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Practical Stability for Discrete Hybrid System with Initial Time Difference

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Abstract. In this paper, we study the discrete hybrid system with different initial times. By employing Lyapunov-like functions, some new comparison theorems and several practical stability criteria for discrete hybrid system with initial time difference are obtained.

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1. Introduction

Generally speaking, a hybrid dynamic system is a system with different kinds of time dynamics, e.g., continuous, discrete or impulsive, in different interacting parts of the systems. As a result, it is more practical to depict the real world, for example, it has vast applications on modeling, design and so on. Naturally, it has caused considerable interesting to investigate the property of the hybrid systems, especially for their stability [1, 2, 5, 6, 11, 15, 16].

There are various concepts of stability, such as absolute stability, asymptotically stability, conditional stability and so on. But there is an evident defect that a system may be stable or asymptotically stable in theory, however, it is unstable in practice actually, because of the stable domain or the attraction domain is not large enough to solve the problem. To settle this contradiction, LaSalle and Lefschetz first brought forth the concept of practical stability in 1961 [7]. Furthermore, Lakshmikantham *et al.* [3] presented a systematic study of practical stability in 1990. In recent years, Lakshmikantham and Vatsala [5] studied the practical stability of hybrid system on time scales, Martynyuk and Sun [12] also gave a fairly comprehensive description of the application of practical stability. Liu and Zhao [9] investigated the practical stability in terms of two measures for impulsive functional differential equations. As for [16], Wang and Wu examined the practical stability in terms of two measures for discrete hybrid system. Liu and Hill [10] considered the uniformly stability and ISS for discrete-time impulsive hybrid systems. Apart from these works, [18, 19] are also the results of this nature. Xuping Xu, Guisheng Zhai and Shouling He [17]

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studied the practical stability for a class of discrete hybrid system–switched system and give an example.

Obviously, most of those works investigated the situation with unchanged starting time, however, it is impossible not to make any error on the starting time. Consequently, it is of great significance to study the nonlinear equations with solutions starting with different initial times. Many scholars have done works with respect to the problems with initial time difference. For example, Song *et al.* [14] focused on the practical stability of nonlinear differential equations with initial time difference, so as McRae [13], Li *et al.* [8] and so on. In the last decade, plenty of results were obtained for hybrid system or practical stability of nonlinear systems or nonlinear systems with initial time difference. Nevertheless, so far as we know, there is hardly any result for the combining of these problems.

In this paper, we study the practical stability of discrete hybrid system with initial time difference which presses closer to our real lives. Lyapunov method is a powerful tool for our research, it provides a convenient way to transform a given complicated differential system into a relatively simpler system. Using Lyapunov functions, we obtain some new comparison theorems first, and on this basis, we get some criteria for several types of practical stability of discrete hybrid system with initial time difference by employing two Lyapunov-like functions. The method of two Lyapunov-like functions offers us a more accurate tool and involuntary we obtain some better new results, see [4, 14]. The paper is organized as follows. In Section 2, we present some necessary definitions and notations relative to discrete hybrid system with initial time difference. In Section 3, we introduce two new comparison theories first, then some sufficient conditions for practical stability for discrete hybrid system with initial time difference are obtained.

2. Preliminary

Consider the discrete hybrid system

$$\begin{aligned} &\Delta x(n) = f(n,x(n),\lambda_r(\tau_r,x_r)), \quad n \in [\tau_r,\tau_{r+1}), \\ &x(\tau_r) = x_r, \quad r = 0,1,2,\ldots \end{aligned} \tag{2.1}$$

where $f \in C[\mathbb{N}^+_{n_0} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $\lambda_r \in C[\mathbb{N}^+_{n_0} \times \mathbb{R}^n, \mathbb{R}^n]$, $\mathbb{N}^+_{n_0} = \{n_0, n_0+1, n_0+2, \ldots\}$, and $x_r \in \mathbb{R}^n$. Let $x(n,n_0,x_0)$ and $y(n,m_0,y_0)$ be the solutions of (2.1) through two initial values (n_0,x_0) and (m_0,y_0), respectively. Without loss of generality, we may assume that $m_0 > n_0$ and $\eta = m_0 - n_0$, where $m_0, n_0 \in [\tau_0, \tau_1)$, $0 \le n_0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_r < \cdots$, $\tau_r \to \infty$ as $r \to \infty$, and denote $S(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho\}$. By the solutions $x(n,n_0,x_0)$ of (2.1) we mean the following:

$$x(n) = x_r(n), \quad \text{for} \quad n \in [\tau_r, \tau_{r+1}), \quad r = 0,1,2,\ldots$$

where $x_r(n) = x_r(n,\tau_r,x_r)$ is the solution of the discrete hybrid system

$$\Delta x_r(n) = f(n,x_r(n),\lambda_r(\tau_r,x_r)), \quad x_r(\tau_r) = x_r,$$

for each $r = 0,1,2,\ldots$ and $\tau_r \le n < \tau_{r+1}$, $n \in \mathbb{N}^+_{n_0}$. The description of the solution $y(n,m_0,y_0)$ is similar to $x(n,n_0,x_0)$ and we omit the details.

