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# Strongly Top Modules

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**Abstract.** Let $R$ be a commutative ring and let $M$ be a top $R$-module. In this article, we investigate some properties of a new class of modules, called strongly top modules. Studying of this family provides an important tool for studying of the prime spectrum of $M$ from the point of view of spectral spaces with different Zariski and quasi Zariski topologies.

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## 1. Introduction

Throughout this article, $R$ denotes a commutative ring with identity and all modules are nonzero and unitary. If $N$ is a subset of an $R$-module $M$, then $N \leq M$ denotes $N$ is an $R$-submodule of $M$. For any ideal $I$ of $R$ containing $\mathrm{Ann}_R(M)$, $\bar{R}$ and $\bar{I}$ denote $R/\mathrm{Ann}(M)$ and $I/\mathrm{Ann}(M)$ respectively. Further $\mathbb{Z}$ (resp. $\mathbb{Q}$) denotes the ring of integers (resp. the field of rational numbers).

**Definition 1.1.** *For $M$ as an $R$-module and $P$, $N$ its submodules, we define*

- *The colon ideal of $M$ into $N$, $(N:M) = \{r \in R | rM \subseteq N\} = \mathrm{Ann}(M/N)$.*
- *A prime submodule of $M$ is a submodule $P \neq M$ such that, whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P:M)$ or $e \in P$. If $p$ is an ideal of $R$, a $p$-prime submodule of $M$ is a prime submodule $P$ of $M$ with $p = (P:M)$* [6].
- *The prime spectrum (or simply, the spectrum) of $M$ is the set of all prime submodules of $M$ and denoted by $\mathrm{Spec}_R(M)$ or $X$.*
- *If $p \in \mathrm{Spec}(R)$, then $\mathrm{Spec}_p(M)$ is the set of all $p$-prime submodules of $M$* [12].
- *The prime radical $\mathrm{rad}(N)$ is defined to be the intersection of all prime submodules of $M$ containing $N$, and in case $N$ is not contained in any prime submodule, $\mathrm{rad}(N)$ is defined to be $M$* [7].

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- *If* $\mathrm{Spec}_R(M) \neq \emptyset$, *the mapping* $\psi : \mathrm{Spec}_R(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ *such that* $\psi(P) = (P : M)/\mathrm{Ann}(M) = \overline{(P : M)}$ *for every* $P \in \mathrm{Spec}_R(M)$, *is called the natural map of* $\mathrm{Spec}_R(M)$ [9].
- *M is said to be primeful if either* $M = (0)$ *or* $M \neq (0)$ *and the natural map of* $X = \mathrm{Spec}_R(M)$ *is surjective* [11]. *M is said to be X-injective if either* $X = \emptyset$ *or* $X \neq \emptyset$ *and the natural map of X is injective* [1].
- *The Zariski topology on* $X = \mathrm{Spec}_R(M)$ *is the topology* $\tau_M$ *described by taking the set* $Z(M) = \{V(N) | N \text{ is a submodule of } M\}$ *as the set of closed sets of X, where* $V(N) = \{P \in X | (P : M) \supseteq (N : M)\}$ [9].
- *The quasi-Zariski topology on* $X = \mathrm{Spec}_R(M)$ *is described as follows: put* $V^*(N) = \{P \in X | P \supseteq N\}$ *and* $Z^*(M) = \{V^*(N) | N \text{ is a submodule of } M\}$. *Then there exists a topology* $\tau^*_M$ *on X having* $Z^*(M)$ *as the set of closed subsets of X if and only if* $Z^*(M)$ *is closed under finite unions. When this is the case,* $\tau^*_M$ *is called the quasi-Zariski topology on X and M is called a top R-module* [12].
- *M is a multiplication module if for every submodule N of M, there exists an ideal I of R such that* $N = IM$ [12].
- *M is a weak multiplication module, if either* $X = \mathrm{Spec}_R(M) = \emptyset$ *or* $X \neq \emptyset$ *and for every prime submodule P of M we have* $P = IM$ *for some ideal I of R* [4].
- *A topological space T is a spectral space if T is homeomorphic to* $\mathrm{Spec}(S)$ *with the Zariski topology for some ring S* [5]. *It is very important to know under what conditions* $\mathrm{Spec}_R(M)$ *is spectral with both topologies* $\tau_M$ *and* $\tau^*_M$. *Some of these conditions have been considered in* [1, 2, 9].

