BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37***(1) (2014), 37–42*

Cline's Formula for the Generalized Drazin Inverse

[1]YIHUA LIAO, [2]JIANLONG CHEN AND [3]JIAN CUI
[1,2,3]Department of Mathematics, Southeast University, Nanjing 210096, China
[1]College of Mathematics and Information Science, Guangxi University, Nanning 530004, China
[1]yhliao@gxu.edu.cn, [2]jlchen@seu.edu.cn, [3]seujcui@126.com

Abstract. It is well known that for an associative ring R, if ab is Drazin invertible then ba is Drazin invertible. In this case, $(ba)^D = b((ab)^D)^2a$. This formula is so-called Cline's formula. In this note, we generalize Cline's formula to the case of the generalized Drazin invertibility.

2010 Mathematics Subject Classification: 15A09, 16S10

Keywords and phrases: Associative ring, generalized Drazin inverse, Drazin inverse.

1. Introduction

Let R be an associative ring with unity $1 \neq 0$. The notation R^{-1} means the group of units of R. Following Koliha and Patricio [14], the *commutant* and *double commutant* of an element a in R are defined by

$$\mathrm{comm}(a) = \{x \in R : ax = xa\} \quad \text{and} \quad \mathrm{comm}^2(a) = \{x \in R : xy = yx \text{ for all } y \in \mathrm{comm}(a)\},$$

respectively. Let $R^{\mathrm{qnil}} = \{a : 1 + ax \in R^{-1} \text{ for every } x \in \mathrm{comm}(a)\}$, and if $a \in R^{\mathrm{qnil}}$ then a is said to be *quasinilpotent* [11]. Let R^{nil} be the set of all nilpotents of R. Clearly, $R^{\mathrm{nil}} \subseteq R^{\mathrm{qnil}}$. Drazin introduced the notion of Drazin inverses in a ring in 1958. Recall that an element $a \in R$ is said to have a *Drazin inverse* [10] if there exists $b \in R$ such that

$$bab = b, \quad b \in \mathrm{comm}(a), \quad a - a^2b \in R^{\mathrm{nil}}.$$

The element b above is unique if it exists and is denoted by a^D. According to [14], $a^D \in \mathrm{comm}^2(a)$; and the nilpotency index of $a - a^2b$ is called the *Drazin index* of a, denoted by $\mathrm{ind}(a)$ (cf. [10]). If $\mathrm{ind}(a) = 1$, then a is *group invertible* and the group inverse of a is denoted by $a^\#$. Cline proved in 1965 [4] that if ab is Drazin invertible then so is ba. In this case, $(ba)^D = b((ab)^D)^2a$. This equation is called Cline's formula. It plays an important role in revealing the relationship between the Drazin inverse of a sum of two elements and the Drazin inverse of a block matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ (cf. [15]). In this note we extend this formula to the case of the generalized Drazin inverse. The concept of the generalized Drazin inverse in a Banach algebra was introduced in 1996 by Koliha [13]. Later, this notion was

Communicated by Jie Du.
Received: July 4, 2011; *Revised:* February 18, 2012.

extended to elements in a ring by Koliha and Patricio (cf. [14]). Recall that an element a of R is *generalized Drazin invertible* [14] in case there is an element $b \in R$ satisfying

$$bab = b, \quad b \in \mathrm{comm}^2(a), \quad a - a^2 b \in R^{\mathrm{qnil}}.$$

Such b, if it exists, is unique; it is called a *generalized Drazin inverse* of a, and will be denoted by a^d. Equivalently, an element $a \in R$ is generalized Drazin invertible if there exists $p^2 = p \in R$ satisfying $p \in \mathrm{comm}^2(a)$, $a + p \in R^{-1}$, and $ap \in R^{\mathrm{qnil}}$. In this situation, $a^d = (a+p)^{-1}(1-p)$ and $p = 1 - a^d a$. The generalized Drazin inverse was deeply investigated in complex Banach algebras and bounded linear operators over a complex Banach space. One may refer to [1, 3, 5, 6, 7, 8, 9], etc. In a Banach algebra, the condition $b \in \mathrm{comm}^2(a)$ in the above definition can be weakened as $b \in \mathrm{comm}(a)$. It seems that it is more difficult to study generalized Drazin inverses in rings, and there are fewer results on this topic. In this note, we extend the Cline's formula to the case of the generalized Drazin invertibility by means of methods of ring theory.

