

Twin Positive Solutions of Second-Order m -Point Boundary Value Problem with Sign Changing Nonlinearities

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Abstract. In this paper, we study second-order m -point boundary value problem

$$\begin{cases} u''(t) + a(t)u'(t) + f(t, u) = 0, & 0 \leq t \leq 1, \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^k a_i u(\xi_i) - \sum_{i=k+1}^{m-2} a_i u(\xi_i), \end{cases}$$

where $a_i > 0 (i = 1, 2, \dots, m-2)$, $0 < \sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i < 1$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $a \in C([0, 1], (-\infty, 0))$ and f is allowed to change sign. We show that there exist two positive solutions by using Leggett-Williams fixed-point theorem. The conclusions in this paper essentially extend and improve some known results.

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1. Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moviseev [5, 6]. Motivated by the study of [5, 6], Gupta [2] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [3, 8–12] for some references along this line.

Recently, Liu *et al.* [9] studied the following three-point boundary value problem (BVP)

$$\begin{cases} u''(t) + a(t)f(u) = 0, & 0 < t < 1, \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases}$$

where $0 < \eta < 1$, $0 < \alpha < 1/\eta$. Authors got the existence of a positive solution by a fixed index point.

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In [10], authors considered three-point BVP

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(t, u) = 0, & 0 < t < 1, \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases}$$

where $0 < \eta < 1$, α is positive constant, $a \in C[0, 1], b \in C([0, 1], (-\infty, 0)), h \in C((0, 1), [0, +\infty))$ and $f \in C((0, 1) \times [0, +\infty), [0, +\infty))$. The existence criteria for positive solutions of the above problem was established by applying the fixed point index theorem under some weaker conditions concerning the first eigenvalue corresponding to the relevant linear operator.

In [8], authors studied the following three-point BVP

$$\begin{cases} u''(t) + a(t)u'(t) + \lambda f(t, u) = 0, & 0 \leq t \leq 1, \\ u'(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases}$$

where $0 < \eta < 1$, α is a positive constant, $a \in C([0, 1], (-\infty, 0)), f \in C([0, 1] \times R, R)$ and there exists $M > 0$ such that $f(t, u) \geq -M$ for $(t, u) \in [0, 1] \times R$. Authors obtained the existence of one positive solution by using Krasnoselskii's fixed point theorem.

Motivated by the results mentioned above, in this paper we study the existence of positive solutions of m -point boundary value problem with sign changing coefficients

$$(1.1) \quad \begin{cases} u''(t) + a(t)u'(t) + f(t, u) = 0, & 0 \leq t \leq 1, \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^k a_i u(\xi_i) - \sum_{i=k+1}^{m-2} a_i u(\xi_i), \end{cases}$$

where $a_i > 0 (i = 1, 2, \dots, m-2), 0 < \sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i < 1, 0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1, a \in C([0, 1], (-\infty, 0))$ and f is allowed to change sign. We show that there exist two positive solutions by using Leggett-Williams fixed-point theorem. Our ideas are similar to be used in [8], but different from that one. By applying Leggett-Williams fixed-point theorem, we get the new results, which are different from the previous results and the conditions are easy to be checked. In particular, we do not need that f is either superlinear or sublinear which was required in [8–10, 12].

In the rest of the paper, we make the following assumptions

- (H₁) $a_i > 0 (i = 1, 2, \dots, m-2), 0 < \sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i < 1, 0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$;
- (H₂) $a \in C([0, 1], (-\infty, 0))$;
- (H₃) $f : [0, 1] \times [0, +\infty) \rightarrow R$ is continuous and there exists $M > 0$ such that $f(t, u) \geq -M$ for $(t, u) \in [0, 1] \times R$.

By a positive solution of BVP (1.1), we understand a function u which is positive on $(0, 1)$ and satisfies the differential equations as well as the boundary conditions in BVP (1.1).

2. Preliminaries and lemmas

In this section, we give some definitions and preliminaries.

