U = 0, \quad \forall \tilde{u}_h \in K_h,$$

$$(2.33) \quad \|\tilde{u} - Q_h \tilde{u}\|_{-t,r,U} \leq C \|\tilde{u}\|_{1,r,U} h^{1+t}, \quad t = 0, 1 \text{ for } \tilde{u} \in W^{1,r}(\Omega).$$

For $\varphi \in W_h$, we shall write

$$(2.34) \quad \phi(\varphi) - \phi(\rho) = -\tilde{\phi}'(\varphi)(\rho - \varphi) = -\phi'(\rho)(\rho - \varphi) + \tilde{\phi}''(\varphi)(\rho - \varphi)^2,$$

where

$$(2.35) \quad \tilde{\phi}'(\varphi) = \int_0^1 \phi'(\varphi + t(\rho - \varphi)) dt, \quad \tilde{\phi}''(\varphi) = \int_0^1 (1-t) \phi''(\rho + t(\varphi - \rho)) dt$$

are bounded functions in $\bar{\Omega}$ [15, 29].

3. Error estimates for the intermediate error

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in K$, we first define the state solution $(p(\tilde{u}), y(\tilde{u}), q(\tilde{u}), z(\tilde{u}))$ associated with \tilde{u} that satisfies

$$(3.1) \quad (A^{-1} p(\tilde{u}), v) - (y(\tilde{u}), \operatorname{div} v) = 0, \quad \forall v \in V,$$

$$(3.2) \quad (\operatorname{div} p(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (f + \tilde{u}, w), \quad \forall w \in W,$$

$$(3.3) \quad (A^{-1} q(\tilde{u}), v) - (z(\tilde{u}), \operatorname{div} v) = -(p(\tilde{u}) - p_d, v), \quad \forall v \in V,$$

$$(3.4) \quad (\operatorname{div} q(\tilde{u}), w) + (\phi'(y(\tilde{u}))z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), \quad \forall w \in W.$$

By Lemma 3.1 in [23], we can obtain the following results:

Lemma 3.1. *Suppose that the assumptions (A1)–(A3) are valid. For any $u_h \in K_h$, there is a positive constant C independent of h such that*

$$(3.5) \quad \|p(u_h) - p_h\|_{\operatorname{div}} + \|y(u_h) - y_h\|_0 \leq Ch,$$

$$(3.6) \quad \|q(u_h) - q_h\|_{\operatorname{div}} + \|z(u_h) - z_h\|_0 \leq Ch.$$

Let $J(\cdot) : K \rightarrow \mathbb{R}$ be a G -differential uniform convex functional near the solution u which satisfies the following form:

$$J(u) = \frac{1}{2} \|p - p_d\|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{\lambda}{2} \|u\|^2,$$

$$J(u_h) = \frac{1}{2} \|p(u_h) - p_d\|^2 + \frac{1}{2} \|y(u_h) - y_d\|^2 + \frac{\lambda}{2} \|u_h\|^2.$$

It can be shown that

$$(J'(u), v) = (\lambda u + z, v),$$

$$(J'(u_h), v) = (\lambda u_h + z(u_h), v),$$

where $(p(u_h), y(u_h), q(u_h), z(u_h))$ is the solution of (3.1)–(3.4) with $\tilde{u} = u_h$. An additional assumption is needed. We assume that the cost function J is strictly convex near the solution u , i.e.,

(A4) For the solution u there exists a neighborhood of u in L^2 such that J is convex in the sense that there is a constant $c > 0$ satisfying:

$$(3.7) \quad (J'(u) - J'(v), u - v) \geq c \|u - v\|_U^2,$$

for all v in this neighborhood of u . The convexity of $J(\cdot)$ is closely related to the second order sufficient optimality conditions of optimal control problems, which are assumed in many studies on numerical methods of the problem. For instance, in many references, the authors assume the following second order sufficiently optimality condition (see [3]): there is $c > 0$ such that $J''(u)v^2 \geq c \|v\|_0^2$.

