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A Non-Vanishing Theorem for Local Cohomology Modules

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Abstract. Assume that $(R,\mathfrak{m})$ is a local Noetherian ring and $\mathfrak{a}$ is an ideal of R. In this paper we introduce a new class of R-modules denoted by weakly finite modules that is a generalization of finitely generated modules and containing the class of Big Cohen-Macaulay modules and $\mathfrak{a}$-cofinite modules. We improve the non-vanishing theorem due to Grothendieck for weakly finite modules. Finally we define the notion $\operatorname{depth}_R M$ and we prove that if M is a weakly finite R-module and $\mathrm{H}^i_{\mathfrak{m}}(M) \neq 0$ for some i, then $\operatorname{depth}_R(M) \leq i \leq \dim M$.

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1. Introduction

Throughout this note the ring R is commutative Noetherian ring with non-zero identity. Let $\mathfrak{a}$ be an ideal of R and let M be a non-zero R-module. Since M is not necessarily finitely generated, by $\dim M$, we mean $\sup\{\dim(R/\mathfrak{p}) | \mathfrak{p} \in \operatorname{Supp}(M)\}$.

Also by $\mathrm{H}^i_{\mathfrak{a}}(M)$, we mean $\varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$, the local cohomology module of M with respect to the ideal $\mathfrak{a}$. Local Cohomology was introduced by Grothendieck and many people have worked on understanding their structure, (non)-vanishing and finiteness properties. For example, Grothendieck's non-vanishing theorem is one of the important theorems in local cohomology that says $\mathrm{H}^{\dim M}_{\mathfrak{m}}(M)$ is non-zero for any finitely generated module M over Noetherian local ring $(R,\mathfrak{m})$. The finiteness assumption on M has crucial rule in the proof of this result, see [1, Theorem 6.1.4]. The study of Grothendieck's non-vanishing theorem for not necessarily finitely generated modules began by the work of Hochster. Let R be an excellent local domain of prime characteristic and let R^+ be the absolute integral closure of R. Hochster showed that $\mathrm{H}^{\dim R}_{\mathfrak{m}}(R^+)$ is non-zero in the case that R is an excellent domain of prime characteristic, see [6, Theorem 6.1]. It is worth to recall from [7, Theorem 5.5] that R^+ is balanced big Cohen-Macaulay in the case that it is excellent and of prime characteristic.

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Also, Sharp proved that Grothendieck's non-vanishing theorem is true for a balanced big Cohen-Macaulay algebra, see [13, Theorem 3.2]. Let us recall from [13, Lemma 2.1], some properties of balanced big Cohen-Macaulay modules.

(1) If M is a balanced big Cohen-Macaulay module, then $\mathrm{Hom}(R/\mathfrak{a},M)$ is finitely generated for some non-zero ideal of R.
(2) If M is a balanced big Cohen-Macaulay module and x is a regular element, then so is M/xM.
(3) If M is a balanced big Cohen-Macaulay module, then $|\mathrm{Ass}(M)| < \infty$, and
(4) If M is a balanced big Cohen-Macaulay module, then $M/\Gamma_{\mathfrak{m}}(M)$ is a balanced big Cohen-Macaulay module too.

In the next section, firstly we define the notion of weakly finite modules that they have the properties similar to big Cohen-Macaulay modules and we show that this class contains finitely generated modules and $\mathfrak{a}$-cofinite modules. Furthermore, we prove that the Grothendieck's non-vanishing theorem is true for this set of modules.

The following is our main result in this paper.

Theorem 1.1. *Let $(R,\mathfrak{m})$ be a local ring and M a non-zero weakly finite R-module. If* $\dim M = n$*, then* $\mathrm{H}^n_{\mathfrak{m}}(M) \neq 0$.

This improves Grothendieck's non-vanishing theorem. Also we define $\mathrm{depth}_R(M)$ similar to $\mathrm{depth}\, R$ and we improve [1, Corollary 6.2.8] for weakly finite modules:

Theorem 1.2. *Let $(R,\mathfrak{m})$ be a local ring and let M be a weakly finite R-module such that $\mathfrak{a}M \neq M$. Then* $\mathrm{depth}_R(M)$ *is the least integer i such that* $\mathrm{H}^i_{\mathfrak{m}}(M) \neq 0$.