Definition 2.1. Let E be a real Banach space over R . A nonempty closed set $P \subset E$ is said to be a cone provided that

- (i) $u \in P, a \geq 0$ implies $au \in P$; and
- (ii) $u, -u \in P$ implies $u = 0$.

Definition 2.2. Given a cone P in a real Banach space E , a functional $\psi : P \rightarrow P$ is said to be increasing on P , provided $\psi(x) \leq \psi(y)$, for all $x, y \in P$ with $x \leq y$.

Definition 2.3. Given a nonnegative continuous functional γ on P of a real Banach space, we define for each $d > 0$ the set

$$P(\gamma, d) = \{x \in P | \gamma(x) < d\}.$$

Definition 2.4. Given a cone P in a real Banach space E , a functional $\alpha : P \rightarrow [0, \infty)$ is said to be nonnegative continuous concave on P , provided $\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)$, for all $x, y \in P$ with $t \in [0, 1]$.

Let $a, b, r > 0$ be constants with P and α as defined above, we note

$$P_r = \{y \in P | \|y\| < r\}, \quad P\{\alpha, a, b\} = \{y \in P | \alpha(y) \geq a, \|y\| \leq b\}.$$

The main tool of this paper is the following well known Leggett-Williams fixed-point theorem.

Theorem 2.1. [4, 7] Assume E be a real Banach space, $P \subset E$ be a cone. Let $A : \bar{P}_c \rightarrow \bar{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(y) \leq \|y\|$, for $y \in \bar{P}_c$. Suppose that there exist $0 < a < b < d \leq c$ such that

- (i) $\{y \in P(\alpha, b, d) | \alpha(y) > b\} \neq \emptyset$ and $\alpha(Ay) > b$, for all $y \in P(\alpha, b, d)$;
- (ii) $\|Ay\| < a$, for all $\|y\| \leq a$;
- (iii) $\alpha(Ay) > b$ for all $y \in P(\alpha, b, c)$ with $\|Ay\| > d$.

Then A has at least three fixed points y_1, y_2, y_3 satisfying

$$\|y_1\| < a, \quad b < \alpha(y_2),$$

and

$$\|y_3\| > a, \quad \alpha(y_3) < b.$$

Lemma 2.1. Assume that (H_1) and (H_2) hold. Then for any $y \in C[0, 1]$ the BVP

$$(2.1) \quad \begin{cases} u''(t) + a(t)u'(t) + y(t) = 0, & 0 \leq t \leq 1, \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^k a_i u(\xi_i) - \sum_{i=k+1}^{m-2} a_i u(\xi_i), \end{cases}$$

has a unique solution

$$(2.2) \quad \begin{aligned} u(t) = & - \int_0^t \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds + \frac{1}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right. \\ & \left. - \sum_{i=1}^k a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right), \end{aligned}$$

where

$$p(t) = \exp\left(\int_0^t a(\tau)d\tau\right).$$

Proof. Suppose $u(t)$ satisfies the BVP (2.1). Since $p(t) = \exp(\int_0^t a(\tau)d\tau)$, we have $p(t) > 0$ and $p(0) = 1$. Now, multiply both sides of equation of (2.1) with $p(t)$, then

$$(p(t)u'(t))' + p(t)y(t) = 0.$$

By the boundary condition $u'(0) = 0$, we have

$$u'(t) = -\frac{1}{p(t)} \int_0^t p(s)y(s)ds$$

and

$$(2.3) \quad u(t) = u(0) - \int_0^t \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds.$$

Thus, together with $u(1) = \sum_{i=1}^k a_i u(\xi_i) - \sum_{i=k+1}^{m-2} a_i u(\xi_i)$, implies that

$$(2.4) \quad u(0) = \frac{1}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds - \sum_{i=1}^k a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right).$$

Hence, combining (2.3) and (2.4), we get (2.2). Conversely, supposing $u(t)$ is given by (2.2), we check that (2.1) holds. Thus, the proof of Lemma 2.1 is completed. ■

Lemma 2.2. Assume that (H_1) and (H_2) hold. Let $y \in C[0, 1]$ and $y(t) \geq 0$ for all $t \in [0, 1]$, the unique solution of the BVP (2.1) satisfies $u(t) \geq 0$.