Theorem 3.1. *Suppose that the assumptions (A1)–(A4) are valid. Let $(p, y, q, z, u) \in (V \times W)^2 \times K$ and $(p_h, y_h, q_h, z_h, u_h) \in (V_h \times W_h)^2 \times K_h$ be the solution of (2.8)–(2.12) and (2.18)–(2.22), respectively. We assume that $\lambda u + z \in H^1(\Omega)$. Then, we have*

$$(3.8) \quad \|u - u_h\|_0 \leq Ch.$$

Proof. For the proof the reader can consult Theorem 3.1 in [23]. ■

Now, we introduce the weighted L^2 -norms which will play a central role in our work to derive L^∞ -error estimates. Let $x_0 \in \bar{\Omega}$ and $\rho > 0$. We define the weight function

$$(3.9) \quad \mu = |x - x_0|^2 + \rho^2, \quad x \in \bar{\Omega}.$$

For any $r \in \mathbb{R}$ we define the r -weighted norm by

$$(3.10) \quad \|v\|_{r,\mu} = \|\mu^{-\frac{r}{2}} v\|_0, \quad v \in L^2(\Omega) \text{ or } (L^2(\Omega))^2.$$

By Lemma 3.1 in [17], we can obtain the following technical results:

Lemma 3.2. *Let μ be given by (3.9), if $v \in (L^2(\Omega))^2$, then*

$$(3.11) \quad \|\nabla \mu^{-1} \cdot v\|_0 \leq C \rho^{-2} \|v\|_{1,\mu}.$$

Lemma 3.3. *If $v \in (L^\infty(\Omega))^2$, then*

$$(3.12) \quad \|v\|_0 \leq C \|v\|_{1,\mu}.$$

Furthermore, we introduce the following relations between weighted L^2 -norms and L^∞ -norms and super-approximability results [31]:

$$(3.13) \quad \|v\|_{1,\mu} \leq C |\ln h|^{\frac{1}{2}} \|v\|_{0,\infty}, \quad v \in L^\infty(\Omega),$$

$$(3.14) \quad \|\mu^{-1}\eta - \pi_h(\mu^{-1}\eta)\|_{-1,\mu} \leq Ch\rho^{-1}\|\eta\|_{1,\mu}, \quad \eta \in V_h.$$

If $v \in W_h$ is a fixed element and $x_0 \in \bar{\Omega}$ is chosen so that $\|v\|_{0,\infty} = |v(x_0)|$, then

$$(3.15) \quad \|v\|_{0,\infty} \leq C_\kappa h^{-1}\rho\|v\|_{1,\mu}, \quad \text{for } \rho \leq \kappa h.$$

Now we recall a priori regularity estimate for the following auxiliary problems:

$$(3.16) \quad -\operatorname{div}(A^*\nabla\xi) + \Phi\xi = F_1, \quad x \in \Omega, \quad \xi|_{\partial\Omega} = 0,$$

$$(3.17) \quad -\operatorname{div}(A\nabla\zeta) + \phi'(y(u_h))\zeta = F_2, \quad x \in \Omega, \quad \zeta|_{\partial\Omega} = 0,$$

where

$$(3.18a) \quad \Phi = \begin{cases} \frac{\phi(y(u_h)) - \phi(y_h)}{y(u_h) - y_h}, & y(u_h) \neq y_h, \\ \phi'(y_h), & y(u_h) = y_h. \end{cases}$$

$$(3.18b)$$

The next lemma gives the desired a priori estimate. (See [22], for example.)

Lemma 3.4. *Let ξ and ζ be the solutions of (3.16) and (3.17), respectively. Assume that Ω is convex, $A \in (W^{1,\infty}(\Omega))^{(2 \times 2)}$, $X^tAX \geq c\|X\|_{\mathbb{R}^2}^2$ for all $X \in \mathbb{R}^2$. Then*

$$(3.19) \quad \|\xi\|_{H^2(\Omega)} \leq C\|F_1\|_{L^2(\Omega)},$$

$$(3.20) \quad \|\zeta\|_{H^2(\Omega)} \leq C\|F_2\|_{L^2(\Omega)}.$$

Let

$$(3.21) \quad \varepsilon_1 := p(u_h) - p_h, \quad r_1 := y(u_h) - y_h,$$

$$(3.22) \quad \varepsilon_2 := q(u_h) - q_h, \quad r_2 := z(u_h) - z_h.$$

From (2.18)–(2.19), (3.1)–(3.2), and (2.34), we have

$$(3.23) \quad (A^{-1}\varepsilon_1, v_h) - (r_1, \operatorname{div}v_h) = 0, \quad \forall v_h \in V_h,$$

$$(3.24) \quad (\operatorname{div}\varepsilon_1, w_h) + (\tilde{\phi}'(y(u_h))r_1, w_h) = 0, \quad \forall w_h \in W_h.$$

Now, we will prove two important theorems.

