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On the Radical Banach Algebras Related to Semigroup Algebras

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Abstract. Let $\mathscr{S}$ be a compactly cancellative foundation semigroup with identity. It is well-known that $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ can be equipped with a multiplication that extends the original multiplication on $M_a(\mathscr{S})$ and makes $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ a Banach algebra. In this paper, among the other things, it is shown that if $\mathscr{S}$ is a nondiscrete compactly cancellative foundation semigroup with an identity, then the radical of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ is infinite-dimensional.

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1. Introduction and notations

Let $\mathscr{S}$ be a locally compact, Hausdorff topological semigroup with identity e. Let $M(\mathscr{S})$ be the space of all complex Borel measures on S. Then $M(\mathscr{S})$ is the continuous dual of $C_0(\mathscr{S})$, the space of all continuous functions on $\mathscr{S}$ vanishing at infinity. The set of all measures $\mu \in M(\mathscr{S})$ for which both the mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ are weakly continuous will denoted by $M_a(\mathscr{S})$, where δ_x denotes the Dirac measure at x. A topological semigroup $\mathscr{S}$ is called a *foundation* semigroup if $\mathscr{S}$ coincides with the closure of $\cup\{\text{supp}(\mu);\ \mu \in M_a(\mathscr{S})\}$. If $\mathscr{S}$ is a foundation topological semigroup, then $M_a(\mathscr{S})$ is a closed L-ideal of $M(\mathscr{S})$ called the semigroup algebra $\mathscr{S}$ [4]. More information on this matter can be found in [1, 4, 5].

A complex-valued function f on $\mathscr{S}$ is said to be $M_a(\mathscr{S})$-measurable if it is μ-measurable for all $\mu \in M_a(\mathscr{S})$. Denote by $L^\infty(\mathscr{S};M_a(\mathscr{S}))$ the space of all bounded $M_a(\mathscr{S})$-measurable functions on $\mathscr{S}$ formed by identifying functions that agree μ-almost every where for all $\mu \in M_a(\mathscr{S})$. Observe that $L^\infty(\mathscr{S};M_a(\mathscr{S}))$ with complex cojugation as involution, the point-wise operations and the norm $\|.\|$ is a commutative C^*-algebra. It is well-known from [11] that if $\mathscr{S}$ is a foundation semigroup with an identity, then $L^\infty(\mathscr{S};M_a(\mathscr{S}))$ can be identified with $M_a(\mathscr{S})^*$. We say that a function $f \in L^\infty(\mathscr{S};M_a(\mathscr{S}))$ vanishes at infinity if for each $\varepsilon > 0$, there is a compact subset K of $\mathscr{S}$ for which $\|f\chi_{\mathscr{S}\setminus K}\| < \varepsilon$, that is, for each $\mu \in M_a(\mathscr{S})$, $|f(x)| < \varepsilon$ for μ-almost all $x \in \mathscr{S} \setminus K$ $(\mu \in M_a(\mathscr{S}))$. Let $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ be

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the C^*-algebra of all $M_a(\mathscr{S})$-measurable functions f on $\mathscr{S}$ such that f vanishes at infinity. Finally, let us recall that $\mathscr{S}$ is said to be compactly cancellative if $C^{-1}D$ and CD^{-1} are compact subsets of $\mathscr{S}$ for all compact subsets C and D of $\mathscr{S}$ [8]. Compactly cancellative foundation semigroups form a large class of locally compact semigroups which includes locally compact groups as elementary examples. As another example, consider the semigroup

$$\mathscr{S}=\{0\}\cup\left\{\frac{1}{n};\ n\geq 1\right\}\cup\left\{\frac{1}{2}+\frac{1}{n};\ n\geq 1\right\}$$

and set

$$\mathscr{B}=\Big\{\{x\};\ x\neq 0\Big\}\cup\left\{\{0\}\cup\left\{\frac{1}{n};\ n\geq k\right\};\ k\geq 1\right\}.$$

Then $\mathscr{S}$ with $\mathscr{B}$ as a base of the topology and the operation $xy=\max\{x,y\}$ defines a compactly cancellative foundation semigroup with identity. For an extensive study of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ in the compactly cancellative foundation semigroup case of $\mathscr{S}$, see [7, 8, 9].