We also need the scalar comparison hybrid system

$$\begin{aligned} &\Delta u(n;k) = g_1(k,u(k),\sigma_r(u_r)), \quad n \in [\tau_r,\tau_{r+1}), \quad k \in [\tau_r,n], \\ &u(\tau_r) = u_r, \quad r = 0,1,2,\ldots \end{aligned} \tag{2.2}$$

$$(2.3)\qquad \begin{aligned} &\Delta v(n;k) = g_2(k, v(k), \mu_r(v_r)), \quad n \in [\tau_r, \tau_{r+1}), \quad k \in [\tau_r, n],\\ &v(\tau_r) = v_r, \quad r = 0,1,2,\ldots \end{aligned}$$

where $g_1,\ g_2 \in C[\mathbb{N}_{n_0}^{+} \times \mathbb{R}^{+} \times \mathbb{R}^{+}, \mathbb{R}]$, and $\sigma_r,\ \mu_r \in C[\mathbb{R}^{+}, \mathbb{R}^{+}]$. By the solution $u(k)$ of (2.2) we mean the following:

$$u(k) = u_r(n;k), \quad \text{for} \quad \tau_r \le n < \tau_{r+1}, \quad \tau_r \le k \le n$$

where $u_r(n;k)$ is the solution of the discrete hybrid system

$$\Delta u_r(n;k) = g_1(k, u_r(k), \sigma_r(u_r)), \quad u_r(\tau_r) = u_r, \quad r = 0,1,2,\ldots.$$

We can describe the solution of system (2.3) similarly to the solution of system (2.2).

We assume the following assumption (H) holds for the system (2.1):

(H) The solution $x(n,n_0,x_0)$ (or $y(n,m_0,y_0)$) of (2.1) exists for all $n \ge n_0$ $(n \ge m_0)$, unique and continuous with respect to the initial data, and $\|x(n,n_0,x_0)\|$ $(\|y(n,m_0,y_0)\|)$ is locally Lipschitzian in x_0 (y_0).

For the sake of convenience, we employ the following notations where $x(n,n_0,x_0)$ and $y(n,m_0,y_0)$ are the solutions of system (2.1) through the initial values (n_0,x_0) and (m_0,y_0), respectively:

(i) $x_r = x_{r-1}(\tau_r, \tau_{r-1}, x_{r-1})$, $x_k = x_r(k, \tau_r, x_r)$ for $\tau_r \le n < \tau_{r+1},\ \tau_r \le k \le n$;

(ii) set $z = z(k) = y(k+\eta, m_0, y_0) - x(k, n_0, x_0)$, which means

$$z(k) = z_r(k), \quad \text{for} \quad \tau_r \le n < \tau_{r+1}, \quad \tau_r \le k \le n$$

$z_0 = y_0 - x_0$ and $z_r = z_r(\tau_r) = y(\tau_r + \eta, m_0, y_0) - x(\tau_r, n_0, x_0)$, $\widetilde{f}(k,z,\eta) = f(k + \eta, z + x(k,n_0,x_0)) - f(k, x(k,n_0,x_0))$, $k \ge n_0$;

(iii) for $n_1 \in [\tau_i, \tau_{i+1})$, $n_2 \in [\tau_j, \tau_{j+1})$, we say that n_1 precedes n_2, denoted by $n_1 \prec n_2$, if $n_1 < n_2$, or if $i < j$;

(iv) two classes of functions
$\mathscr{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+) : a \text{ is strictly increasing and } a(0) = 0\}$,
$\mathscr{CK} = \{a \in C(\mathbb{R}_+^2, \mathbb{R}_+) :\ a(t,u) \in \mathscr{K} \text{ for each } t \in \mathbb{R}_+\}$.

For any $V \in C[\mathbb{N}_{n_0}^{+} \times \mathbb{R}^n, \mathbb{R}^n]$ and any fixed $n \in [n_0, \infty)$, we define

$$\Delta V(k, x(n,k,x_k)) = V(k+1, x(n; k+1, x_k + f(k,x_k))) - V(k, x(n;k,x_k)),$$

for $n_0 \le k \le n$, and $x_k \in \mathbb{R}^n$.