Proof. Obviously,

$$u'(t) = -\frac{1}{p(t)} \int_0^t p(s)y(s)ds < 0.$$

So we have $u(t)$ is a monotone decreasing function for all $t \in [0, 1]$.

This implies that

$$\|u\| = u(0), \quad \min_{t \in [0,1]} u(t) = u(1).$$

So we can get

$$\begin{aligned} u(1) &= -\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds + \frac{1}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right. \\ &\quad \left. - \sum_{i=1}^k a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right) \\ &= \frac{1}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(-\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right. \\ &\quad \left. + \sum_{i=1}^k a_i \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds - \sum_{i=k+1}^{m-2} a_i \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right. \\ &\quad \left. + \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds - \sum_{i=1}^k a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right. \\ &\quad \left. + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right) \\ &\geq \frac{1}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\left(\sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i \right) \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \right) \end{aligned}$$

$$\begin{aligned}
 & - \left(\sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i \right) \int_0^{\xi_k} \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \\
 & = \frac{\sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \int_{\xi_k}^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right) ds \\
 & \geq 0.
 \end{aligned}$$

The proof of Lemma 2.2 is completed. ■

Lemma 2.3. *Let (H_1) and (H_2) hold. If $y \in C^+[0, 1]$, the unique solution of the BVP (2.1) satisfies*

$$\min_{t \in [0,1]} u(t) \geq \gamma \|u\|,$$

where $\gamma = ((\sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i)(1 - \xi_k)) / (1 - \sum_{i=1}^k a_i \xi_k + \sum_{i=k+1}^{m-2} a_i \xi_k)$.

Proof. Clearly

$$u'(t) = -\frac{1}{p(t)} \int_0^t p(s)y(s)ds < 0.$$

This implies that

$$\|u\| = u(0), \quad \min_{t \in [0,1]} u(t) = u(1).$$

It is easy to see that $u'(t_2) \leq u'(t_1)$ for any $t_1, t_2 \in [0, 1]$ with $t_1 \leq t_2$. Hence $u'(t)$ is a decreasing function on $[0, 1]$. This means that the graph of $u(t)$ is concave down on $(0, 1)$. So we have

$$u(\xi_k) - u(1)\xi_k \geq (1 - \xi_k)u(0).$$

Together with $u(1) = \sum_{i=1}^k a_i u(\xi_i) - \sum_{i=k+1}^{m-2} a_i u(\xi_i)$ and $u'(t) \leq 0$ on $[0, 1]$, we get

$$u(0) \leq \frac{u(1) (1 - \sum_{i=1}^k a_i \xi_k + \sum_{i=k+1}^{m-2} a_i \xi_k)}{(\sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i) (1 - \xi_k)} = \frac{u(1)}{\gamma}.$$

The proof of Lemma 2.3 is completed. ■

Lemma 2.4. *Let ω be the unique solution of the following BVP*

$$(2.5) \quad \begin{cases} u''(t) + a(t)u'(t) + 1 = 0, & 0 \leq t \leq 1, \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^k a_i u(\xi_i) - \sum_{i=k+1}^{m-2} a_i u(\xi_i). \end{cases}$$

Then $\omega(t) \leq \Gamma\gamma$, where

$$\Gamma = \frac{1 + \sum_{i=k+1}^{m-2} a_i}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau)d\tau \right) ds \right) \frac{1}{\gamma}.$$

Proof. By Lemma 2.2 and Lemma 2.3, we see $\omega(0) = \max_{t \in [0,1]} \omega(t)$. So we have

$$\begin{aligned}
 \omega(0) &= \frac{1}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds \right. \\
 &\quad \left. - \sum_{i=1}^k a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds \right) \\
 (2.6) \quad &\leq \frac{1 + \sum_{i=k+1}^{m-2} a_i}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds \right) \\
 &= \Gamma\gamma.
 \end{aligned}$$

Therefore, $\omega(t) \leq \Gamma\gamma$. ■

3. The main results

For convenience, we let

$$l = \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds, \quad h = \frac{1 + \sum_{i=k+1}^{m-2} a_i}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds.$$

Let $E = C[0, 1]$, then E is Banach space, with respect to the norm $\|u\| = \sup_{t \in [0,1]} |u(t)|$. We define a cone in E by

$$P = \{u \in E \mid u \geq 0, \min_{t \in [0,1]} u(t) \geq \gamma \|u\|\}.$$

Our main results are following theorems.