Now we equip $X = \mathrm{Spec}_R(M)$ of a top $R$-module $M$ with $\tau_M$, $\tau^*_M$ respectively. Then we have always $\tau_M \subseteq \tau^*_M$ by [9, Theorem 2.1]. $M$ is said to be a strongly top module (s-top for short) if $\tau^*_M = \tau_M$. In fact this family is a proper subclass of top modules (see Definition 3.1 and Example 3.1). In this article, we will provide some useful information about this new class of modules. In Proposition 3.1 we describe the basic properties of s-top modules. For example, we prove every homomorphic image of an s-top module is s-top and we also investigate a necessary and sufficient condition in order that a direct sum of s-top $R$-modules is s-top. In Theorem 3.2 we consider some classes of s-top modules over a PID ring. Moreover, we will study the behavior of s-top modules under localizations (see Proposition 3.2 and Corollary 3.2).

There is a natural question as follows: Is $(X, \tau_M)$ (resp. $(X, \tau^*_M)$) a spectral space? Theorem 3.4 shows that $(X, \tau_M)$, $(X, \tau^*_M)$ are both spectral spaces for some special classes of s-top modules. When $X$ is a spectral space with both topologies $\tau_M$ and $\tau^*_M$, our best desire is to have $\tau_M \neq \tau^*_M$. In Theorem 3.1 we provide some useful characterizations for this case. So whenever $X$ is both spectral with $\tau_M$ and $\tau^*_M$, this theorem shows that $\tau_M \neq \tau^*_M$ if and only if there exists a submodule $N$ of $M$ such that $V^*(N) \neq V(\mathrm{rad}(N))$ if and only if $\psi$:$(X, \tau^*) \to Im(\psi)$, where $\psi$ is the natural map of $X$, is not a closed map. Finally, it may happen $\tau^*_M = \tau_M$ when we have some restrictive conditions on a top module $M$ (for example, see Example 3.1 (a) and Corollary 3.1). This has been a motivation for us to study the family of strongly top modules.

## 2. Preliminaries

In this section we review some properties of prime submodules, top modules, and $X$-injective modules. They are fundamental tools to investigate s-top modules.

**Remark 2.1.** Let $M$ be an $R$-module.

(a) Let $K$ be a submodule of $M$ such that $(K:M)$ is a maximal ideal of $R$. Then $K$ is a prime submodule of $M$.
(b) If $N$ is a maximal submodule of $M$, then $N$ is a prime submodule of $M$ and $(N:M)$ is a maximal ideal of $R$.
(c) Let $p \in \mathrm{Spec}(R)$. Then the prime submodules of $R_p$-module $M_p$ are in a one-to-one correspondence with those prime submodules $N$ of $M$ that satisfy $(N:M) \subseteq p$. We use $f_p$ to denote the natural map $f_p : \mathrm{Spec}_{R_p}(M_p) \longrightarrow \mathrm{Spec}_R(M)$ defined by $Q_p \longmapsto Q$. Clearly, $f_p$ is an injective map by the above arguments.
(d) Let $N$ be a prime submodule of $M$ and $S$ be a multiplicatively closed subset of $R$. Then $S^{-1}(N:_R M) = (S^{-1}N :_{S^{-1}R} S^{-1}M)$ [8].