Definition 2.1. *The solution $x(n) = x(n,n_0,x_0)$ of the system* (2.1) *is said to be*

(S1) *practically stable, if given (λ, A) with $0 < \lambda < A$, there exists a $\sigma(\lambda,A) \ge 0$ such that $\|y_0 - x_0\| < \lambda$, $\eta \le \sigma$ implies $\|y(n+\eta, m_0, y_0) - x(n,n_0,x_0)\| < A$, $n \ge n_0$ for some $n_0 \in \mathbb{N}_{n_0}^{+}$;*

(S2) *uniformly practically stable, if (S_1) holds for all $n_0 \in \mathbb{N}_{n_0}^{+}$;*

(S3) *practically quasi-stable, if for given $(\lambda, B, N) > 0$ and some $n_0 \in \mathbb{N}_{n_0}^{+}$, there exists a $\sigma(\lambda,B,N) \ge 0$ such that $\|y_0 - x_0\| < \lambda$, $\eta \le \sigma$ implies $\|y(n+\eta, m_0, y_0) - x(n,n_0,x_0)\| < B$, $n \ge n_0 + N$;*

(S4) *strongly practically quasi-stable, if (S_1) and (S_3) hold simultaneously;*

(S5) *attractive, if given $\delta \ge 0$, $\varepsilon \ge 0$ and $n_0 \in \mathbb{N}_{n_0}^{+}$, there exists $T = T(n_0, \varepsilon)$, $\sigma = \sigma(\delta, \varepsilon)$, such that $\|y_0 - x_0\| < \delta$, $\eta \le \sigma$ implies $\|y(n+\eta, m_0, y_0) - x(n,n_0,x_0)\| < \varepsilon$, $n \ge n_0 + T$, for some $n_0 \in \mathbb{N}_{n_0}^{+}$;*

(S6) *practically asymptotically stable, if (S_1) and (S_6) hold simultaneously with $\delta = \lambda$;*

(S7) *uniformly practically asymptotically stable, if* (S_7) *holds for all* $n_0 \in \mathbb{N}^+_{n_0}$.

3. Main results

We can now prove a new comparison result for ordinary difference system.

Lemma 3.1. *Let* $x(n,n_0,x_0)$ *and* $y(n,m_0,y_0)$ *be the solutions of* $\Delta x(n) = f(n,x(n))$ *through two initial values* (n_0,x_0) *and* (m_0,y_0) *respectively where* $m_0 > n_0$, *and set* $\eta = m_0 - n_0$, $V \in C[\mathbb{N}^+_{n_0} \times \mathbb{R}^n, \mathbb{R}^+]$, *and* $V(t,u)$ *is locally Lipschitz in u. Assume that for fixed n,* $n_0 \le k \le n$,

(i) $\Delta V(k,h,\eta) = V(k+1,h+\widetilde{f}(k,h,\eta)) - V(k,h) \le g(k,V(k,h),\eta)$ *where* $h = h(k) = y(k+\eta,m_0,y_0) - x(k,n_0,x_0)$, $\widetilde{f}(k,h,\eta) = f(k+\eta,h+x(k,n_0,x_0)) - f(k,x(k,n_0,x_0))$, $k \ge n_0$, $g \in C[\mathbb{N}^+_{n_0} \times \mathbb{R}^+, \mathbb{R}]$, *and* $g(k,u)$ *is increasing in u for each fixed* (n,k);

(ii) $w(n;k,n_0,w_0,\eta)$ *is any solution of* $\Delta w(k) = g(k,w(k)), \quad w(n_0) = w_0$.

Then $V(n_0,y_0-x_0) \le w_0$ *implies* $V(n,y(n+\eta,m_0,y_0) - x(n,n_0,x_0)) \le r(n;n,n_0,w_0,\eta)$.

Proof. Set $m(k) = V(k,y(k+\eta,m_0,y_0) - x(k,n_0,x_0))$, $n_0 \le k \le n$. Then we claim that $m(n_0) \le w_0$ implies that

$$m(k) \le w(n;k,n_0,w_0,\eta), \quad n_0 \le k \le n. \tag{3.1}$$

If (3.1) is not true, then there would exist $k_1 \ge n_0$, such that

$$m(k_1) \le w(n;k_1,n_0,w_0,\eta) \quad \text{but} \quad m(k_1+1) > w(n;k_1+1,n_0,w_0,\eta),$$

we obtain from the above conditions

$$\begin{aligned} g(k_1,w(n;k_1,n_0,w_0,\eta)) &= \Delta w(n;k_1,n_0,w_0,\eta) \\ &= w(n;k_1+1,n_0,w_0,\eta) - w(n;k_1,n_0,w_0,\eta) \\ &< m(k_1+1) - m(k_1) = \Delta m(k_1) \le g(k_1,m(k_1)) \\ &\le g(k_1,w(n;k_1,n_0,w_0,\eta)), \end{aligned}$$

which is contradictive, hence (3.1) is valid. Setting $k = n$, we get the desired result. ∎

Now, we shall prove the corresponding comparison result for the hybrid system (2.1).