Theorem 3.1. *Suppose conditions (H_1) , (H_2) and (H_3) hold and there exist positive constants a, b, c, N with $M\Gamma < a < a + M\Gamma < b < \gamma^2 c, 1/\gamma < N < (cl)/(bh)$ such that*

- (A₁) $f(t, u) < a/h - M$ for $t \in [0, 1], 0 \leq u \leq a$;
- (A₂) $f(t, u) \geq (b/l)N - M$ for $t \in [0, 1], b - M\Gamma \leq u \leq b/\gamma^2$;
- (A₃) $f(t, u) \leq c/h - M$ for $t \in [0, 1], 0 \leq u \leq c$.

Then the BVP (1.1) has at least two positive solutions.

Proof. Let $z = M\omega$. By Lemma 2.4 we have $z(t) = M\omega(t) \leq M\Gamma\gamma \leq a\gamma$. It is easy to see that the BVP (1.1) has a positive solution u if and only if $u + z = \bar{u}$ is a solution of the following BVP

$$(3.1) \quad \begin{cases} u''(t) + a(t)u'(t) = -g(t, u - z), & 0 \leq t \leq 1, \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^k a_i u(\xi_i) - \sum_{i=k+1}^{m-2} a_i u(\xi_i), \end{cases}$$

and $\bar{u} > z$ for $t \in (0, 1)$, where $g : [0, 1] \times R \rightarrow [0, +\infty)$ is defined by

$$g(t, y) = \begin{cases} f(t, y) + M, & (t, y) \in [0, 1] \times [0, +\infty), \\ f(t, 0) + M, & (t, y) \in [0, 1] \times (-\infty, 0). \end{cases}$$

For $v \in P$, define the operator

$$Tv(t) = - \int_0^t \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds$$

$$\begin{aligned}
& + \frac{1}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \right. \\
& - \sum_{i=1}^k a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \\
& \left. + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \right).
\end{aligned}$$

By Lemma 2.1, Lemma 2.2 and Lemma 2.3, we can check $T(P) \subset P$. It is easy to check T is completely continuous by the Arzela-Ascoli theorem.

In the following, we show that all the conditions of Theorem 2.1 are satisfied. Firstly, we define the nonnegative, continuous concave functional $\alpha : P \rightarrow [0, \infty)$ by

$$\alpha(v) = \min_{t \in [0, 1]} v(t)$$

Obviously, for every $v \in P$,

$$\alpha(v) \leq \|v\|.$$

We first assert that if there exists a positive number c such that $T(\bar{P}_c) \subset \bar{P}_c$. If $v \in \bar{P}_c$. When $v(t) \geq z(t)$, we have $0 \leq v(t) - z(t) \leq v(t) \leq c$ and thus $g(t, v(t) - z(t)) = f(t, v(t) - z(t)) + M \geq 0$. By (A_3) we have

$$g(t, v(t) - z(t)) \leq \frac{c}{h} \quad \text{for } t \in [0, 1].$$

When $v(t) < z(t)$, we have $v(t) - z(t) < 0$ and then $g(t, v(t) - z(t)) = f(t, 0) + M \geq 0$. Again by (A_3) we have

$$g(t, v(t) - z(t)) \leq \frac{c}{h} \quad \text{for } t \in [0, 1].$$

In a word, if $v \in \bar{P}_c$, then $g(t, v(t) - z(t)) \leq c/h$ for $t \in [0, 1]$. Then,

$$\begin{aligned}
\|Tv\| & = Tv(0) = \frac{1}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \right. \\
& - \sum_{i=1}^k a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \\
& \left. + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \right) \\
& \leq \frac{1}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \right. \\
& \left. + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \right) \\
& \leq \frac{1 + \sum_{i=k+1}^{m-2} a_i}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \\
& \leq \frac{c}{h} \frac{1 + \sum_{i=k+1}^{m-2} a_i}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds \\
& = c.
\end{aligned}$$