**Remark 2.2.** Let $p \in \mathrm{Spec}(R)$ and $M$ an $R$-module.

(a) The *saturation of a submodule N with respect to p* is the contraction of $N_p$ in $M$ and is denoted by $S_p(N)$. It is known that $S_p(N) = N^{ec} = \{x \in M | tx \in N$ for some $t \in R \setminus p\}$.
(b) $S_p(N) = N$ for every $p$-prime submodule $N$ of $M$.
(c) $S_p(pM)$ is a prime submodule of $M \Leftrightarrow S_p(pM) \neq M \Leftrightarrow (pM:M) = (S_p(pM):M) = p$ [10, Proposition 2.4].

**Remark 2.3.** Let $M$ be an $R$-module.

(a) If $M$ is a top module, then $M$ is an $X$-injective module [12, Theorem 3.5].
(b) If $M$ is an $X$-injective $R$-module. Then

$$\mathrm{Spec}_R(M) = \{S_p(pM) | p \in V(\mathrm{Ann}(M)), S_p(pM) \neq M\}$$

[1, Theorem 3.21].

Let $R$ be a domain. Then $M$ is said to be a torsion (resp. torsion free ) module if $T(M) = M$ (resp $T(M) = 0$), where

$$T(M) = \{m \in M | \exists 0 \neq r \in R \ni rm = 0\}.$$

**Remark 2.4.** Let $R$ be a PID and $M$ an $R$-module. Then the following statements are equivalent:

(a) $M$ is a weak multiplication module;
(b) $M$ is a top module which is a torsion-free or torsion module;
(d) $M$ is an $X$-injective module which is a torsion-free or torsion module;

[1, Corollary 3.19].

## 3. Main results

Throughout the rest of this article, for an $R$-module $M$, $X$ always represents $\mathrm{Spec}_R(M)$.

**Definition 3.1.** *Let M be a top R-module. We say that M is an* strongly top *module (or simply s-top module) if* $\tau_M^* = \tau_M$.

**Example 3.1.**

(a) Every multiplication or finitely generated top $R$-module is an s-top module by [9, Example 1 (c)].

(b) Not every top module is an s-top module. For example, let $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$ for some prime integer $p$. Then $M$ is a top $\mathbb{Z}$-module by [12, Example 3.8]. But it is easy to see that

$$\mathrm{Spec}_{\mathbb{Z}}(M) = \{0 \oplus (\mathbb{Z}/p\mathbb{Z}), \mathbb{Q} \oplus 0\}.$$

Set $P = 0 \oplus (\mathbb{Z}/p\mathbb{Z})$. Then $V^*(P) = \{P\}$. If $V^*(P) \in Z(M)$, then $V^*(P) = V(N)$ for some submodule $N$ of $M$. This implies that

$$(N : M) \subseteq (P : M) = (0 \oplus (\mathbb{Z}/p\mathbb{Z}) : \mathbb{Q} \oplus (\mathbb{Z}/p\mathbb{Z})) = 0.$$

Hence $(N : M) = 0 \subseteq (\mathbb{Q} \oplus 0 : M)$ and therefore $\mathbb{Q} \oplus 0 \in V(N)$. It follows that $V^*(P) = V(N) = \mathrm{Spec}_{\mathbb{Z}}(M)$ which is a contradiction. Thus $V^*(P) \notin Z(M)$, so $\tau^*_M \neq \tau_M$. Therefore $M$ is not an s-top module.

(c) Let $M = \mathbb{Q}$. Then $\mathrm{Spec}_{\mathbb{Z}}(M) = \{0\}$. Hence $M$ is an strongly top module but it is neither finitely generated nor multiplication $\mathbb{Z}$-module.