Theorem 3.1. *Assume that the assumption* (H) *holds,* $x(n,n_0,x_0)$ *and* $y(n,m_0,y_0)$ *are the solutions of* (2.1) *through two initial values* (n_0,x_0) *and* (m_0,y_0), *respectively, and*

(i) $V \in C[\mathbb{N}^+_{n_0} \times \mathbb{R}^n, \mathbb{R}^+]$, $V(n,u)$ *is locally Lipschitz in u, for* $\tau_r \le n < \tau_{r+1}$, $\tau_r \le k \le n$, $r = 0,1,2,\ldots$ *and*

$$\Delta V(k,z,\eta) = V(k+1,z(k)+\widetilde{f}(k,z,\eta)) - V(k,z(k)) \le g(k,V(k,z),\sigma_r(V(\tau_r,z_r)),\eta),$$

where $g \in C[\mathbb{N}^+_{n_0} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}]$, *and* $g(k,u,v)$ *is increasing in u for each fixed* (k,v) *and in v for each fixed* (k,u). *Also,* $\sigma_r(v)$ *is increasing in v for all r;*

(ii) $u(k) = u(k,n_0,u_0,\eta)$ *is any solution of the system* (2.2).

Then $V(n_0,y_0-x_0) \le u_0$ *implies* $V(n,y(n+\eta,m_0,y_0) - x(n,n_0,x_0)) \le u(n;n,n_0,u_0,\eta)$.

Proof. Let $x_r(n,\tau_r,x_r)$ and $y_r(n,\tau_r,y_r)$ be the solutions of (2.1) in the interval $[\tau_r,\tau_{r+1})$ through the initial values (n_0,x_0) and (m_0,y_0), respectively. Set $m(k) = V(k,z)$, then by assumption (i), it is easy to derive the difference inequality

$$\Delta m(k) \le g(k,m(k),\sigma_r(m_r)),$$

where $m_r = V(\tau_r,z_r)$.

Consider the interval $[n_0,\tau_1)$, and $m_0 = m(n_0) = V(n_0,z(n_0)) = V(n_0,y_0-x_0) \le u_0$. For $n_0 \le k \le n < \tau_1$, we have

$$\Delta m(k) \le g(k,m(k),\sigma_0(m_0)).$$

Hence, by Lemma 3.1, we get

$$V(n,z_0(n)) \le u_0(n;n,n_0,V(n_0,z_0)), \quad n \in [n_0,\tau_1),$$

where $u_0(n,n_0,u_0)$ is the solution of

$$\Delta u_0(n) = g(n,u_0(n),\sigma_0(u_0)), \quad u_0(n_0) = u_0, \quad n \in [n_0,\tau_1).$$

Next, consider the interval $[\tau_1,\tau_2)$, choosing $u_1 = u_0(\tau_1,n_0,u_0)$, we have

$$\Delta m(k) \le g(k,m(k),\sigma_1(m_1)), \quad m_1 = m(\tau_1), \quad n \in [\tau_1,\tau_2),$$

where $m_1 = V(\tau_1,z(\tau_1,n_0,x_0)) \le u_0(\tau_1,n_0,u_0) = u_1$, then using assumption (i) and the non-decreasing properties of $g(k,u,v)$ and $\sigma_r(v)$ in v, we have

$$\Delta m(k) \le g(k,m(k),\sigma_1(u_1)), \quad m_1 \le u_1.$$

Therefore, by Lemma 3.1, we have

$$V(n,z_1(n)) \le u_1(n;n,\tau_1,u_1), \quad n \in [\tau_1,\tau_2),$$

where $u_1(n,\tau_1,u_1)$ is the solution of

$$\Delta u_1(n) = g(n,u_1(n),\sigma_0(u_1)), \quad u_1(\tau_1) = u_1, \quad n \in [\tau_1,\tau_2).$$

We repeat the process in the interval $[\tau_r,\tau_{r+1})$, $r = 0,1,2,\dots$ using the special choice of $u_r = u_{r-1}(\tau_r,\tau_{r-1},u_{r-1})$ to get

$$V(n,z_r(n)) \le u_r(n,\tau_r,u_r), \quad n \in [\tau_r,\tau_{r+1}),$$

where $u_r(n,\tau_r,u_r)$ is the solution of

$$\Delta u_r(n) = g(n,u_r(n),\sigma_r(u_r)), \quad u_r(\tau_r) = u_r, \quad n \in [\tau_r,\tau_{r+1}).$$

Then by mathematical induction we get the desired estimate for $n \ge n_0$, and the proof is completed. ∎