2. Grothendieck's non-vanishing Theorem

In this section, we assume that $(R,\mathfrak{m})$ is a local ring. Now we introduce the class of weakly finite modules.

Definition 2.1. *Let $\mathscr{S}$ be the largest class of R-modules satisfying the following four properties:*

(1) *If $M \in \mathscr{S}$, then* $\mathrm{Hom}(R/\mathfrak{m},M)$ *is finitely generated.*
(2) *If M is a non-zero element of $\mathscr{S}$ and x is a regular element, Then $M/xM \in \mathscr{S}$ is non-zero and* $\dim M/xM = \dim M - 1$.
(3) *If $M \in \mathscr{S}$, then $|\mathrm{Ass}(M)| < \infty$, and*
(4) *If $M \in \mathscr{S}$, then $M/\Gamma_{\mathfrak{m}}(M) \in \mathscr{S}$.*

We say that an R-module is weakly finite, if it belongs to $\mathscr{S}$.

Clearly non-zero finitely generated modules and balanced big Cohen-Macaulay modules are weakly finite.

Let $\mathfrak{a}$ be an ideal of $(R,\mathfrak{m})$. Hartshorne defined a (not necessarily finitely generated) module N to be $\mathfrak{a}$-cofinite if the support of N is contained in the variety of $\mathfrak{a}$, and in addition $\mathrm{Ext}^i_R(R/\mathfrak{a},N)$ is a finitely generated R-module for all i. Obviously the set of $\mathfrak{a}$-cofinite modules contains finitely generated modules. Recall from [10] that Grothendieck's non-vanishing theorem holds for the class of $\mathfrak{a}$-cofinite modules. In Lemma 2.1, we will show that $\mathfrak{a}$-cofinite modules are weakly finite too.

Lemma 2.1. *Let* $(R,\mathfrak{m})$ *be a local ring and let* M *be a non-zero* R*-module of dimension* $n > 0$*. If there is an ideal* $\mathfrak{a}$ *of* R *such that* $\mathrm{Supp}(M) \subseteq \mathrm{Var}(\mathfrak{a})$ *and* $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is finitely generated for all* i*, then* M *is weakly finite.*

Proof. We show that $\mathfrak{a}$-cofinite modules satisfy the axiom of weakly finite modules. The first axiom is true because by applying the short exact sequence

$$0 \to \frac{\mathfrak{m}}{\mathfrak{a}} \to \frac{R}{\mathfrak{a}} \to \frac{R}{\mathfrak{m}} \to 0$$

we deduce the long exact sequence $0 \to \mathrm{Hom}(R/\mathfrak{m}, M) \to \mathrm{Hom}(R/\mathfrak{a}, M) \to \cdots$. Since by assumption $\mathrm{Hom}(R/\mathfrak{a}, M)$ is finitely generated and R is Noetherian, therefore $\mathrm{Hom}(R/\mathfrak{m}, M)$ will be finitely generated. Now, we show that if $r \in \mathfrak{m}$ is a non-zero divisor element of $\mathfrak{m}$, then M/rM is non-zero of dimension $n-1$. By applying $\mathrm{Hom}(R/\mathfrak{a}, -)$ to short exact sequence $0 \to M \to M \to M/rM \to 0$, we have the long exact sequence

$$0 \to \mathrm{Hom}(R/\mathfrak{a}, M) \to \mathrm{Hom}(R/\mathfrak{a}, M) \to \mathrm{Hom}(R/\mathfrak{a}, M/rM). \tag{2.1}$$

If $M = rM$, then $\mathrm{Hom}(R/\mathfrak{a}, M) \cong r\mathrm{Hom}(R/\mathfrak{a}, M)$. By Nakayama's Lemma, $\mathrm{Hom}(R/\mathfrak{a}, M) = 0$ and this is a contradiction because

$$\mathrm{Ass}(\mathrm{Hom}(R/\mathfrak{a}, M)) = \mathrm{Ass}(M) \cap \mathrm{Var}(\mathfrak{a}) \neq \emptyset.$$