Thus $Tv \in \bar{P}_c$. Therefore, we have $T(\bar{P}_c) \subset \bar{P}_c$. Especially, if $v \in \bar{P}_a$, then assumption (A_1) yields $g(t, v(t)) - z(t) \leq a/h$ for $t \in [0, 1]$. So, we have $T : \bar{P}_a \rightarrow \bar{P}_a$.

To fulfil condition (i) of Theorem 2.1, let $v(t) = b/\gamma^2$, then $v \in P$, $\alpha(v) = b/\gamma^2 > b$. That is $\{v \in P(\alpha, b, b/\gamma^2) \mid \alpha(v) > b\} \neq \emptyset$. Moreover, if $v \in P(\alpha, b, b/\gamma^2)$, then $\alpha(v) \geq b$, so $b \leq \|v\| \leq b/\gamma^2$. Thus, $b - M\Gamma \leq v(t) - z(t) \leq v(t) \leq b/\gamma^2, t \in [0, 1]$. From assumption (A_2) we get $g(t, v(t)) - z(t) \geq (b/l)N$ for $t \in [0, 1]$. By the definition of α and Lemma 2.3, we have

$$\begin{aligned}
 \alpha(Tv) &= \min_{t \in [0,1]} Tv(t) \geq \gamma \|Tv\| = \gamma Tv(0) \\
 &= \frac{\gamma}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \right. \\
 &\quad - \sum_{i=1}^k a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \\
 &\quad \left. + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_i} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \right) \\
 &\geq \frac{\gamma}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \right. \\
 &\quad - \sum_{i=1}^k a_i \int_0^{\xi_k} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \\
 &\quad \left. + \sum_{i=k+1}^{m-2} a_i \int_0^{\xi_k} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \right) \\
 &= \frac{\gamma}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \right. \\
 &\quad \left. - \left(\sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i \right) \int_0^{\xi_k} \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \right) \\
 &\geq \frac{\gamma}{1 - \sum_{i=1}^k a_i + \sum_{i=k+1}^{m-2} a_i} \left(\int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \right. \\
 &\quad \left. - \left(\sum_{i=1}^k a_i - \sum_{i=k+1}^{m-2} a_i \right) \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \right) \\
 &= \gamma \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau)) - z(\tau) d\tau \right) ds \\
 &\geq \gamma \frac{b}{l} N \int_0^1 \left(\frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds \\
 &= \gamma N b \\
 &> b.
 \end{aligned}$$

Therefore, condition (i) of Theorem 2.1 is satisfied.

Finally, we address condition (iii) of Theorem 2.1. For this we choose $v \in P(\alpha, b, c)$ with $\|Tv\| > b/\gamma^2$. Then from Lemma 2.3, we have

$$\alpha(Tv) = \min_{t \in [0,1]} Tv(t) \geq \gamma \|Tv\| \geq \frac{b}{\gamma} > b.$$

Hence, condition (iii) of Theorem 2.1 holds.

To sum up, all the hypotheses of Theorem 2.1 are satisfied. Hence T has at least three positive fixed points v_1, v_2 and v_3 such that

$$\|v_1\| < a, \quad b < \alpha(v_2),$$

and

$$\|v_3\| > a, \quad \alpha(v_3) < b.$$

Further, $u_i = v_i - z (i = 1, 2, 3)$ are solutions of the BVP (1.1). Moreover,

$$v_2(t) \geq \gamma \|v_2\| \geq \gamma \alpha(v_2) \geq \gamma b > \gamma M\Gamma \geq z(t), \quad t \in [0, 1],$$

$$v_3(t) \geq \gamma \|v_3\| \geq \gamma \alpha(v_3) \geq \gamma a > \gamma M\Gamma \geq z(t), \quad t \in [0, 1].$$