Theorem 3.2. *Assume that*

(i) $V_1 \in C[\mathbb{N}^+_{n_0} \times S(\rho),\mathbb{R}^+]$, *$V_1(n,x)$ is locally Lipschitz in x, for $(n,x) \in \mathbb{N}^+_{n_0} \times S(\rho)$, $a_1 \in \mathscr{CK}$, $V_1(n,x) \le a_1(n,\|x\|)$ and*

$$\Delta V_1(k,x(n),\eta)) \le g_1(k,V_1(k,x(n)),\sigma_r(V(\tau_r,x(\tau_r))),\eta), \quad \tau_r \le k \le n < \tau_{r+1},$$

where $g_1 \in C[\mathbb{N}^+_{n_0} \times \mathbb{R}^+ \times \mathbb{R}^+,\mathbb{R}]$;

(ii) $V_2 \in C[\mathbb{N}^+_{n_0} \times S(\rho) \cap S^c(\lambda),\mathbb{R}^+]$, *$V_2(t,x)$ is locally Lipschitz in x, for $(n,x) \in \mathbb{N}^+_{n_0} \times S(\rho) \cap S^c(\lambda)$, $b \in \mathscr{K}$, $b(\|x\|) \le V_2(n,x) \le a_2(\|x\|)$, a_2 and*

$$\Delta V_1(k,x(n),\eta) + \Delta V_2(k,x(n),\eta) \le g_2(k,V_1(k,x(n)+V_2(k,x(n))),\sigma_r(V(\tau_r,x(\tau_r))),\eta),$$

where $g_2 \in C[\mathbb{N}^+_{n_0} \times \mathbb{R}^+ \times \mathbb{R}^+,\mathbb{R}]$, and $\tau_r \le n < \tau_{r+1}$, $\tau_r \le k \le n$;

(iii) *$0 < \lambda < A < \rho$, and $a_1(n_0,\lambda) + a_2(\lambda) < b(A)$ for some $n_0 \in \mathbb{N}^+_{n_0}$;*

(iv) *$u_0 < a_1(n_0,\lambda)$ implies $u(n;k,n_0,u_0) < a_1(n_0,\lambda)$ for $n \ge n_0$, where $u(n;k,n_0,u_0)$ is any solution of the comparison system* (2.2). *$v_0 < a_1(n_0,\lambda) + a_2(\lambda)$ implies $v(n;k,n_0,w_{10}) < b(A)$ for $n \ge n_0$, where $v(n;k,n_0,w_{20})$ is any solution of the comparison system* (2.3).

Then the solution $x(n) = x(n,n_0,x_0)$ of (2.1) *is practically stable.*

Proof. We claim that for given (λ, A), there exists a $\sigma(\lambda, A) \geq 0$, such that $\|y_0 - x_0\| < \lambda$, $\eta \leq \sigma$ implies

$$\|y(n+\eta, m_0, y_0) - x(n, n_0, x_0)\| < A,\ n \geq n_0,$$

where $x(n, n_0, x_0)$ and $y(n, m_0, y_0)$ are the solutions of (2.1) through two initial values (n_0, x_0) and (m_0, y_0).

If it is not true, the solutions $x(n, n_0, x_0)$ and $y(n, m_0, y_0)$ of system (2.1) that for every given $\sigma(\lambda, A) \geq 0,\ \eta \leq \sigma$, there exists $n_2 \succ n_1 \succ n_0$, such that

$$\|y(n_1+\eta, m_0, y_0) - x(n_1, n_0, x_0)\| = \lambda,\ \|y(n_2+\eta, m_0, y_0) - x(n_2, n_0, x_0)\| = A,$$

and

$$\lambda \leq \|y(n+\eta, m_0, y_0) - x(n, n_0, x_0)\| \leq A, \quad n_1 \leq n \leq n_2.$$

We can easily obtain from the above that for $n_0 \leq n \leq n_1$, $\|z(n)\| \leq \lambda \leq \rho$. Then using condition (ii), we get by Theorem 3.1

$$V_1(n, z(n)) \leq r_1(n, n_0, V_1(n_0, y_0 - x_0), \eta), \quad n_0 \leq n \leq n_1, \tag{3.2}$$

where $r_1(n, n_0, V_1(n_0, y_0 - x_0), \eta)$ is the maximal solution of the comparison system (2.2) through $(n_0, V_1(n_0, y_0 - x_0))$. Similarly, by condition (iii), we have

$$V_1(n, z(n)) + V_2(n, z(n)) \leq r_2(n, n_1, V_1(n_1, z(n_1)) + V_2(n_1, z(n_1)), \eta), \quad n_1 \leq n \leq n_2, \tag{3.3}$$

where $r_2(n, n_1, V_1(n_1, z(n_1)) + V_2(n_1, z(n_1)), \eta)$ is the maximal solution of the comparison system (2.3) through $(n_1, V_1(n_1, z(n_1)) + V_2(n_1, z(n_1)))$. Since $\|y_0 - x_0\| < \lambda < \rho$, it follows from (ii) that

$$V_1(n_0, y_0 - x_0) \leq a_1(n_0, \|y_0 - x_0\|) < a_1(n_0, \lambda). \tag{3.4}$$