Theorem 3.2. *Let (p, y, q, z) and $(p(u_h), y(u_h), q(u_h), z(u_h))$ be the solution of (2.8)–(2.12) and (3.1)–(3.4), respectively. Suppose that the assumptions (A1)–(A4) are fulfilled. Then, we have*

$$(3.25) \quad \|P_h y(u_h) - y_h\|_0 + \|P_h z(u_h) - z_h\|_0 \leq Ch^2.$$

Proof. Here, we only prove $\|P_h y(u_h) - y_h\|_0 \leq Ch^2$, the other part of (3.25) can be estimated in the same way. By (2.24), we can rewrite (3.23)–(3.24) as

$$(3.26) \quad (A^{-1}\varepsilon_1, v_h) - (P_h y(u_h) - y_h, \operatorname{div}v_h) = 0, \quad \forall v_h \in V_h,$$

$$(3.27) \quad (\operatorname{div}\varepsilon_1, w_h) + (\tilde{\phi}'(y(u_h))r_1, w_h) = 0, \quad \forall w_h \in W_h.$$

Let $\tau = P_h y(u_h) - y_h$ and ξ be the solution of (3.16) with $F_1 = P_h y(u_h) - y_h$, then it follows from (2.28), (3.23)–(3.24), (3.16), and (3.26)–(3.27) that

$$(3.28) \quad \begin{aligned} \|\tau\|_0^2 &= (\tau, -\operatorname{div}(A^*\nabla\xi) + \Phi\xi) = (\operatorname{div}\varepsilon_1, \xi) + (\tilde{\phi}'(y(u_h))r_1, \xi) \\ &= (\operatorname{div}\varepsilon_1, \xi - P_h\xi) + (\tilde{\phi}'(y(u_h))r_1, \xi - P_h\xi). \end{aligned}$$

We then estimate the two terms on the right side of (3.28). First, from Lemma 3.1 and (2.25) it follows that

$$(3.29) \quad (\operatorname{div} \varepsilon_1, \xi - P_h \xi) \leq \|\varepsilon_1\|_{\operatorname{div}} \cdot \|\xi - P_h \xi\|_0 \leq Ch^2 \|\xi\|_2 \leq Ch^2 \|\tau\|_0.$$

Now, we estimate the second term

$$(3.30) \quad (\tilde{\phi}'(y(u_h))r_1, \xi - P_h \xi) \leq C \|r_1\|_0 \cdot \|\xi - P_h \xi\|_0 \leq Ch^2 \|\tau\|_0.$$

Inserting (3.29) and (3.30) into (3.28) and we can deduce that $\|\tau\|_0 \leq Ch^2$, from which the theorem follows immediately. \blacksquare

Theorem 3.3. *Let (p, y, q, z) and $(p(u_h), y(u_h), q(u_h), z(u_h))$ be the solution of (2.8)–(2.12) and (3.1)–(3.4), respectively. Suppose that the assumptions (A1)–(A4) are fulfilled. Then, we have*

$$(3.31) \quad \|\pi_h p(u_h) - p_h\|_{0,\infty} + \|\pi_h q(u_h) - q_h\|_{0,\infty} \leq C |\ln h|^{\frac{1}{2}} h^{\frac{1}{2}}.$$

Proof. Let us denote $\sigma = \pi_h p(u_h) - p_h$. Note that

$$(3.32) \quad \begin{aligned} \|\sigma\|_{1,\mu}^2 &\leq C(A^{-1}\sigma, \mu^{-1}\sigma) \\ &\leq C\{(A^{-1}\sigma, \mu^{-1}\sigma - \pi_h(\mu^{-1}\sigma)) + (A^{-1}\varepsilon_1, \pi_h(\mu^{-1}\sigma)) \\ &\quad + (A^{-1}(\pi_h p(u_h) - p(u_h)), \pi_h(\mu^{-1}\sigma))\} \\ &\leq C\{h\rho^{-1}\|\sigma\|_{1,\mu} + (A^{-1}\varepsilon_1, \pi_h(\mu^{-1}\sigma))\} \\ &\quad + |\ln h|^{\frac{1}{2}}(1 + h\rho^{-1}) \sup_T \|\pi_h p(u_h) - p(u_h)\|_{0,\infty,T} \cdot \|\sigma\|_{1,\mu}, \end{aligned}$$

using ε -Cauchy inequality and for $h\rho^{-1}$ sufficiently small, we then have

$$(3.33) \quad \|\sigma\|_{1,\mu}^2 \leq C(A^{-1}\varepsilon_1, \pi_h(\mu^{-1}\sigma)) + C|\ln h| \sup_T \|\pi_h p(u_h) - p(u_h)\|_{0,\infty,T}^2.$$