2. Main results

Let $\mathscr{S}$ be a compactly cancellative foundation semigroup with identity. Given any $\mu\in M_a(\mathscr{S})$ and $f\in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$, define the complex-valued functions $fo\mu$ and μof on $\mathscr{S}$ by $fo\mu(x)=\mu(L_xf)$ and $\mu of(x)=\mu(R_xf)$, where $L_xf(y)=f(xy)$ and $R_xf(y)=f(yx)$ for all $x,y\in\mathscr{S}$. It is known that $fo\mu$ and μof are in $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ with $\|fo\mu\|\leq\|f\|\|\mu\|$ and $\|\mu of\|\leq\|f\|\|\mu\|$. For $f\in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ and $F\in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ we define $Ff\in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ as a linear functional on $M_a(\mathscr{S})$ by $\langle Ff,\mu\rangle=\langle F,\mu of\rangle$, see [8, Proposition 3.2]. We define the Arens product of G and F, denoted by $G.F$ to be the functional defined by $\langle G.F,f\rangle=\langle G,Ff\rangle$ for $f\in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$. Equipped with this multiplication, $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ is a Banach algebra and this multiplication agrees on $M_a(\mathscr{S})$ with the given product [8].

Theorem 2.1. *Let $\mathscr{S}$ be a compactly cancellative foundation semigroup with an identity. Then $C_0(\mathscr{S})^\perp$ is a closed two-sided ideal of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ and $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*/C_0(\mathscr{S})^\perp$ is isometrically isomorphic as an algebra to $M(\mathscr{S})$.*

Proof. By [9, Theorem 3.6], $C_0(\mathscr{S})^\perp$ is a weak* closed two-sided ideal of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$, and so $C_0(\mathscr{S})^\perp$ is a norm closed ideal of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. From Banach space theory, there is an isometric linear space isometric between $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*/C_0(\mathscr{S})^\perp$ and $C_0(\mathscr{S})^*$ [10]. In addition, there is an isometric linear space isomorphism between $C_0(\mathscr{S})^*$ and $M(\mathscr{S})$. The composite isometric isomorphism T is defined by $T(F+C_0(\mathscr{S})^\perp)=\mu$, where $\langle F,f\rangle=\int f(x)d\mu(x)$ for all $f\in C_0(\mathscr{S})$. It remains for us to see that T is an algebra isomorphism when $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*/C_0(\mathscr{S})^\perp$ is given the quotient space multiplication induced from the multiplication in $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ and multiplication in $M(\mathscr{S})$ is convolution. For $F_1,F_2\in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$, we put $\mu_1=T(F_1+C_0(\mathscr{S})^\perp)$ and $\mu_2=T(F_2+C_0(\mathscr{S})^\perp)$. Let $\mu_3=T(F_1.F_2+C_0(\mathscr{S})^\perp)$. Then for each $f\in C_0(\mathscr{S})$, $F_2f\in C_0(\mathscr{S})$. Indeed, any $f\in C_0(\mathscr{S})$ can be written in the form $f=\mu oh$ with $\mu\in M_a(\mathscr{S})$ and $h\in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$, see [8, Proposition 2.6]. On the other hand, $F_2f=F_2\mu oh=\mu oF_2h$. By [9, Proposition 3.1], $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ is a left introverted subspace of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$. This shows that $F_2h\in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$. Hence $F_2f=\mu oF_2h\in C_0(\mathscr{S})$ again by [8, Proposition 2.6]. It is easy to see that $F_2f(x) = \langle F_2, L_xf\rangle$ for all $x \in \mathscr{S}$. Now, let $f \in C_0(\mathscr{S})$. We have

$$\int f(z)d\mu_3(z) = \langle F_1.F_2, f\rangle = \langle F_1, F_2f\rangle = \int \langle F_2, L_xf\rangle d\mu_1(x)$$
$$= \int\int f(xy)d\mu_1(x)d\mu_2(y) = \int f(z)d\mu_1 * \mu_2(z).$$

Since this holds for all $f \in C_0(\mathscr{S})$, we conclude that $\mu_3 = \mu_1 * \mu_2$ and so T defines an isometric algebra isomorphism from $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*/C_0(\mathscr{S})^\perp$ onto $M(\mathscr{S})$. ∎