On the other hand, using the following inequalities:

$$\begin{aligned} n-1 &= \dim\left(\mathrm{Hom}_R(R/\mathfrak{a}, M)/r\mathrm{Hom}_R(R/\mathfrak{a}, M)\right) \\ &\leq \dim\left(\mathrm{Hom}_R(R/\mathfrak{a}, M/rM)\right) \\ &\leq \dim\left(M/rM\right) \leq n-1, \end{aligned}$$

we have $\dim M/rM = n-1$. Clearly, $\mathrm{Supp}(M/rM) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M/rM)$ is finitely generated for all i. Now, to show that $\mathrm{Ass}(M)$ is finite, we have

$$\mathrm{Ass}\left(\mathrm{Hom}(R/\mathfrak{a}, M)\right) = \mathrm{Ass}(M) \cap \mathrm{Var}(\mathfrak{a}).$$

Since $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$ we conclude that $\mathrm{Ass}\left(\mathrm{Hom}(R/\mathfrak{a}, M)\right) = \mathrm{Ass}(M)$. Now by assumption $\mathrm{Hom}(R/\mathfrak{a}, M)$ has finitely many associated primes and therefore $|\mathrm{Ass}(M)| < \infty$.

By using the exact sequence (2.1), in combination with the fact that $\mathrm{Ext}^i(R/\mathfrak{a}, M)$ is finitely generated, we get that M/rM is weakly finite. Respectively, using the short exact sequence $0 \to \Gamma_{\mathfrak{m}}(M) \to M \to M/\Gamma_{\mathfrak{m}}(M) \to 0$, we conclude that $M/\Gamma_{\mathfrak{m}}(M)$ is weakly finite too. Note that $\Gamma_{\mathfrak{m}}(M)$ is $\mathfrak{a}$-cofinite, see[11, Corollary 1.8] ∎

In the next lemma, we prove that similar to finitely generated modules, if M is weakly finite, then the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(M)$ are not finitely generated when $i > 0$.

Lemma 2.2. *Let* n *be an integer and* M *be a weakly finite* R*-module. If* $\mathrm{H}^n_{\mathfrak{m}}(M)$ *is non-zero and finitely generated, then* $n = 0$*.*

Proof. By definition of weakly finite R-modules, one has $|\mathrm{Ass}(M)| < \infty$. Now, the proof follows by straight forward modification of [1, Exercise 6.1.6]. ∎

In the main theorem, we need to have Artinianness of $\mathrm{H}^i_{\mathfrak{m}}(M)$ to use the concept of attached primes. In the following theorem we prove that if M is weakly finite, therefore $\mathrm{H}^i_{\mathfrak{m}}(M)$ is Artinian.

Theorem 2.1. *Let* $(R,\mathfrak{m})$ *be a local ring and let* M *be a weakly finite* R*-module. Then* $\mathrm{H}^i_{\mathfrak{m}}(M)$ *is Artinian for all* $i \in \mathbb{N}_0$*.*

Proof. We use induction on i. Assume that $i=0$. Obviously $\Gamma_{\mathfrak{m}}(M)$ is $\mathfrak{a}$-torsion. Also $(0:_{\Gamma_{\mathfrak{m}}(M)}\mathfrak{a})$ is Artinian, because by applying the functor $\mathrm{Hom}_R(R/\mathfrak{a},-)$ to the sequence $0\to\Gamma_{\mathfrak{m}}(M)\to M$, we deduce the monomorphism $\mathrm{Hom}_R(R/\mathfrak{a},\Gamma_{\mathfrak{m}}(M))\hookrightarrow\mathrm{Hom}_R(R/\mathfrak{a},M)$. By assumption $\mathrm{Hom}(R/\mathfrak{a},M)$ is finitely generated and therefore by Noetherianness of R, $\mathrm{Hom}_R(R/\mathfrak{a},\Gamma_{\mathfrak{m}}(M))$ will be finitely generated.

Since $(0:_{\Gamma_{\mathfrak{m}}(M)}\mathfrak{a})\simeq\mathrm{Hom}_R(R/\mathfrak{a},\Gamma_{\mathfrak{m}}(M))$, then $(0:_{\Gamma_{\mathfrak{m}}(M)}\mathfrak{a})$ is Artinian. Now, by Melkersson's criterion [1, Theorem 7.1.2], $\Gamma_{\mathfrak{m}}(M)$ is Artinian. We assume inductively that $i>0$ and we have shown that $\mathrm{H}^j_{\mathfrak{m}}(N)$ is Artinian for every weakly finite R-module N and $j<i$. It is well known that $\mathrm{H}^i_{\mathfrak{m}}(M)\cong\mathrm{H}^i_{\mathfrak{m}}(M/\Gamma_{\mathfrak{m}}(M))$ by [1, Corollary 2.1.7]. Therefore we can assume that M is an $\mathfrak{m}$-torsion-free R-module. By the fact that $|\mathrm{Ass}(M)|<\infty$, we deduce that $\mathfrak{m}$ contains an element r which is a non-zero divisor on M. Using exact sequence