For the first term of the right hand of (3.33), integrating in polar coordinates, we obtain $\|\mu^{-1}\|_0 \leq C\rho^{-1}$, thus using Equation (3.23), we obtain

$$(3.34) \quad \begin{aligned} (A^{-1}\varepsilon_1, \pi_h(\mu^{-1}\sigma)) &= (r_1, \operatorname{div} \circ \pi_h(\mu^{-1}\sigma)) = (r_1, P_h \circ \operatorname{div}(\mu^{-1}\sigma)) \\ &= (\tau, \operatorname{div}(\mu^{-1}\sigma)) = (\tau, \nabla \mu^{-1}\sigma) + (\tau, \mu^{-1} \operatorname{div} \sigma) \\ &\leq \|\tau\|_0 \cdot \|\nabla \mu^{-1}\sigma\|_0 + \|\tau\|_0 \cdot \|\mu^{-1}\|_0 \cdot \|\operatorname{div} \sigma\|_{0,\infty} \\ &\leq Ch^2(\rho^{-2}\|\sigma\|_{1,\mu} + \rho^{-1} \cdot \|\operatorname{div} \sigma\|_{0,\infty}). \end{aligned}$$

Using (3.27) and definition of P_h , we can easily see that

$$(3.35) \quad P_h \circ \operatorname{div} \varepsilon_1 = -P_h [\tilde{\phi}'(y(u_h))r_1],$$

then, using (2.31), we can see that

$$(3.36) \quad \operatorname{div} \sigma = \operatorname{div} \circ \pi_h \varepsilon_1 = P_h \circ \operatorname{div} \varepsilon_1 = -P_h [\tilde{\phi}'(y(u_h))r_1],$$

thus we have

$$(3.37) \quad \|\operatorname{div} \sigma\|_{0,\infty} \leq \|\tilde{\phi}'(y(u_h))r_1\|_{0,\infty} \leq \|r_1\|_{0,\infty} \leq Ch,$$

where we used the priori estimate $\|r_1\|_{0,\infty} \leq Ch$, which was demonstrated in [28]. Inserting (3.37) to (3.34) yields the bound

$$(3.38) \quad (A^{-1}\varepsilon_1, \pi_h(\mu^{-1}\sigma)) \leq Ch^2 \rho^{-2} \|\sigma\|_{1,\mu} + Ch^3 \rho^{-1}.$$

For the second term of the right side of (3.33), let \hat{T} be the reference element of T , using the transformation formula (6.8) in [4] and Bramble-Hilbert Lemma, we have

$$(3.39) \quad \begin{aligned} \|p(u_h) - \pi_h p(u_h)\|_{0,\infty,T} &\leq C \|\hat{p}(u_h) - \widehat{\pi_h p}(u_h)\|_{0,\infty,\hat{T}} \leq C |\hat{p}(u_h)|_{1,\hat{T}} \\ &\leq Ch_T |p(u_h)|_{1,T} \leq Ch |y(u_h)|_{2,T} \leq Ch \|y(u_h)\|_2. \end{aligned}$$

Inserting (3.38) and (3.39) into (3.33), and using ε -Cauchy inequality, we have

$$(3.40) \quad \|\sigma\|_{1,\mu}^2 \leq C(\varepsilon)h^2 |\ln h| + \varepsilon \|\sigma\|_{1,\mu}^2.$$

Let $h\rho^{-2} = C^{-2}$, that is to say $\rho = Ch^{\frac{1}{2}}$. Combining (3.15) and (3.40), we then have

$$(3.41) \quad \|\sigma\|_{0,\infty} \leq Ch^{-\frac{1}{2}} \|\sigma\|_{1,\mu} \leq Ch^{\frac{1}{2}} |\ln h|^{\frac{1}{2}}.$$

The proof of $\|\pi_h q(u_h) - q_h\|_{0,\infty} \leq h^{\frac{1}{2}} |\ln h|^{\frac{1}{2}}$ is quite similar with above and we omitted here. ■

4. L^∞ -error estimates

In this section, we will give the L^∞ -error estimates of optimal order both for the control variable and the state variables.