Theorem 2.2. *Let $\mathscr{S}$ be a nondiscrete and compactly cancellative foundation semigroup with an identity. Then $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ is not semisimple and is not commutative.*

Proof. Since $\mathscr{S}$ is not discrete, it is an immediate consequence of the Hahn-Banach theorem that $C_0(\mathscr{S})^\perp \neq \{0\}$. Now if $F \in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$, let F' be an extension of F to $L^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ such that $\|F\| = \|F'\|$. By [11], $L^\infty(\mathscr{S};M_a(\mathscr{S}))$ can be identified with $M_a(\mathscr{S})^*$. Since $M_a(\mathscr{S})$ is weak* dense in $M_a(\mathscr{S})^{**}$ [10], so that we can find a net $\{\mu_\alpha\}$ in $M_a(\mathscr{S})$ such that $\mu_\alpha \to F'$ in the weak* topology of $M_a(\mathscr{S})^{**}$. We conclude that $\mu_\alpha \to F$ in the weak* topology of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. For $G \in C_0(\mathscr{S})^\perp$ and $f \in C_0(\mathscr{S})$, we have

$$\langle F.G, f\rangle = \langle F, Gf\rangle = \lim_\alpha \langle \mu_\alpha, Gf\rangle = \lim_\alpha \langle G, \mu_\alpha of\rangle = 0,$$

since $\mu_\alpha of \in C_0(\mathscr{S})$ for all α, see [8, Proposition 2.1]. This shows that $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ $C_0(\mathscr{S})^\perp = \{0\}$. By [3, Proposition 1.5.6], we have $0 \neq C_0(\mathscr{S})^\perp \subseteq \mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*)$ and consequently $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ is not semisimple.

It remains for us to see that $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ is not commutative. Suppose that $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ is commutative. Let $F \in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. Clearly, the map $G \mapsto F.G = G.F$ is weak* weak* continuous on $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. This says that $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ is Arens regular. By [8, Theorem 4.3], $\mathscr{S}$ is discrete which is contradiction. ∎

Corollary 2.1. *Let $\mathscr{S}$ be a nondiscrete compactly cancellative foundation semigroup with an identity. Then the radical of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ is infinite-dimensional.*

Proof. For any integer n, there are n mutually disjoint relatively compact open subsets $U_1,...,U_n$ in $\mathscr{S}$, whose union is not all of $\mathscr{S}$. For $1 \leq i \leq n$, 1_{U_i} denotes the characteristic function of U_i. Since 1_{U_1} is not in the closure of $C_0(\mathscr{S})$, there exists $F_1 \in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ such that $\langle F_1, 1_{U_1}\rangle = 1$ but $\langle F_1, f\rangle = 0$ for every $f \in C_0(\mathscr{S})$. For $1 \leq i \leq n-1$, $\langle 1_{U_1},...,1_{U_i}\rangle \oplus C_0(\mathscr{S})$ is a closed subspace of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$, see [10, Theorem 1.42]. [10, Theorem 3.5] furnishes then a $F_{i+1} \in C_0(\mathscr{S})^\perp$ such that $\langle F_{i+1}, 1_{U_{i+1}}\rangle = 1$ and $\langle F_{i+1}, 1_{U_j}\rangle = 0$ for all $1 \leq j \leq i$. Clearly $\{F_1,...,F_n\}$ is a linearly independent subset of $C_0(\mathscr{S})^\perp$. By Theorem 2.2 and its proof, the radical $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ is an infinite-dimensional subspace of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. ∎

By a semicharacter on $\mathscr{S}$ we mean a non-zero function χ in $B(\mathscr{S})$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x,y \in \mathscr{S}$. We denote the set of all continuous semicharacters on $\mathscr{S}$ by $\hat{\mathscr{S}}$. Let A be a closed subalgebra of $M(\mathscr{S})$. By a multiplicative linear functional on A we mean a non-zero functional $h \in A^*$ such that $\langle h, \mu * \nu\rangle = \langle h, \mu\rangle\langle h, \nu\rangle$ for all $\mu, \nu \in A$. The set of all multiplicative linear functionals on A is denoted by $\hat{A}$. There exists a one-to-one mapping τ of $\hat{\mathscr{S}}$ onto $M(\hat{\mathscr{S}})$ such that $\hat{\chi}(\mu) = \int \chi(x)d\mu(x)$ for all $\hat{\chi} \in M_a(\hat{\mathscr{S}})$ where $\tau(\chi) = \hat{\chi}$ is in $\hat{\mathscr{S}}$, see [4, Theorem 5.3].