**Remark 3.1.** Let $M$ be an $R$-module. We denote the set $\{p \in \mathrm{Spec}(R) | S_p(pM) \neq M\}$ by $\Omega(M)$. Clearly, $\Omega(M) = \{p \in V(\mathrm{Ann}(M) | S_p(pM) \neq M\}$ by Remark 2.2 (c). Note that if $M$ is a primeful $R$-module, then $\Omega(M) = V(\mathrm{Ann}(M))$ by [11, Theorem 2.1] so that $\Omega(M)$ is homomorphic with $\mathrm{Spec}\,\bar{R}$.

The following theorem provides some important characterizations about s-top modules.

**Theorem 3.1.** *Let $M$ be an $R$-module, then the following statements are equivalent:*

(a) *$M$ is an s-top $R$-module;*
(b) *$V^*(N) = V(\mathrm{rad}(N))$ for every submodule $N$ of $M$;*
(c) *$M$ is a top module and for any $p \in \Omega(M)$ and for every family $\{p_i\}_{i \in I}$, where $p_i \in \Omega(M)$, we have $\cap_{i \in I} p_i \subseteq p \Longrightarrow \cap_{i \in I} S_{p_i}(p_i M) \subseteq S_p(pM)$;*
(d) *$M$ is a top module and $\psi$:$(X, \tau^*_M) \to Im(\psi)$ is a closed map, where $\psi$ is a natural map of $X$.*

*Proof.* (a) $\Leftrightarrow$ (b) Let $M$ be an s-top $R$-module and $N$ a submodule of $M$. Since $\tau_M = \tau^*_M$, $Y = V^*(N)$ is a closed subset of $(X, \tau_M)$. This implies that $Y = cl(Y)$, where $cl(Y)$ is the topological closure of $Y$ in $(X, \tau_M)$. But $cl(Y) = V(\cap_{P \in Y} P)$ by [9, Proposition 5.1]. On the other hand, $\cap_{P \in Y} P = rad(N)$. By the above arguments, we have $V^*(N) = V(\mathrm{rad}(N))$. The reverse implication follows from the fact that $\tau_M \subseteq \tau^*_M$.

(b) $\Leftrightarrow$ (c) Assume that $V^*(N) = V(\mathrm{rad}(N))$ for every submodule $N$ of $M$. Clearly, $M$ is a top module. Now let $p \in \Omega(M)$, $\{p_i\}_{i \in I}$ a family of elements of $\Omega(M)$ and $\cap_{i \in I} p_i \subseteq p$. We show that $\cap_{i \in I} S_{p_i}(p_i M) \subseteq S_p(pM)$. We have $S_p(pM) \neq M$. Hence $S_p(pM)$ is a $p$-prime submodule of $M$. By assumption, it follows that $S_p(pM) \in V(\cap_{i \in I} S_{p_i}(p_i M))$. Since $V(rad(\cap_{i \in I} S_{p_i}(p_i M))) = V(\cap_{i \in I} S_{p_i}(p_i M))$, we have $S_p(pM) \in V(\mathrm{rad}(\cap_{i \in I} S_{p_i}(p_i M)))$. This implies that $S_p(pM) \in V^*(\cap_{i \in I} S_{p_i}(p_i M))$ by (b). Thus $\cap_{i \in I} S_{p_i}(p_i M) \subseteq S_p(pM)$. Conversely, let $N$ be a submodule of $M$ and set $\Gamma = \{p \in \Omega(M) | N \subseteq S_p(pM)\}$. Then by Remark 2.3 (b), $\mathrm{rad}(N) = \bigcap_{p \in \Gamma} S_p(pM)$. It turns out that $V^*(N) = V(\mathrm{rad}(N))$ as required.