$$0\to M\to M\to M/rM\to 0$$

we arrive to the exact sequence

$$\mathrm{H}^{i-1}_{\mathfrak{m}}(M/rM)\to\mathrm{H}^i_{\mathfrak{m}}(M)\to\mathrm{H}^i_{\mathfrak{m}}(M)$$

of local cohomology modules. Since M/rM is weakly finite, it follows from the inductive hypothesis that $\mathrm{H}^{i-1}_{\mathfrak{m}}(M/rM)$ is Artinian. So that $(0:_{\mathrm{H}^i_{\mathfrak{m}}(M)}r)$ is Artinian. Since $\mathrm{H}^i_{\mathfrak{m}}(M)$ is (r)-torsion, it follows from [1, Theorem 7.1.2] $\mathrm{H}^i_{\mathfrak{m}}(M)$ is Artinian. ∎

This theorem is a generalization of the following corollaries.

Corollary 2.1 (See [1, Theorem 7.1.3]). *Let $(R,\mathfrak{m})$ be a local ring. Let M be a finitely generated R-module. Then the R-module $\mathrm{H}^i_{\mathfrak{m}}(M)$ is Artinian for each i.*

Corollary 2.2 (See [10, Lemma 2.7]). *Let M be an $\mathfrak{a}$-cofinite R-module. Then for any maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, the R-module $\mathrm{H}^i_{\mathfrak{m}}(M)$ is Artinian for all i.*

As we mentioned above, in the next result, we use the concept of attached prime ideals. Recall that in a commutative ring R, an R-module M is said to be secondary if $M\neq 0$ and, for each $a\in R$, the endomorphism $f_a:M\to M$ defined by $f_a(m)=am$ is either surjective or nilpotent. In this case, $\mathfrak{p}=\sqrt{\mathrm{Ann}(M)}$ is a prime ideal and M is said to be $\mathfrak{p}$-secondary. A secondary representation of an R-module M is an expression of M as a finite sum of secondary submodules.

A prime ideal $\mathfrak{p}$ is called an attached prime ideal of M if M has a $\mathfrak{p}$-secondary quotient. The set of the attached prime ideals of M is denoted by $\mathrm{Att}(M)$. Artinian modules and injective modules have secondary representation and so one can consider the set of attached primes of these modules.

According to [1, Exercise 7.2.9], an Artinian R-module M is non-zero if and only if $\mathrm{Att}(M)$ is non-empty. For an R-module M, the subset $\mathrm{W}(M)$ of R is defined by $\mathrm{W}(M)=\{a\in R|M\neq aM\}$. It is well-known that

$$\mathrm{W}(M)=\bigcup_{\mathfrak{p}\in\mathrm{Att}(M)}\mathfrak{p}.$$

Now we are ready to prove our main result.

Theorem 2.2 (The non-vanishing Theorem). *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let M be a non-zero weakly finite R-module. If* $\dim M=n$*, then* $\mathrm{H}^n_{\mathfrak{m}}(M)\neq 0$.

Proof. We give a proof by using induction on n. In the case $n=0$ we have nothing to prove, because $\mathrm{H}^0_{\mathfrak{m}}(M)=M\neq 0$. Assume that $n\geq 1$ and the result has been proved for $n-1$. Note that $\mathrm{H}^n_{\mathfrak{m}}(M/\Gamma_{\mathfrak{m}}(M))\cong \mathrm{H}^n_{\mathfrak{m}}(M)$ and $\dim(M/\Gamma_{\mathfrak{m}}(M))=n$. Thus we can assume that M is $\mathfrak{m}$-torsion free. Now assume the contrary that $\mathrm{H}^n_{\mathfrak{m}}(M)=0$. Note that $\mathrm{Ass}(M)$ is finite. Then the equality $\Gamma_{\mathfrak{m}}(M)=0$ implies that there exists $r\in\mathfrak{m}\setminus \mathrm{Z}(M)$ such that M/rM is a non-zero weakly finite R-module of dimension $n-1$. Thus by induction hypothesis $\mathrm{H}^{n-1}_{\mathfrak{m}}(M/rM)\neq 0$. By our assumption $\mathrm{H}^n_{\mathfrak{m}}(M)=0$, so the exact sequence $0\to M\xrightarrow{r} M\to M/rM\to 0$, induces the exact sequence