Example 2.1. Let $\mathscr{S}$ be the additive semigroup $\mathbb{Z}^+$ of all nonnegative integer numbers. Then $\mathscr{S}$ with the discrete topology is a compactly cancellative foundation semigroup with identity. A character χ of $\mathbb{Z}^+$ is plainly determined by the number $\chi(1)$, since $\chi(n) = \chi(1)^n$ ($n \in \mathbb{Z}^+$), and $\chi(1)$ can be any number in $\mathbb{T}$. Then clearly $\hat{\mathscr{S}}$ separates the points of $\mathscr{S}$.

Let $\mathscr{S}$ be a compactly cancellative foundation semigroup with an identity. A function $f \in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ is said to be almost periodic if the set $\{L_x f;\ x \in \mathscr{S}\}$ of left translates of f is norm relatively compact in $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$. The set of all almost periodic functions on $\mathscr{S}$ is denoted by $AP(\mathscr{S})$.

Theorem 2.3. *Let $\mathscr{S}$ be a compactly cancellative foundation semigroup with an identity which is not compact. Further, suppose that $\mathscr{S}$ is commutative and $\hat{\mathscr{S}}$ separates the points of $\mathscr{S}$. Then*

$$AP(\mathscr{S})^\perp \nsubseteq \mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*).$$

Proof. Assume that $AP(\mathscr{S})^\perp \subseteq \mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*)$. By [4, Theorem 5.9], $M(\mathscr{S})$ is semisimple. It follows from Theorem 2.1 and [3, Theorem 1.5.21] that

$$\mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*) \subseteq C_0(\mathscr{S})^\perp.$$

We conclude that $AP(\mathscr{S})^\perp \subseteq C_0(\mathscr{S})^\perp$, and consequently $C_0(\mathscr{S}) \subseteq AP(\mathscr{S})$. However, since $\mathscr{S}$ is not compact, $C_0(\mathscr{S}) \cap AP(\mathscr{S}) = \{0\}$ is a consequence of the theory of almost periodic functions on semigroups [2]. ∎

Remark 2.1.

(i) Let $\mathscr{S}$ be a compactly cancellative foundation semigroup with an identity. By [9, Theorem 3.3], $M_a(\mathscr{S})$ is a closed ideal in $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. Further, suppose that $\mathscr{S}$ is commutative and $\hat{\mathscr{S}}$ separates the points of $\mathscr{S}$. By [4, Theorem 5.9], $M_a(\mathscr{S})$ is semisimple. Since $\mathrm{rad}(M_a(\mathscr{S})) = M_a(\mathscr{S}) \cap \mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*)$, see [3, Theorem 1.5.4], we conclude that $M_a(\mathscr{S}) \cap \mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*) = \{0\}$. Now, let $\mathscr{S}$ be a compact abelian group. Then $\hat{\mathscr{S}}$ separates the points of $\mathscr{S}$ [6]. Consequently, if $\mathscr{S}$ is a compact abelian group, then

$$\mathrm{rad}(L^1(\mathscr{S})^{**}) \cap L^1(\mathscr{S}) = \{0\}.$$

(ii) Let $\mathscr{S}$ be a nondiscrete and compactly cancellative foundation semigroup with an identity. By Theorem 2.2 and its proof, it is easy to see that

$$\mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*) = \{F;\ L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^* F = \{0\}\}.$$

(iii) Let $\mathscr{S}$ be a compact foundation semigroup with identity. Let $f \in M_a\hat{(\mathscr{S})}$, $\mu \in M_a(\mathscr{S})$. Clearly $\mu o f \in M_a\hat{(\mathscr{S})}$. It follows that $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^* M_a\hat{(\mathscr{S})}^\perp = \{0\}$, and so $M_a\hat{(\mathscr{S})}^\perp \subseteq \mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*)$. But $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*/M_a\hat{(\mathscr{S})}^\perp$ is semisimple. We conclude that

$$M_a\hat{(\mathscr{S})}^\perp = \mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*).$$