(c) $\Leftrightarrow$ (d) Let the situation be as in part (c). We show that $\psi$:$(X, \tau^*) \to Im(\psi)$ is a closed map. To see this, let $Y$ be an arbitrary subset of $Im(\psi)$. It is enough to show that $\psi^{-1}(cl(Y)) \subseteq cl(\psi^{-1}(Y))$, where $cl(Y)$ refers to the closure of $Y$ in $Im(\psi)$. (Note that if $W$ and $W'$ are two topological spaces and $f : W \to W'$ is a bijective map, then $f$ is closed if and only if $f^{-1}(cl(Z)) \subseteq cl(f^{-1}(Z))$ for every subset $Z$ of $W'$.) By Remark 2.3 (b), there exists a family $\{\bar{p}_i\}_{i \in I}$ of prime ideals of $\mathrm{Spec}(\bar{R})$ such that $Y = \{\bar{p}_i : i \in I, p_i \in \Omega(M)\}$. Now let

$P \in \psi^{-1}(cl(Y))$ so that $\psi(P) \in cl(Y)$. Then by Remark 2.3 (b), $P = S_p(pM)$, $p \in \Omega(M)$. Also by Remark 2.2 (c), $(S_p(pM) : M) = p$. Hence by [9, Proposition 5.1], we have

$$\psi(P) = \psi(S_p(pM)) = \bar{p} \in cl(Y) = V(\cap_{i \in I} \bar{p}_i) \cap Im(\psi),$$

where $V(\cap_{i \in I} \bar{p}_i)$ is the closure of $Y$ in $\mathrm{Spec}(\bar{R})$. (We note that the closure of $Y$ in $Im(\psi)$ is equal to the intersection of closure $Y$ in $\mathrm{Spec}(\bar{R})$ and $Im(\psi)$.) Thus $p \supseteq \cap_{i \in I} p_i$. So by assumption, we have $S_p(pM) \supseteq \cap_{i \in I} S_{p_i}(p_i M)$. Therefore $P \in V^*(\cap_{i \in I} S_{p_i}(p_i M))$. On the other hand, $cl(\psi^{-1}(Y)) = V^*(\cap_{i \in I} S_{p_i}(p_i M))$. It follows that $\psi^{-1}(cl(Y)) \subseteq cl(\psi^{-1}(Y))$ as desired. We have similar arguments for the reverse implication and the proof is completed. ∎

**Corollary 3.1.** *Let $M$ be an $R$-module. Then $M$ is an s-top module in the following cases:*

(a) *$M$ is weak multiplication and $|\mathrm{Spec}(R)| < \infty$. However, not every weak multiplication module is an s-top module;*
(b) *$M$ is top, flat, and $|\mathrm{Spec}(R)| < \infty$;*
(c) *$M$ is top and $|Max(R)| = |\mathrm{Spec}(R)| < \infty$;*
(d) *$M$ is top, primeful, and $\Omega(M)$ is a discrete subspace with its Zariski topology.*

*Proof.* (a) The first assertion is straightforward by using Theorem 3.1 (b). To see the second assertion, set $M = \oplus_{i \in I} \mathbb{Z}/p_i\mathbb{Z}$, where $\Lambda = \{p_i\}_{i \in I}$ is the set of all prime integers. Then $\mathrm{Spec}_{\mathbb{Z}}(M) = \{p_i M | i \in I\}$ so that $M$ is a weak multiplication module. Now let $N = \mathbb{Z}/p_0\mathbb{Z}$ with $p_0 \in \Lambda$. Then $V^*(N) = \{pM | p \in \Lambda, p \neq p_0\}$ and $V(\mathrm{rad}(N)) = X$ so that $M$ is not an s-top module by Theorem 3.1 (b).

(b) By part (a) and [1, Theorem 3.15 (d)].

(c) The proof is straightforward by using Theorem 3.1 (c).

(d) By Remark 3.1, $\Omega(M)$ is homomorphic with $Im(\psi)$. Now the result follows from Theorem 3.1 (d). ∎

**Remark 3.2.** A family $(M_i)_{i \in I}$ of $R$-modules is said to be *prime-compatible* if, for all $i \neq j$ in $I$, there does not exist a prime ideal $p$ in $R$ with $\mathrm{Spec}_p(M_i)$ and $\mathrm{Spec}_p(M_j)$ both nonempty. Further the following hold (see [12]).