2. Main results

We begin with the following result known as Jacobson's lemma.

Lemma 2.1. *Let $a,\ b \in R$. If $1 + ab$ is invertible, then $1 + ba$ is invertible and $(1+ba)^{-1} = 1 - b(1+ab)^{-1}a$.*

Theorem 2.1. (Cline's Formula). *Let $a,\ b \in R$. If ab is generalized Drazin invertible, then so is ba, and*

$$(ba)^d = b((ab)^d)^2 a.$$

Proof. Let $\alpha = ab,\ \beta = ba,\ p = 1 - \alpha^d \alpha$ and $q = 1 - b\alpha^d a$. Then $p \in \mathrm{comm}^2(\alpha),\ \alpha + p \in R^{-1}$ and $\alpha p \in R^{\mathrm{qnil}}$. In what follows, we prove that (i) $\beta + q \in R^{-1}$; (ii) $\beta q \in R^{\mathrm{qnil}}$; and (iii) $q^2 = q \in \mathrm{comm}^2(\beta)$.

First, we note that $1 + (a - \alpha^d a)b = \alpha + (1 - \alpha^d \alpha) = \alpha + p \in R^{-1}$. By Lemma 2.1,

$$\beta + q = \beta + (1 - b\alpha^d a) = 1 + b(a - \alpha^d a) \in R^{-1}.$$

So we obtain (i).

To prove (ii), we write $c = \beta q$. Then

$$c = ba(1 - b\alpha^d a) = ba - bab\alpha^d a = b(1 - \alpha\alpha^d)a = bpa.$$

Let $z \in R$ with $cz = zc$. Next we show that $1 - zc \in R^{-1}$. From $cz = zc$, we have $cz^2 = z^2 c$, i.e.,

$$bpaz^2 = z^2 bpa.$$

Multiplying this equation by a on the left and by b on the right yields

$$\alpha p(az^2 b) = abpaz^2 b = az^2 bpab = (az^2 b)\alpha p.$$

Thus $az^2 b \in \mathrm{comm}(\alpha p)$. Since $\alpha p \in R^{\mathrm{qnil}}$, we have

$$1 - \alpha p(az^2 b) = 1 - (\alpha pa)(z^2 b) \in R^{-1}.$$

In view of Lemma 2.1, $1 - (z^2 b)\alpha pa$ is invertible. Since $c^2 = bpabpa = bp\alpha pa = b\alpha pa$ and $cz = zc$, it follows that

$$(1 - zc)(1 + zc) = (1 + zc)(1 - zc) = 1 - z^2 c^2 \in R^{-1}.$$

Hence $1 - zc \in R^{-1}$, as required.

To show (iii), we first prove that q is an idempotent. Indeed,

$$q^2 = (1-b\alpha^d a)(1-b\alpha^d a) = 1-2b\alpha^d a + b\alpha^d \alpha\alpha^d a = 1-b\alpha^d a = q.$$

Note that

$$\beta q = ba(1-b\alpha^d a) = ba - bab\alpha^d a = ba - b\alpha^d aba = (1-b\alpha^d a)\beta = q\beta. \tag{2.1}$$

Let $y \in R$ be such that $y\beta = \beta y$, i.e.,

$$y(ba) = (ba)y. \tag{2.2}$$

Then we obtain

$$(ayb)ab = ab(ayb)$$

by multiplying by a on the left and by b on the right. Thus $ayb \in \mathrm{comm}(\alpha)$. Since $\alpha^d \in \mathrm{comm}^2(\alpha)$, $ayb \in \mathrm{comm}(\alpha^d)$. It follows that $ayb \in \mathrm{comm}(\alpha^d - (\alpha^d)^2)$, i.e.,

$$ayb(\alpha^d - (\alpha^d)^2) = (\alpha^d - (\alpha^d)^2)ayb.$$

Then by Equation (2.2),

$$bayb(\alpha^d - (\alpha^d)^2)a = ybab(\alpha^d - (\alpha^d)^2)a = yb\alpha(\alpha^d - (\alpha^d)^2)a = yb\alpha\alpha^d a - yb\alpha^d a$$

and

$$b(\alpha^d - (\alpha^d)^2)ayba = b(\alpha^d - (\alpha^d)^2)abay = b(\alpha^d - (\alpha^d)^2)\alpha ay = b\alpha^d\alpha ay - b\alpha^d ay.$$

Hence,

$$yb\alpha\alpha^d a - yb\alpha^d a = b\alpha^d\alpha ay - b\alpha^d ay.$$

Applying Equation (2.2) to this result, we obtain after a calculation

$$yq - y\beta q = qy - \beta qy.$$

Combining this with Equations (2.1) and (2.2), one has

$$(1-\beta q)yq(1-\beta q) = (1-\beta q)qy(1-\beta q).$$

By (ii), $\beta q \in R^{\mathrm{qnil}}$. So $1-\beta q \in R^{-1}$. Hence, $yq = qy$, and so (iii) follows.