Theorem 4.1. *Let (p, y, q, z, u) and $(p_h, y_h, q_h, z_h, u_h)$ be the solution of (2.8)–(2.12) and (2.18)–(2.22), respectively. Suppose that the assumptions (A1)–(A4) are fulfilled. Then, we have*

$$(4.1) \quad \|u - u_h\|_{0,\infty} \leq Ch,$$

$$(4.2) \quad \|y - y_h\|_{0,\infty} + \|z - z_h\|_{0,\infty} \leq Ch,$$

$$(4.3) \quad \|p - p_h\|_{0,\infty} + \|q - q_h\|_{0,\infty} \leq Ch^{\frac{1}{2}} |\ln h|^{\frac{1}{2}}.$$

Proof.

Part I. By (2.3)–(2.4), (2.25)–(2.26), (3.25), (3.31), and the classical imbedding theorem $H^2(\Omega) \subset C(\bar{\Omega})$, we can see that

$$(4.4) \quad \begin{aligned} \|y - y_h\|_{0,\infty} &\leq \|y - y(u_h)\|_{0,\infty} + \|y(u_h) - y_h\|_{0,\infty} \\ &\leq C \|y - y(u_h)\|_{C(\bar{\Omega})} + \|y(u_h) - P_h y(u_h)\|_{0,\infty} + \|P_h y(u_h) - y_h\|_{0,\infty} \\ &\leq C \|y - y(u_h)\|_2 + Ch + \|P_h y(u_h) - y_h\|_{0,\infty} \\ &\leq C (\|u - u_h\|_0 + h + h^{-1} \|P_h y(u_h) - y_h\|_0) \\ &\leq Ch, \end{aligned}$$

and

$$(4.5) \quad \begin{aligned} \|z - z_h\|_{0,\infty} &\leq \|z - z(u_h)\|_{0,\infty} + \|z(u_h) - z_h\|_{0,\infty} \\ &\leq C \|z - z(u_h)\|_{C(\bar{\Omega})} + \|z(u_h) - P_h z(u_h)\|_{0,\infty} + \|P_h z(u_h) - z_h\|_0 \\ &\leq C \|z - z(u_h)\|_2 + Ch + \|P_h z(u_h) - z_h\|_{0,\infty} \\ &\leq C (\|u - u_h\|_0 + h + h^{-1} \|P_h z(u_h) - z_h\|_0) \\ &\leq Ch. \end{aligned}$$

Part II. Note that the projection $\Pi_{[\alpha,\beta]}$ defined in (2.13) is Lipschitz continuous. Then, from (2.14) and (2.23), we obtain that

$$(4.6) \quad |u - u_h| = \left| \Pi_{[\alpha,\beta]} \left(-\frac{1}{\lambda} z \right) - \Pi_{[\alpha,\beta]} \left(-\frac{1}{\lambda} z_h \right) \right| \leq \frac{|z - z_h|}{\lambda},$$

hence, combining (4.5) and (4.6), we have

$$(4.7) \quad \|u - u_h\|_{0,\infty} \leq C \|z - z_h\|_{0,\infty} \leq Ch.$$

Part III. By (2.3)–(2.4), (2.29), (3.25), (3.31), and the classical imbedding theorem $W^{2,3}(\Omega) \subset W^{1,\infty}(\Omega)$ and $W^{0,3}(\Omega) \subset W^{2,3}(\Omega)$, we can see that

$$(4.8) \quad \begin{aligned} \|p - p_h\|_{0,\infty} &\leq \|p - p(u_h)\|_{0,\infty} + \|p(u_h) - p_h\|_{0,\infty} \\ &\leq C \|\nabla y - \nabla y(u_h)\|_{0,\infty} + \|p(u_h) - \pi_h p(u_h)\|_{0,\infty} \\ &\quad + \|\pi_h p(u_h) - p_h\|_{0,\infty} \\ &\leq C \left(\|y - y(u_h)\|_{2,3} + h + h^{\frac{1}{2}} |\ln h|^{\frac{1}{2}} \right) \\ &\leq C \left(\|u - u_h\|_{0,3} + h + h^{\frac{1}{2}} |\ln h|^{\frac{1}{2}} \right) \\ &\leq C \left(\|u - u_h\|_{0,\infty} + h + h^{\frac{1}{2}} |\ln h|^{\frac{1}{2}} \right) \\ &\leq Ch^{\frac{1}{2}} |\ln h|^{\frac{1}{2}}, \end{aligned}$$