(a) Every homomorphic image of a top $R$-module is top.
(b) Let $(M_i)_{i \in I}$ be a family of $R$-modules and $M = \oplus_{i \in I} M_i$. $M$ is a top module if and only if $(M_i)_{i \in I}$ is a family of prime-compatible top $R$-modules.
(c) Every submodule of a top $R$-module is not necessarily a top $R$-module.
(d) Let $R$ be a domain with the field of fractions $Q$ and let $M'$ be an $R$-module. Then the $R$-module $Q \oplus M'$ is a top module if and only if $M'$ is a torsion top module.
(e) Let $\{I_\lambda | \lambda \in \Lambda\}$ be a family of ideals of $R$. Then $M = \bigoplus_{\lambda \in \Lambda} R/I_\lambda$ is a top module if and only if the ideals $I_\lambda$ ($\lambda \in \Lambda$) are comaximal.

In Proposition 3.1 we investigate the above properties for s-top modules.

An $R$- module $M$ is called a primeless if it has no prime submodules [12].

**Proposition 3.1.**

(a) *Every homomorphic image of an s-top module is s-top.*
(b) *If $M = \oplus_{i \in I} M_i$ is an s-top module, then $(M_i)_{i \in I}$ is a family of prime-compatible s-top modules. But the converse is not true in general.*
(c) *Every submodule of an s-top $R$-module is not necessarily s-top.*

(d) *Let $R$ be a domain and $(M_i)_{i\in I}$ be a family of $R$-modules such that $M_t$ is a torsion free $R$ module for some $t\in I$. Then $M=\oplus_{i\in I}M_i$ is an s-top module if and only if* $\mathrm{Spec}_R(M)=\{T(M)\}$. *In particular if $R$ is a domain with the field of fractions $Q$ and $M'$ is an $R$-module, then the $R$-module $M=Q\oplus M'$ is an s-top module if and only if $M'$ is a primeless $R$-module.*

(e) *Let $\Lambda$ be a finite index set and let $I_\lambda$ ($\lambda\in\Lambda$) be comaximal ideals of the ring $R$. Then $M=\bigoplus_{\lambda\in\Lambda}R/I_\lambda$ is an s-top $R$-module.*

*Proof.* (a) Let $M$ be an s-top $R$-module and $N$ a submodule of $M$. Let $K/N$ be a submodule of $M/N$. By Theorem 3.1 (b), it is enough to prove that $V(\mathrm{rad}(K/N))=V^*(K/N)$. To see this, let $L\in V(\mathrm{rad}(K/N))$. Then $L=Q/N$, where $N\subseteq Q\in X$. This implies that

$$(Q/N:M/N)\supseteq(\bigcap_{N\subseteq P\in V^*(K)}P/N:M/N)=\bigcap_{N\subseteq P\in V^*(K)}(P/N:M/N).$$

Therefore, we have $(Q:M)\supseteq(\bigcap_{N\subseteq P\in V(K)}P:M)$ by [10, Result 1]. It follows that $Q\in V(\mathrm{rad}(K))$. Since $M$ is an s-top $R$-module, $V^*(K)=V(\mathrm{rad}(K))$ by Theorem 3.1 (b) so that $Q\in V^*(K)$. Hence $V(\mathrm{rad}(K/N))\subseteq V^*(K/N)$. The reverse inclusion is clear.

(b) The first assumption follows from part (a) and Remark 3.2 (b). To see the second assertion, set $M=\oplus_{i\in I}\mathbb{Z}/p_i\mathbb{Z}$, where $\Lambda=\{p_i\}_{i\in I}$ is the set of all prime integers. It is easy to see that $M_i$'s, where $M_i=\mathbb{Z}/p_i\mathbb{Z}$, are prime-compatible s-top $\mathbb{Z}$-modules. But $M$ is not an s-top module as we saw in the proof of Corollary 3.1 (a).