Theorem 2.4. *Let $\mathscr{S}$ be a compactly cancellative foundation semigroup with an identity. Further, suppose that $M_a(\mathscr{S})$ is a semisimple Banach algebra. Then $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*/\mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*) \cong M_a(\mathscr{S})$ if and only if $\mathscr{S}$ is a discrete semigroup.*

Proof. Let $\mathscr{S}$ be a discrete semigroup. By [8, Proposition 3.4], we have $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^* \cong M_a(\mathscr{S})$. It follows that

$$L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*/\operatorname{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*) \cong M_a(\mathscr{S}).$$

Suppose $\mathscr{S}$ is not discrete. Let $\mathscr{U}$ denote the family of relatively compact neighborhoods of e and regard $\mathscr{U}$ as a directed set in the usual way: $U \succeq V$ if $U \subseteq V$. Since $\mathscr{S}$ is a foundation semigroup, we can find a probability measure $e_U \in M_a(\mathscr{S})$ such that $e_U(U) = 1$ for all $U \in \mathscr{U}$. It is easy to see that $\{e_U\}_{U\in\mathscr{U}}$ is a bounded approximate identity for $M_a(\mathscr{S})$ [4]. By the Banach-Alaoglu's theorem, without loss of generality, we may assume that $e_\alpha \to E$ in the weak* topology of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. It is known that E is a right identity for $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ [3]. We conclude that $E.F - F \in \operatorname{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*)$ for all $F \in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. Thus $E + \operatorname{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*)$ is an identity for $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*/\operatorname{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*)$. By assumption, $M_a(\mathscr{S})$ has an identity, say μ. Since $\{e_U\}_{U\in\mathscr{U}}$ is a bounded approximate identity for $M_a(\mathscr{S})$, $e_U = e_U * \mu \to \mu$ in the norm topology. It is not hard to see that $e_U \to \delta_e$ in the $\sigma(M(\mathscr{S}),C_0(\mathscr{S}))$ topology of $M(\mathscr{S})$. It follows that $\delta_e = \mu \in M_a(\mathscr{S})$. This is a contradiction, see [4, Exercise 3.10]. ∎

Let $\mathscr{S}$ be a locally compact foundation semigroup with an identity. If $i : C_0(\mathscr{S}) \to L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ is the inclusion map, then the restriction $i^*(F)$ of $F \in L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ to the subspace $C_0(\mathscr{S})$ of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ determines a quotient mapping $i^* : L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^* \to M(\mathscr{S})$. Notice that i^* is the identity on $M_a(\mathscr{S})$.

Theorem 2.5. *Let $\mathscr{S}$ be a compactly cancellative foundation semigroup with identity. Then $\mathscr{S}$ is compact if there is a finite-dimensional right ideal $\mathscr{I}$ in $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ such that $i^*(\mathscr{I}) \cap M(\mathscr{S}) \neq \{0\}$.*