and

$$(4.9) \quad \begin{aligned} \|q - q_h\|_{0,\infty} &\leq \|q - q(u_h)\|_{0,\infty} + \|q(u_h) - q_h\|_{0,\infty} \\ &\leq \|\nabla(z - z(u_h)) + p - p(u_h)\|_{0,\infty} + \|q(u_h) - \pi_h q(u_h)\|_{0,\infty} \\ &\quad + \|\pi_h q(u_h) - q_h\|_{0,\infty} \\ &\leq C \left(h + h^{\frac{1}{2}} |\ln h|^{\frac{1}{2}} \right) \\ &\leq Ch^{\frac{1}{2}} |\ln h|^{\frac{1}{2}}. \end{aligned}$$

Thus, we completed the proof. ■

5. Numerical examples

In this section, we are going to validate the L^∞ -error estimates for the error in the control, state, and co-state numerically. The optimization problem were dealt numerically with codes developed based on AFEPACK. The package is freely available and the details can be found at [18].

Our numerical example is the following optimal control problem:

$$(5.1) \quad \min_{u \in K} \left\{ \frac{1}{2} \|p - p_d\|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{1}{2} \|u\|^2 \right\}$$

$$(5.2) \quad \operatorname{div} p + y^5 = u + f, \quad p = -A \nabla y, \quad x \in \Omega, \quad y|_{\partial\Omega} = 0,$$

$$(5.3) \quad \operatorname{div} q + 5y^4 z = y - y_d, \quad q = -A(\nabla z + p - p_d), \quad x \in \Omega, \quad z|_{\partial\Omega} = 0.$$

In our examples, we choose the domain $\Omega = [0, 1] \times [0, 1]$ and $A = I$. We present two examples to illustrate the theoretical results for the optimal control problem.

Example 5.1. First, we will consider the case where the constrained set is given by $K = \{u \in L^\infty(\Omega) : u \geq 0\}$. We set the known function as follows:

$$\begin{aligned}
 y &= x_1x_2(1-x_1)(1-x_2), \\
 z &= 2x_1x_2(1-x_1)(1-x_2), \\
 u &= \max(-z, 0), \\
 f &= 2x_2(1-x_2) + 2x_1(1-x_1) + y^5 - u, \\
 y_d &= y + 4x_2(1-x_2) + 4x_1(1-x_1) - 5y^4z, \\
 p &= -((1-2x_1)x_2(1-x_2), (1-2x_2)x_1(1-x_1)), \\
 q &= 2p_d = -(2(1-2x_1)x_2(1-x_2), 2(1-2x_2)x_1(1-x_1)).
 \end{aligned}$$

Table 1. The numerical errors on uniformly rectangle mesh grid

h	Errors									
	$\ u - u_h\ _{0,\infty}$	rate	$\ p - p_h\ _{0,\infty}$	rate	$\ y - y_h\ _{0,\infty}$	rate	$\ q - q_h\ _{0,\infty}$	rate	$\ z - z_h\ _{0,\infty}$	rate
1/16	8.519e-03	-	4.259e-03	-	8.519e-03	-	1.283e-01	-	1.815e-01	-
1/32	4.377e-03	0.96	2.189e-03	0.97	4.377e-03	0.96	9.286e-02	0.48	1.313e-01	0.47
1/64	2.221e-03	0.98	1.110e-03	0.98	2.221e-03	0.98	6.641e-02	0.49	9.392e-02	0.48
1/128	1.119e-03	0.99	5.595e-04	1.00	1.119e-03	0.99	4.723e-02	0.50	6.679e-02	0.49

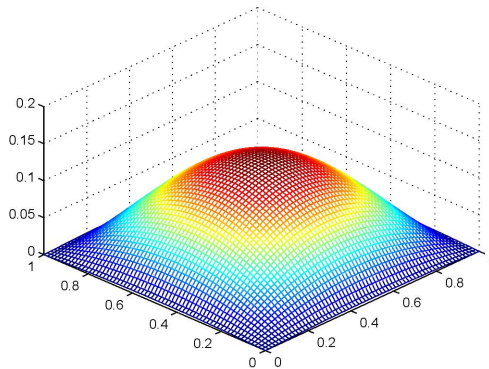


Figure 1. The numerical solution on the 64×64 rectangle mesh grids.