Therefore, $\beta = ba$ has a generalized Drazin inverse and

$$(ba)^d = \beta^d = (\beta+q)^{-1}(1-q).$$

Further, let $t = a - \alpha^d a$. Then we have

$$1+tb = \alpha + p \quad \text{and} \quad 1+bt = \beta + q.$$

So

$$(\beta+q)^{-1} = (1+bt)^{-1} = 1 - b(1+tb)^{-1}t = 1 - b(\alpha+p)^{-1}t.$$

Note that

$$\alpha^d = (\alpha+p)^{-1}(1-p) = (\alpha+p)^{-1}\alpha\alpha^d.$$

Then

$$\begin{aligned}(ba)^d &= (\beta+q)^{-1}(1-q) = [1-b(\alpha+p)^{-1}t]b\alpha^d a \\ &= b\alpha^d a - b(\alpha+p)^{-1}\alpha\alpha^d a + b(\alpha+p)^{-1}\alpha^d\alpha\alpha^d a \\ &= b(\alpha^d)^2 a = b((ab)^d)^2 a.\end{aligned}$$

This completes the proof. ∎

Cline formula for the Drazin inverse was investigated variously, one can refer to [12] and [15]. However, the proof of Theorem 2.1 can be slightly modified to obtain the following result.

Corollary 2.1. *Let $a,\ b \in R$. If ab is Drazin invertible with* $\operatorname{ind}(ab) = k$*, then ba is Drazin invertible with $k-1 \le \operatorname{ind}(ba) \le k+1$, and*

$$(ba)^D = b((ab)^D)^2 a.$$

Proof. Let $c = b((ab)^D)^2 a$. In view of Theorem 2.1, ba is generalized Drazin inverse with $(ba)^d = c$. By hypothesis, $(ab)^k = (ab)^{k+1}(ab)^D$, which implies that

$$[ba - (ba)^2 c]^{k+1} = b[(ab)^k - (ab)^{k+1}(ab)^D]a = 0.$$

Thus $ba - (ba)^2 c \in R^{\mathrm{nil}}$, and so $(ba)^D = c = b((ab)^D)^2 a$. From the above argument, one also has

$$\operatorname{ind}(ba) \le k+1 = \operatorname{ind}(ab) + 1.$$

By symmetry, $\operatorname{ind}(ab) \le \operatorname{ind}(ba) + 1$. Hence, $\operatorname{ind}(ba) \ge k-1$, and so we obtain the required result. ∎

Letting $k = 1$ in Corollary 2.1, we get a result for the group inverse.

Corollary 2.2. *Let $a,\ b \in R$. If ab is group invertible, then one of the following holds*:

(1) *ba is invertible;*
(2) *ba is group invertible with $(ba)^{\#} = b((ab)^{\#})^2 a$;*
(3) *ba is Drazin invertible with* $\operatorname{ind}(ba) = 2$*, and $(ba)^D = b((ab)^{\#})^2 a$.*

Remark 2.1.

(1) For $a,\ b \in R$, if ab is invertible, then ba need not be invertible. For instance, let $\mathbb{N}$ be the set of positive integers and $\mathbb{R}$ be the set of real numbers. Denote by $R = \mathbb{CFM}_{\mathbb{N}}(\mathbb{R})$ the ring of column finite $\mathbb{N} \times \mathbb{N}$ matrices over $\mathbb{R}$. Let E_{ij} be an element of R with (i,j)-entry is 1 and 0 elsewhere. Take

$$A = \sum_{i=1}^{\infty} E_{i(i+1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \in R, \quad B = \sum_{i=1}^{\infty} E_{(i+1)i} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \in R.$$

Then AB is the identity, but BA is not invertible since its first row are zero vector. However, in this case BA is group invertible, and $(BA)^{\#} = B(AB)^{-2}A$. (This shows that the indices of ab and ba need not be equal.)