Applying (3.2), (3.4) and condition (v), we have

$$V_1(n_1, z(n_1)) \leq a_1(n_0, \lambda).$$

From (iii), $V_2(n_1, z(n_1)) \leq a_2(\|z(n_1)\|) = a_2(\lambda)$. Consequently,

$$V_1(n_1, z(n_1)) + V_2(n_1, z(n_1)) < a_1(n_0, \lambda) + a_2(\lambda).$$

Then we combine (3.3) with condition (v) to get

$$V_1(n_2, z(n_2)) + V_2(n_2, z(n_2)) \leq r_2(n_2, n_1, V_1(n_1, z(n_1)) + V_2(n_1, z(n_1))) < b(A).$$

But, by condition (iii), we get

$$V_1(n_2, z(n_2)) + V_2(n_2, z(n_2)) \geq V_2(n_2, z(n_2)) \geq b(\|z(n_2)\|) = b(A),$$

which is a contradiction, so the claim is proved. ∎

From definition 2.1, we can get the following result easily.

Corollary 3.1. *Assume that the assumptions of Theorem 3.2 holds, instead of* $a_1(n, \|x\|) \in \mathscr{CK}$*, the condition* $a_1(\|x\|) \in \mathscr{K}$ *holds, then the solution* $x(n) = x(n, n_0, x_0)$ *of* (2.1) *is uniformly practically stable.*

Theorem 3.3. *Suppose that the assumption of Theorem 3.2 hold except that the condition* (ii) *being replaced by* (ii)* *and* (iii)*:

(ii)* $w(n,x) \in C[\mathbb{N}_{n_0}^+ \times S(\rho), \mathbb{R}^+]$, *$w(n,x)$ is locally Lipschitz in x; for $(n,x) \in \mathbb{N}_{n_0}^+ \times S(\rho)$, $w(n,x) \geq b_0(\|x\|)$, where $b_0 \in \mathscr{K}$, $w(n,x) \in \mathscr{CK}$, and there exists a M such that $\Delta w(n,z) \leq M$.*

(iii)* $V_1 \in C[\mathbb{N}_{n_0}^+ \times S(A), \mathbb{R}^+]$, *and* $V_1(n,x)$ *is locally Lipschitz in* x; *for* $(n,x) \in \mathbb{N}_{n_0}^+ \times S(A)$, $V_1(n,x) \le a_1(n,\|x\|)$, $a_1 \in \mathscr{CK}$, *and*

$$\Delta V_1(n,z,\eta) + p(w(n,x)) \le g_1(n, V_1(n,z,\eta), \sigma_r(V(\tau_r,z_r),\eta), \quad \tau_r \le n < \tau_{r+1}, \tag{3.5}$$

where $p \in \mathscr{K}$, $g_1 \in C[\mathbb{N}_{n_0}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}]$, *and* $g_1(n,v,\eta)$ *is nondecreasing in* v.

Then the solution $x(n) = x(n,n_0,x_0)$ *of system* (2.1) *is practically asymptotical stable.*

Proof. Firstly, the solution $x(n) = x(n,n_0,x_0)$ of (2.1) is practical stable by Theorem 3.2. For each given (λ_1, A_1), $0 < \lambda_1 < A_1$, there exists a $\sigma_1(\lambda_1,A_1) \ge 0$ such that $\|y_0 - x_0\| < \lambda_1$, $\eta \le \sigma_1$ implies

$$\|y(n+\eta,m_0,y_0) - x(n,n_0,x_0)\| < A_1, \quad n \ge n_0.$$

We need to show that $\|y(n+\eta,m_0,y_0) - x(n,n_0,x_0)\| \to 0$, as $n \to \infty$, when $\|x_0 - y_0\| < \lambda_1$, $\eta \le \sigma_1$. Since $w(n,x) \in \mathscr{CK}$ and $w(n,x) \ge b_0(\|x\|)$, we only to show that

$$\lim_{n\to\infty} w(n, y(n+\eta,m_0,y_0) - x(n,n_0,x_0)) = 0$$

when $\|y_0 - x_0\| < \lambda_1$, $\eta \le \sigma_1$.

