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On Dynamics of Lotka-Volterra Type Operators

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Abstract. In the present paper, we study dynamics of Lotka-Volterra (LV) type operators defined in finite dimensional simplex. We introduce a new class of LV-type operators, called *M*LV type. Some concrete examples are also provided. We prove that trajectories of such kind of operators converge, and moreover, we find an estimation for fixed points of the introduced operators.

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1. Introduction

It is known that Lotka-Volterra (LV) systems typically model the time evolution of conflicting species in biology [1, 6, 16]. On the other hand, the use of LV discrete-time systems is a well-known subject of applied mathematics [4, 8]. They were first introduced in a biomathematical context by Moran [11], and later popularized in [9, 10]. Since then, LV systems have proved to be a rich source of analysis for the investigation of dynamical properties and modelling in different domains (see for example, [5, 7]). Typically in all these applications, the LV systems are taken quadratic. It is natural to investigate non-quadratic LV systems. It is well known that even for one species the dynamics may be extremely complex, and it may be very difficult to predict the detailed asymptotic behavior. In [4, 10] it was introduced, generalization of the LV systems, to model the interaction among biochemical populations. In [14] it is established new sufficient conditions for global asymptotic stability of the positive equilibrium in some LV-type discrete models. The mentioned papers show importance the study of limiting behavior of discrete LV type operators. One the other hand, one of the most important questions from a biological point of view concerns the conditions under which long term survival of all the species is assured. Therefore, in the paper our aim is to provide some sufficient conditions for the stability of LV type operators. Namely, we introduce a new class of LV type operators, called *M*LV type. We then prove that trajectories of such kind of operators converge, and moreover, we find an estimation for fixed points of the introduced operators.

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2. Preliminaries

In this section we are going to provide some necessary notions and definitions.

Let

$$S^{m-1} = \left\{ x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : \sum_{k=1}^{m} x_k = 1, x_k \geq 0 \right\}$$

be the $(m-1)$-dimensional simplex. One can see that the points $e_k = (\delta_{1k}, \delta_{2k}, \dots, \delta_{mk})$ are the extremal points of the simplex S^{m-1}, where δ_{ik} is the Kronecker's symbol.

Let $I = \{1, 2, \dots, m\}$ and α be an arbitrary subset of I. The set $\Gamma_\alpha = \{x \in S^{m-1} : x_k = 0,\ k \notin \alpha\}$ is called *a face* of the simplex. *A relatively interior* $ri\Gamma_\alpha$ of the face Γ_α is defined by $ri\Gamma_\alpha = \{x \in \Gamma_\alpha : x_k > 0, k \in \alpha\}$.

Given a mapping $\mathbf{f} : x \in S^{m-1} \to (f_1(x), f_2(x), \dots, f_m(x)) \in \mathbb{R}^m$ in what follows, we are interested in the following operator defined by

$$(Vx)_k = x_k(1 + f_k(x)), \quad k = \overline{1,m}\ \ x \in S^{m-1}. \tag{2.1}$$

Proposition 2.1. *Let V be an operator given by* (2.1). *The following conditions are equivalent:*

(i) *The operator V is continuous in S^{m-1} and $V(S^{m-1}) \subset S^{m-1}$. Moreover, $V(ri\Gamma_\alpha) \subset ri\Gamma_\alpha$ for all $\alpha \subset I$.*

(ii) *The mapping $\mathbf{f} \equiv (f_1, f_2, \dots, f_m) : S^{m-1} \to \mathbb{R}^m$ satisfies the following conditions:*
   - 1^0 $\mathbf{f}$ *is continuous in S^{m-1};*
   - 2^0 *for every $x \in S^{m-1}$ one has $f_k(x) \geq -1$, for all $k = \overline{1,m}$;*
   - 3^0 *for every $x \in S^{m-1}$ one has $\sum_{k=1}^{m} x_k f_k(x) = 0$;*
   - 4^0 *for every $\alpha \subset I$ one holds $f_k(x) > -1$ for all $x \in ri\Gamma_\alpha$ and $k \in \alpha$.*

Proof. (i)⇒(ii). The continuity of V implies 1^0. Take $x \in S^{m-1}$ and it yields that (a) $(Vx)_k \geq 0$; (b) $\sum_{k=1}^{m} (Vx)_k = 1$. Hence, from (a) it follows that $x_k(1 + f_k(x)) \geq 0$ which implies 2^0. From (b) one has

$$\sum_{k=1}^{m} x_k + \sum_{k=1}^{m} x_k f_k(x) = 1$$

which immediately yields 3^0.