$$\mathrm{H}^{n-1}_{\mathfrak{m}}(M)\xrightarrow{r}\mathrm{H}^{n-1}_{\mathfrak{m}}(M)\to \mathrm{H}^{n-1}_{\mathfrak{m}}(M/rM)\to 0.$$

Therefore $\mathrm{H}^{n-1}_{\mathfrak{m}}(M/rM)\cong \mathrm{H}^{n-1}_{\mathfrak{m}}(M)/r\mathrm{H}^{n-1}_{\mathfrak{m}}(M)$ and so $\mathrm{H}^{n-1}_{\mathfrak{m}}(M)\neq r\mathrm{H}^{n-1}_{\mathfrak{m}}(M)$. This means that $\mathfrak{m}\setminus \mathrm{Z}(M)\subseteq \mathrm{W}(\mathrm{H}^{n-1}_{\mathfrak{m}}(M))$. Thus, $\mathfrak{m}\subseteq \mathrm{Z}(M)\cup \mathrm{W}(\mathrm{H}^{n-1}_{\mathfrak{m}}(M))$. In view of Lemma 2.1, we see that $\mathrm{H}^{n-1}_{\mathfrak{m}}(M)$ is Artinian. Then it has finite number of attached primes. Hence $\mathfrak{m}\in \mathrm{Att}(\mathrm{H}^{n-1}_{\mathfrak{m}}(M))$. Assume that $\mathrm{Att}(\mathrm{H}^{n-1}_{\mathfrak{m}}(M))=\{\mathfrak{m},\mathfrak{p}_1,\cdots,\mathfrak{p}_t\}$. Choose $c\in\mathfrak{m}\setminus\cup_{i=1}^t\mathfrak{p}_i\cup \mathrm{Z}(M)$. Then

$$\mathrm{H}^{n-1}_{\mathfrak{m}}(M/cM)\cong \mathrm{H}^{n-1}_{\mathfrak{m}}(M)/c\mathrm{H}^{n-1}_{\mathfrak{m}}(M).$$

Since

$$\mathrm{Att}(H^{n-1}_{\mathfrak{m}}(M/cM))=\mathrm{Supp}(R/cR)\cap \mathrm{Att}(H^{n-1}_{\mathfrak{m}}(M))=\{\mathfrak{m}\},$$

thus $\mathrm{H}^{n-1}_{\mathfrak{m}}(M/cM)$ is of finite length and by Lemma 2.2 we conclude that $n=1$. Now consider the following exact sequence

$$0\to \mathrm{H}^0_{\mathfrak{m}}(M)\xrightarrow{c}\mathrm{H}^0_{\mathfrak{m}}(M)\to \mathrm{H}^0_{\mathfrak{m}}(M/cM)\to \mathrm{H}^1_{\mathfrak{m}}(M).$$

By our assumption $\mathrm{H}^0_{\mathfrak{m}}(M)=0=\mathrm{H}^1_{\mathfrak{m}}(M)$ and hence $\mathrm{H}^0_{\mathfrak{m}}(M/cM)=0$ that is a contradiction. Therefore $\mathrm{H}^n_{\mathfrak{m}}(M)\neq 0$. ∎

Corollary 2.3 (See [1, 6.1.4])**.** *Let M be a non-zero finitely generated R-module of dimension n. Then* $\mathrm{H}^n_{\mathfrak{m}}(M)\neq 0$.

Recall that an R-module M is said to be minimax if it has a finitely generated submodule, say N, such that M/N is Artinian. In the next example we show that the non-vanishing theorem does not hold for minimax modules.