(2) Jacobson's lemma states that for any $a,\ b \in R$, $1+ab$ is invertible if and only if $1+ba$ is invertible. In [2, 16], the authors generalized Jacobson's lemma to the Drazin invertibility and showed that for $a,\ b \in R$, if $1+ab$ is Drazin invertible with $\operatorname{ind}(1+ab) = k$, then $1+ba$ is Drazin invertible with $\operatorname{ind}(1+ba) = k$; Zhuang *et al.* [16] presented Jacobson's lemma for the generalized Drazin invertibility, and proved that $1+ab$ is generalized Drazin invertible if and only if so is $1+ba$.

A special case of Theorem 2.1 is an application of Cline's formula to a Banach algebra $\mathscr{A}$. In a Banach algebra it is enough to require that the generalized Drazin inverse a^d of an element a merely commutes with a. To show how much difference this makes, we give an alternative proof of Cline's formula recalling that an element w of a Banach algebra is quasinilpotent if and only if $\|w^n\|^{1/n} \to 0$ as $n \to \infty$.

Corollary 2.3. *Let $a, b \in \mathscr{A}$ and let ab be generalized Drazin invertible. Then so is ba, and*

$$(ba)^d = b((ab)^d)^2 a.$$

Proof. Let $c = b((ab)^d)^2 a$. To show that $c = (ba)^d$, we need to prove:

$$\text{(i) } bac = cba, \quad \text{(ii) } bac^2 = c, \quad \text{(iii) } w = ba - (ba)^2 c \in \mathscr{A}^{\mathrm{qnil}}:$$

(i) $bac = bab((ab)^d)^2 a = b(ab)^d a$, and $cba = b((ab)^d)^2 aba = b(ab)^d a$.

(ii) $bac^2 = (bac)c = b(ab)^d ab((ab)^d)^2 a = b((ab)^d)^2 a = c$.

(iii) Write $p = 1 - ab(ab)^d$; then p is idempotent and $pab \in \mathscr{A}^{\mathrm{qnil}}$. Then $w = bpa$, and induction shows that

$$w^{n+1} = b(pab)^n a, \quad n \geq 1.$$

Hence

$$\|w^{n+1}\|^{1/n} \leq \|b\|^{1/n} \|(pab)^n\|^{1/n} \|a\|^{1/n} \to 0 \quad \text{as} \quad n \to \infty,$$

and also $\|w^n\|^{1/n} \to 0$. ∎

3. Cline's formula for rectangular matrices and operators

For positive integers m, n, let $R^{m\times n}$ be the set of all $m \times n$ matrices over the ring R. Let $k = m+n$. Given matrices $A \in R^{m\times n}$ and $B \in R^{n\times m}$, we define two $k \times k$ matrices C, D by

$$C = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}. \tag{3.1}$$

We observe that

$$CD = \begin{pmatrix} 0 & 0 \\ 0 & AB \end{pmatrix}, \quad DC = \begin{pmatrix} BA & 0 \\ 0 & 0 \end{pmatrix}.$$

Since CD and DC belong to the same ring $R^{k\times k}$, Theorem 2.1 applies to give the following result: If CD is generalized Drazin invertible, then so is DC, and Cline's formula holds:

$$(DC)^d = D((CD)^d)^2 C. \tag{3.2}$$

This leads to Cline's formula for rectangular matrices.

Corollary 3.1. *Let $A \in R^{m\times n}$ and $B \in R^{n\times m}$. If $AB \in R^{m\times m}$ is generalized Drazin invertible, then so is $BA \in R^{n\times n}$, and*

$$(BA)^d = B((AB)^d)^2 A. \tag{3.3}$$

Proof. Write $k = m+n$. A direct verification of the conditions for the generalized Drazin inverse shows that any $m \times m$ matrix T over R is generalized Drazin invertible if and only if $\left(\begin{smallmatrix} T & 0 \\ 0 & 0 \end{smallmatrix}\right) \in R^{k\times k}$ is, while

$$\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}^d = \begin{pmatrix} T^d & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly, $S \in R^{n\times n}$is generalized Drazin invertible if and only if $\left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right) \in R^{k\times k}$ is, and

$$\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}^d = \begin{pmatrix} 0 & 0 \\ 0 & S^d \end{pmatrix}.$$