Let $x \in ri\Gamma_\alpha$, then $Vx \in ri\Gamma_\alpha$ which with (2.1) and $x_k > 0$ for all $k \in \alpha$ implies that $f_k(x) > -1$ for all $k \in \alpha$ this means 4^0.

The implication (ii)⇒(i) is evident. ∎

We say that an operator V defined by (2.1) is *Lotka-Volterra (LV) type* if one of the conditions of Proposition 2.1 is satisfied. The corresponding mapping $\mathbf{f}$ is called *generating mapping* for V. From Proposition 2.1 we immediately infer that any LV type operator maps the simplex S^{m-1} into itself. By $\mathscr{V}$ we denote the set of all LV type operators. Note that for $f_k(x) = \exp\left\{r_k - \sum_{j=1}^{n} a_{kj} x_j\right\}$ the mapping (2.1) has been investigated in [4, 10]. Other particular cases were studied in [2, 13] (see also [3] for review). Note that some biological interpretations of LV-type operators have been provided in [12]. Some examples are also given there.

Given $x^0 \in S^{m-1}$, then the sequence $\left\{x^0, Vx^0, V^2x^0, \dots, V^nx^0, \dots\right\}$ is called *a trajectory of V starting from the point x^0*, where $V^{n+1}x^0 = V(V^nx^0), n = 1, 2, \dots$ By $\omega(x^0)$ we denote the set of all limiting points of such a trajectory.

A point $x \in S^{m-1}$ is called *fixed* if $Vx = x$ and by $Fix(V)$ we denote the set of all fixed points of V. A point $x \in S^{m-1}$ is called *r-periodic* if $V^r x = x$ and $V^i x \neq x$ for all $i \in \overline{1, r-1}$.

3. $M-$Lotka-Volterra type operators

In this section we introduce a class of LV type operators, called M-LV, and study their asymptotic behavior.

Given $x \in S^{m-1}$ put

$$M(x) = \left\{ i \in I : x_i = \max_{k=\overline{1,m}} x_k \right\},$$

here as before $I = \{1, \dots, m\}$.

We say that an LV type operator given by (2.1) is called M_1-*Lotka-Volterra* (for shortness M_1LV) (resp. M_0-*Lotka-Volterra* (M_0LV)) if for each $x \in S^{m-1}$ and for all $k \in M(x)$, $j = \overline{1,m}$ the functional $\varphi(x) = x_k - x_j$ is increasing (resp. decreasing and $\varphi(V^n(x)) \geq 0$ for all $n \geq 0$) along the trajectory of V starting from the point x, i.e. $\varphi(V^n x) \leq \varphi(V^{n+1} x)$, $n \geq 0$ (resp. $\varphi(V^n x) \geq \varphi(V^{n+1} x)$, $n \geq 0$).

By $\mathcal{V}\mathcal{M}_1$ and $\mathcal{V}\mathcal{M}_0$ we denote the sets of all M_1LV and M_0LV type operators, respectively.

Remark 3.1. It immediately follows from the definition that

$$\mathcal{V}\mathcal{M}_1 \cap \mathcal{V}\mathcal{M}_0 = \{Id\},$$

where $Id : S^{m-1} \to S^{m-1}$ is an identity mapping.

Proposition 3.1. *Let V_0 and V_1 be M_1LV (resp. M_0LV) type operators. Then the following conditions are satisfied:*

(i) *The operator $V_1 \circ V_0$ is M_1LV (res. M_0LV) type.*
(ii) *For each $\lambda \in [0,1]$ the operator $(1-\lambda)V_0 + \lambda V_1$ is M_1LV (resp. M_0LV) type.*

Proof. Without loss of generality we may suppose that the operators V_0 and V_1 are M_1LV type. Then for each $x \in S^{m-1}$ and for all $k \in M(x)$, $j = \overline{1,m}$ we have

$$x_k - x_j \leq (V_0 x)_k - (V_0 x)_j \leq (V_1(V_0 x))_k - (V_1(V_0 x))_j$$

which implies that $V_1 \circ V_2 \in \mathcal{V}\mathcal{M}_1$.

Now for all $\lambda \in [0,1]$ one finds

$$\begin{aligned} x_k - x_j &= (1-\lambda)(x_k - x_j) + \lambda(x_k - x_j) \\ &\leq (1-\lambda)((V_0 x)_k - (V_0 x)_j) + \lambda((V_1 x)_k - (V_1 x)_j) \\ &= ((1-\lambda)V_0 x + \lambda V_1 x)_k - ((1-\lambda)V_0 x + \lambda V_1 x)_j, \end{aligned}$$

that yields the required assertion.