So $u_2 = v_2 - z, u_3 = v_3 - z$ are two positive solutions of the BVP (1.1). This completes the proof. ■

Theorem 3.2. *Suppose conditions $(H_1), (H_2)$ and (H_3) hold and there exist positive constants a_i, b_i, N with $M\Gamma < a_i < a_i + M\Gamma < b_i < \gamma^2 a_{i+1}, 1/\gamma < N < (a_{i+1}l)/(b_i h), (i = 1, 2, \dots, n - 1)$ such that*

$$(A_4) \quad f(t, u) < a_i/h - M \text{ for } t \in [0, 1], 0 \leq u \leq a_i;$$

$$(A_5) \quad f(t, u) \geq (b_i/l)N - M \text{ for } t \in [0, 1], b_i - M\Gamma \leq u \leq b_i/\gamma^2.$$

Then, the BVP (1.1) has at least $2(n - 1)$ positive solutions.

Proof. When $n = 2$, it is that Theorem 3.1 holds (with $c_1 = a_2$), so we can get at least two positive solutions u_2 and u_3 such that $u_2 \geq \gamma a_1, u_3 \geq \gamma b_1$. Following the identical fashion, by the induction method we immediately complete the proof. ■

Remark 3.1. Comparing paper [8], our boundary value conditions extend their boundary value conditions. Furthermore, Our results are new and different from the results in [8]. In particular, the following condition in [8] are not need in our paper

$$\lim_{u \rightarrow \infty} \frac{f(t, u)}{u} = +\infty \quad \text{uniformly on } [0, 1].$$

Remark 3.2. Comparing paper [9, 10], our nonlinear terms f is allowed to change sign. Meanwhile, we do not need f is superlinear or sublinear. So our conclusions in this paper essentially extend and improve the known results in [9, 10].

4. Example

In the section, we present a simple example to explain our results.

Example 4.1. Consider the following four-point boundary value problem with sign changing coefficients

$$(4.1) \quad \begin{cases} u''(t) - u'(t) + \varphi(u) - \frac{1}{320} = 0, & 0 \leq t \leq 1, \\ u'(0) = 0, \quad u(1) = u(\frac{1}{4}) - \frac{1}{2}u(\frac{1}{2}), \end{cases}$$

where $a_1 = 1$, $a_2 = 1/2$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, $a(t) = -1$,

$$\varphi(u) = \begin{cases} \frac{1}{5}u, & 0 \leq u \leq \frac{3}{300}, \\ \frac{1497}{5}u - \frac{1496}{500}, & \frac{3}{300} \leq u \leq \frac{4}{300}, \\ 1, & \frac{4}{300} \leq u \leq \frac{16}{135}, \\ \frac{135}{1604}u + \frac{1588}{1604}, & \frac{16}{135} \leq u \leq 12, \\ \frac{u}{6}, & u \geq 12. \end{cases}$$

By simple calculation, we get $l \approx 4/5$, $h \approx 4$, $\gamma = 3/8$, $\Gamma \approx 32/3$. We choose $a = 3/300$, $b = 5/300$, $c = 12$, $N = 6$, $M = 1/320$. Obviously, $M\Gamma < a < a + M\Gamma < b < \gamma^2 c$, $1/\gamma < N < (cl)/(bh)$ and $f(u) = \varphi(u) - 1/320 \geq -M$. Moreover,

- (i) for $0 \leq u \leq 3/300$, we have $f(u) = \varphi(u) - 1/320 \leq (1/5)(3/300) - M < (1/4)(3/300) - M = a/h - M$;
- (ii) for $b - M\Gamma = 4/300 \leq u \leq 16/135 = b/\gamma^2$, we have $f(u) = \varphi(u) - 1/320 = 1 - M \geq 1/8 - M = b/lN - M$;
- (iii) for $0 \leq u \leq 12$, we have $f(u) = \varphi(u) - 1/320 \leq 135/1604 \times 12 + 1588/1604 - M = 2 - M \leq 3 - M = c/h - M$.

By Theorem 3.1, we know the BVP (4.1) has at least two positive solutions.

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