Define matrices C and D as in Equation (3.1). Setting $S = AB$ and $T = BA$ in the above argument and applying Equation (3.2), we obtain the result by matrix calculation:

$$\begin{pmatrix} (BA)^d & 0 \\ 0 & 0 \end{pmatrix} = (DC)^d = D((CD)^d)^2 C = \begin{pmatrix} B((AB)^d)^2 A & 0 \\ 0 & 0 \end{pmatrix}.$$

∎

Let $\mathscr{B}(X,Y)$ denote the set of all bounded linear operators between Banach spaces $X$and Y, and let $\mathscr{B}(X) = \mathscr{B}(X,X)$. We observe that if $A \in \mathscr{B}(X,Y)$ and $B \in \mathscr{B}(Y,X)$, then the operator matrices C and D defined by (3.1) belong to the algebra $\mathscr{B}(X \oplus Y)$. Using these matrices we obtain the operator case of Cline's formula.

Corollary 3.2. *Let X,Y be Banach spaces, let $A \in \mathscr{B}(X,Y)$ and $B \in \mathscr{B}(Y,X)$. If $AB \in \mathscr{B}(Y)$ is generalized Drazin invertible, then so is $BA \in \mathscr{B}(X)$, and Cline's formula (3.3) holds.*

Acknowledgement. The authors are highly grateful to the referee for valuable suggestions and providing many results which led to substantial improvements of the paper. In particular, Corollary 2.3, Corollary 3.2 and the proof of Corollary 3.1 were suggested to the authors by the referee. This research was supported by the National Natural Science Foundation of China (11371089, 11326062), the Specialized Research Fund for the Doctoral Program of Higher Education (20120092110020) and the Natural Science Foundation of Jiangsu Province (BK2010393).

References

[1] N. Castro González and J. J. Koliha, New additive results for the g-Drazin inverse, *Proc. Roy. Soc. Edinburgh Sect. A* **134** (2004), no. 6, 1085–1097.

[2] N. Castro-González, C. Mendes-Araújo and P. Patricio, Generalized inverses of a sum in rings, *Bull. Aust. Math. Soc.* **82** (2010), no. 1, 156–164.

[3] N. Castro-González and J. Y. Vélez-Cerrada, Elements of rings and Banach algebras with related spectral idempotents, *J. Aust. Math. Soc.* **80** (2006), no. 3, 383–396.

[4] R. E. Cline, An application of representation for the generalized inverse of a matrix, MRC Technical Report, **592**, 1965.

[5] D. S. Cvetković-Ilić, The generalized Drazin inverse with commutativity up to a factor in a Banach algebra, *Linear Algebra Appl.* **431** (2009), no. 5-7, 783–791.

[6] D. S. Cvetković-Ilić, D. S. Djordjević and Y. Wei, Additive results for the generalized Drazin inverse in a Banach algebra, *Linear Algebra Appl.* **418** (2006), no. 1, 53–61.

[7] C. Deng and Y. Wei, New additive results for the generalized Drazin inverse, *J. Math. Anal. Appl.* **370** (2010), no. 2, 313–321.

[8] D. S. Djordjević and P. S. Stanimirović, On the generalized Drazin inverse and generalized resolvent, *Czechoslovak Math. J.* **51(126)** (2001), no. 3, 617–634.

[9] D. S. Djordjević and Y. Wei, Additive results for the generalized Drazin inverse, *J. Aust. Math. Soc.* **73** (2002), no. 1, 115–125.

[10] M. P. Drazin, Pseudo-inverses in associative rings and semigroups, *Amer. Math. Monthly* **65** (1958), 506–514.

[11] R. Harte, On quasinilpotents in rings, *Panamer. Math. J.* **1** (1991), 10–16.

[12] R. E. Hartwig and J. Shoaf, Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices, *J. Austral. Math. Soc. Ser. A* **24** (1977), no. 1, 10–34.

[13] J. J. Koliha, A generalized Drazin inverse, *Glasgow Math. J.* **38** (1996), no. 3, 367–381.

[14] J. J. Koliha and P. Patricio, Elements of rings with equal spectral idempotents, *J. Aust. Math. Soc.* **72** (2002), no. 1, 137–152.

[15] P. Patrício and R. E. Hartwig, Some additive results on Drazin inverses, *Appl. Math. Comput.* **215** (2009), no. 2, 530–538.

[16] G. F. Zhuang, J. L. Chen and J. Cui, Jacobson's lemma for the generalized Drazin inverse, *Linear Algebra Appl.* **436** (2012), no. 3, 742–746.