By the similar argument one can prove the statements for the case of M_0LV type operators. ∎

Corollary 3.1. *The sets $\mathcal{V}\mathcal{M}_1$ and $\mathcal{V}\mathcal{M}_0$ are convex.*

Let us provide some examples of M_1LV and M_0LV type operators, respectively.

Example 3.1. Let us consider an operator $V_{\varepsilon,\ell}$ defined by

$$(V_{\varepsilon,\ell}x)_k = x_k\left(1+\varepsilon\left(x_k^\ell - \sum_{i=1}^m x_i^{\ell+1}\right)\right), \quad k=\overline{1,m} \tag{3.1}$$

where $0<\varepsilon\le 1$ and $\ell\in\mathbb{N}$. One can show that $V_{\varepsilon,\ell}$ is an M_1LV type operator.

Example 3.2. Let us consider an operator $W_{\varepsilon,\ell}: S^{m-1}\to S^{m-1}$ defined by

$$(W_{\varepsilon,\ell}x)_k = x_k\left(1+\varepsilon\left(\sum_{i=1}^m x_i^{\ell+1} - x_k^\ell\right)\right), \quad k=\overline{1,m} \tag{3.2}$$

where $0<\varepsilon\le 1$ and $\ell\in\mathbb{N}$. Then $W_{\varepsilon,\ell}$ is an M_0LV type operator.

Observe that by means of the provided examples and Proposition 3.1 one can construct lots of nontrivial examples of M_1LV and M_0LV type operators, respectively. To study stability properties of M_0LV and M_1LV type operators we need the following auxiliary result.

Lemma 3.1. *If for a sequence $\{x^{(n)}\}_{n=0}^\infty\subset S^{m-1}$ and some $k\in I$ the limits*

$$\lim_{n\to\infty}\left(x_k^{(n)}-x_j^{(n)}\right), \quad \forall\, j=\overline{1,m}, \tag{3.3}$$

exist, where $x^{(n)}=\left(x_1^{(n)},x_2^{(n)},\dots,x_m^{(n)}\right)$, then the sequence $\left\{x^{(n)}\right\}_{n=0}^\infty$ converges.

Proof. The convergence of the sequences $\left\{x_k^{(n)}-x_j^{(n)}\right\}_{n=0}^\infty$, for all $j=\overline{1,m}$, implies the convergence of a sequence $\left\{\sum_{j=1}^m\left(x_k^{(n)}-x_j^{(n)}\right)\right\}_{n=0}^\infty$. Then the equality

$$mx_k^{(n)} = \sum_{j=1}^m\left(x_k^{(n)}-x_j^{(n)}\right)+\sum_{j=1}^m x_j^{(n)} = \sum_{j=1}^m\left(x_k^{(n)}-x_j^{(n)}\right)+1.$$

implies the convergence of the sequence $\left\{x_k^{(n)}\right\}_{n=0}^\infty$. From (3.3) we obtain the convergence of $\left\{x_j^{(n)}\right\}_{n=0}^\infty$, for all $j=\overline{1,m}$ which yields the required assertion. ∎

Now we are ready to prove stability property of M_0LV and M_1LV type operators.

Theorem 3.1. *Let V be a M_1LV (resp. M_0LV) type operator. Then the trajectory $\{V^n x\}_{n=0}^\infty$ converges for every $x\in S^{m-1}$, i.e. $\omega(x)$ is a single point and $\omega(x)\in Fix(V)$.*

Proof. Let V be a M_1LV type operator. Then for some $k\in M(x)$ and all $j=\overline{1,m}$ we have

$$x_k - x_j \le (Vx)_k-(Vx)_j\le\cdots\le (V^nx)_k-(V^nx)_j\le\cdots\le 1$$

Therefore the sequence $\left\{(V^nx)_k-(V^nx)_j\right\}_{n=0}^\infty$ converges. It follows from Lemma 3.1 that the trajectory $\{V^nx\}_{n=0}^\infty$ converges.

By the similar way the statement can be proved for a M_0LV case. ∎

From this theorem we conclude that M_0LV and M_1LV type operators do not have periodic points.

Lemma 3.2. *Let V be a M_1LV type operator. Then for every $x\in S^{m-1}$ and for all $n\in\mathbb{N}$ one has*

$$M(V^nx)=M(x).$$

Moreover, if there exists $\lim_{n\to\infty}V^nx=x^*$ then $M(x^*)=M(x)$.