Now, we claim that $\lim_{n\to\infty} w(n, y(n+\eta,m_0,y_0) - x(n,n_0,x_0)) = 0$, when $\|y_0 - x_0\| < \lambda_1$, $\eta \le \sigma_1$. If it is not true, then there exist two different sequences $\{k_n\}$, $\{k_n^*\}$, such that

$$w(k_i, y(k_i+\eta,m_0,y_0) - x(k_i,n_0,x_0)) = \gamma/2, \tag{3.6}$$

$$w(k_i^*, y(k_i^*+\eta,m_0,y_0) - x(k_i^*,n_0,x_0)) = \gamma, \tag{3.7}$$

and

$$\gamma/2 \le w(k, y(k+\eta,m_0,y_0) - x(k,n_0,x_0)) \le \gamma, \quad k \in [k_i,k_i^*], \quad i = 1,2,\ldots$$

or

$$w(k_i, y(k_i+\eta,m_0,y_0) - x(k_i,n_0,x_0)) = \gamma,$$
$$w(k_i^*, y(k_i^*+\eta,m_0,y_0) - x(k_i^*,n_0,x_0)) = \gamma/2,$$

and

$$\gamma/2 \le w(k, y(k+\eta,m_0,y_0) - x(k,n_0,x_0)) \le \gamma, \quad k \in [k_i^*,k_i], \quad i = 1,2,\ldots.$$

For similarity of these two cases, we can only consider the first one. Since $\Delta w(n,z) \le M$, we have

$$\sum_{k_i}^{k_i^*} \Delta w(n,z) \le M(k_i^* - k_i).$$

Combining (3.6) with (3.7), we obtain $k_i^* - k_i \ge \gamma/2M$ for each i.

Let

$$G(n,z,\eta) = V_1(n,z,\eta) + \sum_{n_0}^{n} p(w(n,z(n))).$$

As a result of assumption (3.5), we have

$$\Delta G(n,z,\eta) \le \Delta V_1(n,z,\eta) + p(w(n,z(n))) \le g_1(n,V_1(n,z),\eta) \le g_1(n,G(n,z),\eta), \quad n \in [\tau_r,\tau_{r+1}).$$

By Theorem 3.1, we get

$$G(n,z,\eta) \le r_1(n,n_0,V_1(n_0,y_0-x_0),\eta), \quad n \ge n_0,$$

where $r_1(n,n_0,V_1(n_0,y_0-x_0),\eta)$ is the maximal solution of the comparison system (2.3) through $(n_0,V_1(n_0,y_0-x_0))$. Then we can get for sufficiently large n

$$\begin{aligned}
0 &\leq V_1(n,y(n+\eta,m_0,y_0)-x(n,n_0,x_0)) \\
&\leq r_1(n,n_0,V_1(n_0,y_0-x_0),\eta)) - \sum_{n_0}^{n} p(w(n,z(n))) \\
&\leq a_1(n_0,\lambda) - c(\gamma/2) \sum_{1\leq i\leq n} (k_i^* - k_i) \\
&\leq a_1(n_0,\lambda) - c(\gamma/2)n\gamma/2M < 0.
\end{aligned}$$

This is a contradiction, hence we can get

$$\lim_{n\to\infty} w(n,y(n+\eta,m_0,y_0)-x(n,n_0,x_0)) = 0,$$

when $\|y_0-x_0\| < \lambda_1$, $\eta \leq \sigma_1$. That is to say

$$\lim_{n\to\infty} y(n+\eta,m_0,y_0)-x(n,n_0,x_0) = 0,$$

when $\|y_0-x_0\| < \lambda_1$, $\eta \leq \sigma_1$. So the proof is completed. ∎

Corollary 3.2. *Under the assumptions of Theorem 3.3, Let $a_1(n,\|x\|) \in \mathscr{CK}$ be substituted for $a_1(\|x\|) \in \mathscr{K}$. Then the solution $x(n) = x(n,n_0,x_0)$ of* (2.1) *is uniformly practically asymptotically stable.*

Theorem 3.4. *Assume that*

(i) *$V \in C[\mathbb{N}_{n_0}^+ \times S(\rho),\mathbb{R}^+]$, $V(n,x)$ is locally Lipschitz in x, for $(n,x) \in \mathbb{N}_{n_0}^+ \times S(\rho)$,*

$$b(n,\|x\|) \leq V(n,x) \leq a(n,\|x\|),$$

a, $b \in \mathscr{CK}$ and

$$\Delta V(n,z,\eta) \leq g_1(n,V(n,z),\eta), \tag{3.8}$$

where $\tau_r \leq n < \tau_{r+1}$, $r = 0,1,2,\ldots$ and $g_1 \in C[\mathbb{N}_{n_0}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\mathbb{R}]$;

(ii) *$r(n,n_0,u_0)$ is any solution of the comparison system* (2.2).