Example 2.1. Let $(R,\mathfrak{m})$ be a local domain of dimension 1. Then by [12, p. 658], $Q(A)$, field of fractions of R, is minimax. Note that $Q(A)=E_R(R/0)$ and

$$\mathrm{Ass}_R(Q(A))=\mathrm{Ass}(E_R(R/0))=\{0\}.$$

Therefore $\mathrm{Supp}_R(Q(A))=\{0,\mathfrak{m}\}$, and so $\dim_R Q(A)=1$. But since $Q(A)$ is injective, then we have $\mathrm{H}^1_{\mathfrak{m}}(Q(A))=0$.

Corollary 2.4. *Let $\varphi: R\to S$ be a homomorphism of rings, such that $(R,\mathfrak{m})$ is local and S is finite as a module over R. Let $\mathfrak{a}$ be an ideal of R and M be an R-module such that* $\mathrm{Supp}(M)\subseteq \mathrm{Var}(\mathfrak{a})$ *and* $\mathrm{Ext}^i_S(S/\mathfrak{a}S,M)$ *is finitely generated S-module. Then* $\mathrm{H}^{\dim_R M}_{\mathfrak{m}}(M)\neq 0$.

Proof. In view of [2, Proposition 2], we see that $\mathrm{Ext}^n_R(R/\mathfrak{a},M)$ is finitely generated for all n. The claim is now clear by Theorem 2.2. ∎

It is celebrated result that if $(R,\mathfrak{m})$ is a local ring and M a non-zero finitely generated R-module and for some integer i, $\mathrm{H}^i_{\mathfrak{m}}(M)\neq 0$, therefore $\operatorname{depth} M\leq i\leq \dim M$. (see [1, Corollary 6.2.8]).

If M is an R-module, one can define

$$\operatorname{depth}_R(M) := \inf\{i \in \mathbb{N}_0 | \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}.$$

It is easy to see that

$$\operatorname{depth}_R(M) = 0 \Leftrightarrow \operatorname{Hom}_R(R/\mathfrak{m}, M) \neq 0 \Leftrightarrow \mathfrak{m} \in \operatorname{Ass}_R(M)$$

and if M has finitely many associated primes(and not necessarily finitely generated), then these are equivalent to $Z(M) = \mathfrak{m}$.

In the next theorem we show that this statement is also true for weakly finite modules.

Theorem 2.3. *Let* $(R, \mathfrak{m})$ *be a local ring and let* M *be a weakly finite* R*-module such that* $\mathfrak{a}M \neq M$*. Then* $\operatorname{depth}_R(M)$ *is the least integer* i *such that* $\operatorname{H}^i_{\mathfrak{m}}(M) \neq 0$.

Proof. Set $t = \operatorname{depth}_R(M)$. The proof is by induction on t. If $t = 0$, then $\operatorname{Hom}(R/\mathfrak{m}, M) \neq 0$ and so $(0 :_M \mathfrak{m})$ is non-zero. Therefore $\operatorname{H}^0_{\mathfrak{m}}(M) \neq 0$. Now we assume that $t > 0$ and the result has been proved for each weakly finite R-module N with $\mathfrak{a}N \neq N$ and $\operatorname{depth}_R(N) < n$. Since $t > 0$ and $|\operatorname{Ass}(M)| < \infty$, there exists $r \in \mathfrak{m}$ such that r is a non-zero divisor on M. Set $N = M/rM$. It is obvious that $\operatorname{depth}_R(N) = t-1$, $N \neq \mathfrak{m}N$ and N is weakly finite. Therefore $\operatorname{H}^{t-1}_{\mathfrak{m}}(N) \neq 0$ and the exact sequence $0 \to M \xrightarrow{r} M \to N \to 0$ induces, for each i, an exact sequence

$$\operatorname{H}^{i-1}_{\mathfrak{m}}(M) \to \operatorname{H}^{i-1}_{\mathfrak{m}}(N) \to \operatorname{H}^i_{\mathfrak{m}}(M) \xrightarrow{r} \operatorname{H}^i_{\mathfrak{m}}(M).$$

This shows that if $i < t$, then $\operatorname{H}^i_{\mathfrak{m}}(M) = 0$ and we have the exact sequence $0 \to \operatorname{H}^{t-1}_{\mathfrak{m}}(N) \to \operatorname{H}^t_{\mathfrak{m}}(M)$, and since $\operatorname{H}^{t-1}_{\mathfrak{m}}(N) \neq 0$, it follows that $\operatorname{H}^t_{\mathfrak{m}}(M) \neq 0$. ∎