In this numerical tests, the numerical results are presented in Table 1, it is obvious that the results of Theorem 4.1 (L^∞ -error estimates) remain in our data. The profiles of the numerical solution on the 64×64 rectangle are plotted in Figure 1.

Example 5.2. In the next example, we set the constrained set $K = \{u \in L^\infty(\Omega) : \alpha(x) \leq u \leq \beta(x)\}$. We assume that

$$\alpha(x_1, x_2) = 0.02 + 0.04 \frac{|x_1 - x_2|}{\sqrt{2}},$$

$$\beta(x_1, x_2) = 0.04 + 0.06 \frac{|1 - x_1 - x_2|}{\sqrt{2}}.$$

Then not only the constraints depend on the coordinates (x_1, x_2) , but also there are some weak discontinuities in both constraints.

Now, we define the optimal state function by

$$y = x_1 x_2 (1 - x_1)(1 - x_2),$$

thus the state variable p can be given by

$$p = -((1 - 2x_1)x_2(1 - x_2), (1 - 2x_2)x_1(1 - x_1)),$$

and the source function f is given by

$$f = \begin{cases} f_1 + y^5 - \alpha(x), & \text{if } u_f < \alpha(x), \\ f_1 + y^5 - u_f, & \text{if } u_f \in [\alpha(x), \beta(x)], \\ f_1 + y^5 - \beta(x), & \text{if } u_f > \beta(x), \end{cases}$$

with $f_1(x_1, x_2) = 2x_1(1 - x_1) + 2x_2(1 - x_2)$ and $u_f(x_1, x_2) = 2x_1 x_2 (1 - x_1)(1 - x_2)$. Due to the state equation (5.2), we obtain for the exact control function u as follows:

$$u = \begin{cases} \alpha(x), & \text{if } u_f < \alpha(x), \\ u_f, & \text{if } u_f \in [\alpha(x), \beta(x)], \\ \beta(x), & \text{if } u_f > \beta(x). \end{cases}$$

For the optimal co-state function z , we find

$$z = -2x_1 x_2 (1 - x_1)(1 - x_2),$$

then the desired state variables can be given by

$$\begin{aligned} p_d &= -((1 - 2x_1)x_2(1 - x_2), (1 - 2x_2)x_1(1 - x_1)), \\ y_d &= y + 4x_1(1 - x_1) + 4x_2(1 - x_2) - 5y^4 z. \end{aligned}$$

Table 2. The numerical errors on uniformly rectangle mesh grid

h	Errors									
	$\ u - u_h\ _{0,\infty}$	rate	$\ p - p_h\ _{0,\infty}$	rate	$\ y - y_h\ _{0,\infty}$	rate	$\ q - q_h\ _{0,\infty}$	rate	$\ z - z_h\ _{0,\infty}$	rate
1/16	7.735e-03	-	4.260e-03	-	8.519e-03	-	1.284e-01	-	1.815e-01	-
1/32	3.937e-03	0.97	2.189e-03	0.96	4.377e-03	0.96	9.288e-02	0.47	1.313e-01	0.47
1/64	1.988e-03	0.99	1.111e-03	0.98	2.221e-03	0.98	6.642e-02	0.49	9.392e-02	0.48
1/128	9.991e-04	0.99	5.595e-04	1.00	1.119e-03	1.00	4.723e-02	0.49	6.679e-02	0.49

The profiles of the numerical solution are presented in Figure 2. From the error data on the uniform refined rectangle meshes, as listed in Table 2, it can be seen that the L^∞ -estimates results remain in our data.

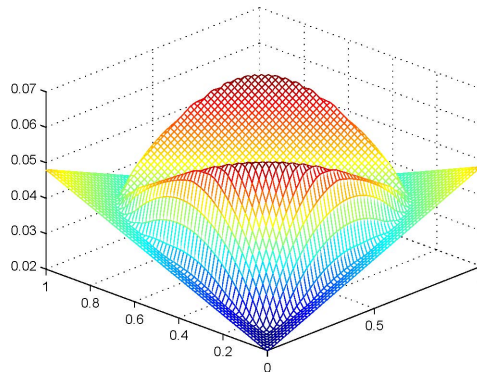


Figure 2. The numerical solution on the 64×64 rectangle mesh grids.

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