Proof. Take any $x \in S^{m-1}$, and assume that $k \in M(x)$. Since V is M_1LV type then for every $j = \overline{1,m}$ one has

$$0 \leq x_k - x_j \leq (Vx)_k - (Vx)_j \leq \cdots \leq (V^n x)_k - (V^n x)_j \leq \cdots \tag{3.4}$$

which implies that $k \in M(V^n x)$ i.e.

$$M(x) \subset M(V^n x). \tag{3.5}$$

Now let us show $M(V^n x) \subset M(x)$. Assume from the contrary, i.e. there is $k_0 \in M(V^n x)$ that $k_0 \notin M(x)$. Take any $k_1 \in M(x)$, then from (3.5) we infer that $k_1 \in M(V^n x)$ which means $(V^n x)_{k_1} - (V^n x)_{k_0} = 0$. On the other hand, from (3.4), (3.5) one has

$$0 < x_{k_1} - x_{k_0} \leq (Vx)_{k_1} - (Vx)_{k_0} \leq \cdots \leq (V^n x)_{k_1} - (V^n x)_{k_0} = 0$$

which is a contradiction, hence $M(V^n x) \subset M(x)$. Thus, we have $M(V^n x) = M(x)$, for any $n \in \mathbb{N}$.

Now assume that $\{V^n x\}_{n=0}^{\infty}$ converges to x^*. Then from (3.4) one has

$$x_k - x_j \leq x_k^* - x_j^* \tag{3.6}$$

for all $k \in M(x)$ and $j = \overline{1,m}$. Then (3.6) yields that

$$M(x) \subset M(x^*). \tag{3.7}$$

Now let us establish $M(x^*) \subset M(x)$. Again, assume from the contrary, i.e. there is $k_0 \in M(x^*)$ that $k_0 \notin M(x)$. Then we use the same argument as above, i.e. for any $k_1 \in M(x)$ it follows from (3.6), (3.7) that

$$0 < x_{k_1} - x_{k_0} \leq x_{k_1}^* - x_{k_0}^* = 0.$$

Again, the last contradiction shows that $M(x^*) \subset M(x)$, which yields the required equality. ∎

Note that in general a similar result as Lemma 3.2 is not satisfied for M_0LV type operators.

Theorem 3.2. *Let V be an M_1LV type operator. Then the centers of all faces of the simplex are fixed points of V and*

$$|Fix(V)| \geq 2^m - 1, \tag{3.8}$$

here as before $|A|$ stands for the cardinality of a set A.

Proof. Let V be an M_1LV type operator and $x^0 = \left(x_1^0, \ldots, x_m^0\right)$ be the center of the face Γ_α, i.e.

$$x_k^0 = \begin{cases} \frac{1}{|\alpha|}, & k \in \alpha \\ 0, & k \notin \alpha. \end{cases}$$

where $\alpha \subset I$.

It is clear that $M(x^0) = \alpha$. According to Theorem 3.1 the trajectory $\{V^n x^0\}_{n=0}^{\infty}$ converges to some point x^* which is a fixed point of V. Since the face Γ_α is invariant w.r.t. V then $x^* \in \Gamma_\alpha$. According to Lemma 3.2 we have

$$M(x^*) = M(x^0) = \alpha. \tag{3.9}$$

which means $x^* \in ri\Gamma_\alpha$. On the other hand, it follows from (3.9) that all non null coordinates of x^* are maximal, it means $x^* = x^0$. So, x^0 is a fixed point of V.

One can see that the number of faces of the simplex is

$$\sum_{i=1}^{m} C_m^i = 2^m - 1,$$

so one gets (3.8). ∎

We note that the operator $V_{\varepsilon,\ell}$ given by (3.1) was first considered in [15], in a particular case, when $\varepsilon = 1$, $l = 1$. There, it was established that for every $x^0 \in S^{m-1}$ the trajectory $\{V_{1,1}^n x^0\}_{n=0}^{\infty}$ starting from any $x^0 \in S^{m-1}$ always converges. Since the operator (3.1) is also M_1LV type then according to Theorem 3.1 for every $x^0 \in S^{m-1}$ the trajectory $\{V_{\varepsilon,\ell}^n x^0\}_{n=0}^{\infty}$ always converges for all $0 < \varepsilon \leq 1$ and $\ell \in \mathbb{N}$.

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