Then the practical stability properties of the solution of comparison system (2.2) *implies the corresponding practical stability properties of system* (2.1)*, respectively.*

Proof. Firstly, we prove practical stability of system (2.1). Suppose that the solution of (2.2) is practically stable, i.e., for given (λ,A), with $0 < \lambda < A$, there exists a $\sigma_1 = \sigma_1(\lambda,A) > 0$ such that $u_0 < a(\lambda)$, $\eta \leq \sigma_1$ implies

$$u(n,n_0,u_0,\eta) < b(A),\ n \geq n_0. \tag{3.9}$$

We claim that the solutions of (2.1) is practically stable. If not, there exists solutions $x(n,n_0,x_0)$ and $y(n,m_0,y_0)$ of (2.1). Although $\|y_0-x_0\| \leq \lambda$, $\eta \leq \sigma_1$ but there exists a $k_1 > n_0$ such that

$$\|y(k_1+\eta,m_0,y_0)-x(k_1,n_0,x_0)\| = A, \tag{3.10}$$

$$\|y(k+\eta,m_0,y_0)-x(k,n_0,x_0)\| < A, \quad n_0 \leq k < k_1. \tag{3.11}$$

Let $u_0 = a(n_0,\|y_0-x_0\|)$. Then by Theorem 3.1 we obtain

$$V(n,z(n)) \leq r(n,n_0,u_0,\eta), \quad n_0 \leq k < k_1. \tag{3.12}$$

From $(3.10)-(3.14)$, we have

$$b(A)=b(\|y(k_1+\eta,m_0,y_0)-x(k_1,n_0,x_0)\|)\le V(k_1,y(k_1+\eta,m_0,y_0)-x(k_1,n_0,x_0))$$
$$\le r(k_1,n_0,a(\|y_0-x_0\|),\eta)<b(A).$$

This is a contradiction, i.e., the solution of (2.1) is practically stable.

Next, we shall prove that the system (2.1) is strongly practically stable for $(\lambda,A,B,N)>0$. To do this, we need to prove only the practical quasi-stability of system (2.1). Suppose that (2.2) is strongly practically stable for $(a(\lambda),b(B),N)$, i.e.,

$$u_0<a(\lambda)\quad \text{implies}\quad u(n,n_0,u_0)<b(B),\ n\ge n_0+N.$$

We claim that the solution of system (2.1) is practical quasi-stability. If it is not true, there exists solutions $x(n,n_0,x_0)$ and $y(n,m_0,y_0)$ of (2.1), although $\|y_0-x_0\|\le\lambda$, $\eta\le\sigma_1$, but there exists a $k_2>n_0+N$, such that

$$\|y(k_2+\eta,m_0,y_0)-x(k_2,n_0,x_0)\|=A,$$
$$\|y(k+\eta,m_0,y_0)-x(k,n_0,x_0)\|<A,\quad n_0+N\le k<k_2.$$

From the foregoing argument, we get for all $n\ge n_0+N$,

$$b(B)=b(\|y(k_2+\eta,m_0,y_0)-x(k_2,n_0,x_0)\|)\le V(k_2,y(k_2+\eta,m_0,y_0)-x(k_2,n_0,x_0))$$
$$\le r(k_2,n_0,a(\|y_0-x_0\|),\eta)<b(B),$$

which is a contradiction, hence the system (2.1) is practical quasi-stability.

At last, suppose that system (2.2) is practically asymptotically stable, i.e., for any given ε, there exists a $T=T(n_0,\varepsilon)$, such that

$$0<u_0<\lambda\quad \text{implies}\quad u(n,n_0,u_0,\eta)<b(\varepsilon),\quad n\ge n_0+T.$$

In order to prove the practical asymptotical stability of (2.1), we only need to prove that it is attractive. Setting $y(n+\eta,m_0,y_0)-x(n,n_0,x_0)<\lambda$, then it follows from condition (i) and (3.12) that

$$b(\|y(n+\eta,m_0,y_0)-x(n,n_0,x_0)\|)\le V(n,y(n+\eta,m_0,y_0)-x(n,n_0,x_0))$$
$$\le r(n,n_0,u_0,\eta)<b(\varepsilon),\quad n\ge n_0+T.$$

Since $b\in\mathscr{K}$, then

$$\|y(n+\eta,m_0,y_0)-x(n,n_0,x_0)\|<\varepsilon,\quad n\ge n_0+T,$$

that is to say, (2.1) is practically asymptotically stable, and our proof is completed. ∎

4. Conclusion

In this paper, we develop some new comparison principles for discrete hybrid system with initial time difference, and generalized the concepts of practical stability of common discrete hybrid systems with unchanged initial time to the discrete hybrid system which has different initial values for each solution. Meanwhile, several stability criteria relative to initial time difference are obtained.

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