Corollary 2.5. *Let* $(R, \mathfrak{m})$ *be a local ring and let* M *be an* $\mathfrak{a}$*-cofinite (respectively finitely generated)R-module. Then, if for some integer* i, $\operatorname{H}^i_{\mathfrak{m}}(M) \neq 0$, *we have*

$$\operatorname{depth}_R(M) \leq i \leq \dim(M).$$

Proof. Just apply the Grothendieck's Vanishing theorem, the non-vanishing theorem and the above theorem. ∎

3. Cohomological dimension

Let $\mathfrak{a}$ be an ideal of the ring R. The cohomological dimension of M with respect to $\mathfrak{a}$ is defined as

$$\operatorname{cd}(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} | \operatorname{H}^i_{\mathfrak{a}}(M) \neq 0\}.$$

Grothendieck has shown that $\operatorname{cd}(\mathfrak{a}, M)$ has a lower bound and an upper bound $\operatorname{depth} M$ and $\dim M$ respectively. The cohomological dimension has been studied by several authors. In [5] and [8], Faltings and Huneke–Lyubeznik provide bounds for cohomological dimension.

Theorem 3.1. *Let* M *be an* R*-module. Then the following hold:*

(1) $\operatorname{cd}(\mathfrak{a}, M) \leq \sup\{\operatorname{cd}(\mathfrak{a}, N) | N$ *is a finitely generated submodule of* $M\}$.
(2) *If* M *is weakly finite (respectively* $\mathfrak{a}$*-cofinite), then for any* $\mathfrak{p} \in \operatorname{Supp}(M)$, $\operatorname{cd}(\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$ $= \sup\{\operatorname{cd}(\mathfrak{p}R_{\mathfrak{p}}, L) | L$ *is a finitely generated submodule of* $M_{\mathfrak{p}}\}$.

Proof. For part (1), see [3, Theorem 1.1]. For part (2), just consider the following (in)equalities

$$\begin{aligned}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) &= \operatorname{cd}(\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}) \\ &\leq \sup\{\operatorname{cd}(\mathfrak{p}R_{\mathfrak{p}}, L) | L \text{ is a finitely generated submodule of } M_{\mathfrak{p}}\}\end{aligned}$$

$$\leq \sup\{\dim_{R_\mathfrak{p}}(L) | L \quad \text{is a finitely generated submodule of} \quad M_\mathfrak{p}\}$$
$$\leq \dim_{R_\mathfrak{p}}(M_\mathfrak{p}). \qquad \blacksquare$$

Theorem 3.2. *If M is an R-module with finite cohomological dimension with respect to $\mathfrak{a}$, then $\operatorname{cd}(\mathfrak{a}, M) \leq \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Moreover, if M is weakly finite (respectively $\mathfrak{a}$-cofinite), then for any $\mathfrak{p} \in \operatorname{Supp}(M)$ there exists $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\operatorname{cd}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) = \operatorname{cd}(\mathfrak{p}R_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p})$.*

Proof. First part is from [3, Theorem 1.3]. For the second part, consider the following (in)equalities.

$$\begin{aligned}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{cd}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) &\leq \sup\{\operatorname{cd}(\mathfrak{p}R_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) | \mathfrak{p} \supseteq \mathfrak{q} \in \operatorname{Supp}(M)\} \\ &\leq \sup\{\dim(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) | \mathfrak{p} \supseteq \mathfrak{q} \in \operatorname{Supp}(M)\} \\ &\leq \dim_{R_\mathfrak{p}}(M_\mathfrak{p}). \qquad \blacksquare\end{aligned}$$

Corollary 3.1 (See [3, Theorem 1.4])**.** *Let N and M be R-modules and M weakly finite (respectively $\mathfrak{a}$-cofinite). If $\operatorname{Supp}_R(N) \subseteq \operatorname{Supp}_R(M)$, then*

$$\operatorname{cd}(\mathfrak{m}, N) \leq \operatorname{cd}(\mathfrak{m}, M).$$

Proof. The assertion follows from the following (in)equalities.

$$\operatorname{cd}(\mathfrak{m}, N) \leq \dim(N) \leq \dim(M) = \operatorname{cd}(\mathfrak{m}, M). \qquad \blacksquare$$

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