Proof. Suppose that $\mathscr{S}$ is non-compact. Assume towards a contradiction that $\mathscr{I}$ is a finite-dimensional right ideal in $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ such that $i^*(\mathscr{I}) \cap M(\mathscr{S}) \neq \{0\}$. If $x \in \mathscr{S}$, let G be an extension of δ_x (regarded as a functional on $C_0(\mathscr{S})$) to $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))$ such that $\|G\| = \|\delta_x\|$ [10]. Then, for every $F \in \mathscr{I}$, we have $F\delta_x = Fi^*(G) = F.G \in \mathscr{I}$. This shows that $\mathscr{I}$ is a right translation invariant subspace of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. Take $F \in \mathscr{I}$ such that $i^*(F) \neq 0$ and $\|i^*(F)\| = 1$. Take $\nu \in M_a(\mathscr{S})$ such that $i^*(F) * \nu \neq 0$. Otherwise, $i^*(F) = 0$. Thus, without loss of generality, we may assume that $i^*(F) \in M_a(\mathscr{S})$. Since $\mathscr{I}$ is a finite-dimensional subspace of $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$, $\mathscr{X} := \{i^*(F) * \delta_x;\ x \in \mathscr{S}\}$ is finite-dimensional. Let $\dim(\mathscr{X}) = n$. Let $i^*(F) * \delta_{x_1}, ..., i^*(F) * \delta_{x_n}$ generate $\mathscr{X}$ as a subspace of $M_a(\mathscr{S})$. It is evident that the mapping $\varphi : \mathbb{C}^n \to \mathscr{X}$ defined by $\varphi(c_1, ..., c_n) = \sum_{j=1}^n c_j i^*(F) * \delta_{x_j}$ is a homeomorphism [10]. Hence, there is a constant $c > 0$ such that each $\mu \in \mathscr{X}$ can be written as $\sum_{j=1}^n c_j i^*(F) * \delta_{x_j}$ with $c_1, ..., c_n \in \mathbb{C}$ and $\sum_{j=1}^n |c_j| \leq c\|\mu\|$. Choose $\varepsilon \in (0,1)$ with $\varepsilon(1+c) < 1$. Let K be a compact subset of $\mathscr{S}$ such that $|i^*(F)|(K) > 1 - \varepsilon$. Since the semigroup is non-compact, there exists $x \in \mathscr{S}$ such that Kx is disjoint from $Kx_1 \cup \cdots \cup Kx_n$. Clearly

$$1 - \varepsilon < |i^*(F) * \delta_x|(Kx) \leq \sum_{j=1}^n |\alpha_j||i^*(F) * \delta_{x_j}|(Kx) < c\varepsilon.$$

We conclude that $\varepsilon(1+c) > 1$ which is contradiction. ∎

Theorem 2.6. *Let $\mathscr{S}$ be a compactly cancellative foundation semigroup with an identity. Let $\mathscr{I}$ be a right ideal of $M_a(\mathscr{S})$ of dimension $n \geq 1$. Then $\mathscr{I}\operatorname{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*) \subset \mathscr{I}$.*

Proof. Let $F \in \mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$, $\mu \in \mathscr{I}$ and $\{e_\alpha\}_{\alpha\in J}$ be a bounded approximate identity for $M_a(\mathscr{S})$ [4]. For $\alpha \in J$ we have $e_\alpha.F \in M_a(\mathscr{S})$, since $M_a(\mathscr{S})$ is an ideal in $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$ (see [8, Proposition 3.3]). Since $\mathscr{I}$ is finite-dimensional, $\mathscr{I}$ is a closed right ideal in $M_a(\mathscr{S})$, see [10, Theorem 1.21]. Clearly $\|\mu * e_\alpha.F - \mu.F\| \to 0$ and $\mu * e_\alpha.F \in \mathscr{I}$ for all $\alpha \in J$. We conclude that $\mu.F \in \mathscr{I}$ and so $\mathscr{I} L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^* \subseteq \mathscr{I}$. We assume that a contrario that $\mathscr{I} = \mathscr{I} L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$. If $\mathscr{I}$ is cyclic, say $\mathscr{I} = \mu M_a(\mathscr{S})$, then

$$\mathscr{I} = \mathscr{I}\,\mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*) = \mu\,\mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*).$$

We must have $\mu = \mu.F$ for some $F \in \mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*)$. By [3, Corollary 1.5.3], we have $\mu = 0$ and thus $\mathscr{I} = 0$ which is a contradiction. Now suppose that $\mathscr{I} = \mu_1 M_a(\mathscr{S}) + \cdots + \mu_n M_a(\mathscr{S})$, and that the Theorem holds for right $L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*$-modules with $n-1$ generators. Since $\mathscr{I}/\mu_1 M_a(\mathscr{S})$ has $n-1$ generators, and since

$$\mathscr{I}/\mu_1 M_a(\mathscr{S})\,\mathrm{rad}(L_0^\infty(\mathscr{S};M_a(\mathscr{S}))^*) = \mathscr{I}/\mu_1 M_a(\mathscr{S}),$$

it follows that $\mathscr{I}/\mu_1 M_a(\mathscr{S}) = \{0\}$. This shows that $\mathscr{I} = \mu_1 M_a(\mathscr{S})$. Therefore, by cyclic case, $\mathscr{I} = 0$. This is contradiction. ∎

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