(c) Let $p$ be a prime integer. Let $\mathbb{Z}_{(p)}=S^{-1}\mathbb{Z}$, where $S=\mathbb{Z}\setminus(p)$. Consider the $\mathbb{Z}_{(p)}$-module $M=(\mathbb{Q}/p\mathbb{Z}_{(p)})\oplus\mathbb{Q}$. Then $\mathrm{Spec}(M)=\{(\mathbb{Q}/p\mathbb{Z}_{(p)})\oplus(\mathbf{0})\}$ is a singleton set so that $M$ is an s-top $\mathbb{Z}_{(p)}$-module. Now consider the submodule $N=(\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)})\oplus\mathbb{Q}$ of $M$. By [1, Example 4.12], $N$ is not an s-top $\mathbb{Z}_{(p)}$-module.

(d) Let $M$ be an s-top module and let $j\in I$ with $j\neq t$. We show that $\mathrm{Spec}_R(M_j)=\emptyset$. To see this, let $P_j\in\mathrm{Spec}_R(M_j)$. Then by [1, Proposition 3.7 (b)], $K=(\bigoplus_{i\in I_{i\neq j}}M_i)\bigoplus P_j\in\mathrm{Spec}_R(M)$. Also we have $L=(\mathbf{0})\bigoplus(\bigoplus_{i\in I_{i\neq t}}M_i)\in\mathrm{Spec}_R(M)$. This implies that $K\in V(L)$ so that $K\supseteq L$. Thus $P_j\supseteq M_j$, a contradiction. It follows that $\mathrm{Spec}_R(M)=\{L\}$ by [1, Proposition 3.7]. But $L=S_{(0)}(\mathbf{0})=T(M)$ by [10, Corollary 3.7], so $\mathrm{Spec}_R(M)=\{T(M)\}$ as desired. The reverse implication is clear.

(e) By Remark 3.2, part (e), $M$ is a top $R$-module so that $M$ is $X$-injective and $\mathrm{Spec}_R(M)=\{p_\lambda M|p_\lambda\in V(I_\lambda),\lambda\in\Lambda\}$ by Remark 2.3. Now since $\Lambda$ is a finite set, it follows that $V^*(N)=V(\mathrm{rad}(N))$ for every submodule $N$ of $M$. Hence $M$ is an s-top module by Theorem 3.1 (b). ∎

**Remark 3.3.** Proposition 3.1 (b) shows a property which holds for top modules but it is not valid for s-top modules.

Let $M$ be an $R$-module. Then $M$ is called a content module if for every $x\in M$, $x\in c(x)M$, where $c(x)=\cap\{I|I$ is an ideal of $R$ such that $x\in IM\}$. Every projective module is a content module and for a content module $M$, $\cap_{i\in J}(A_iM)=(\cap_{i\in J}A_i)M$ for every family $\{A_i|i\in J\}$ of ideals $A_i$ of $R$ [11, p. 140].

**Theorem 3.2.** *Let $R$ be a PID and $M$ be an $R$-module. Then $M$ is an s-top $R$-module in each of the following cases:*

(a) *$M$ is a non faithful top module;*

(b) *M is a content top module which is a torsion (or torsion-free) module. In particular M is a projective torsion (or torsion-free) top module;*
(c) *M is content X-injective module which is a torsion (or torsion-free) module;*
(d) *M is content weak multiplication module.*

*Proof.* (a) Let $\{p_i\}_{i\in I}$ be a family of elements of $\Omega(M)$ such that $\cap_{i\in I} p_i \subseteq p$, where $p \in \Omega(M)$. Since $\mathrm{Ann}(M) \neq (0)$, $|\mathrm{Spec}(\bar{R})| < \infty$. Thus $I$ is a finite set, so $p_k = p$ for some $k \in I$. This implies that $S_{p_k}(p_k M) = S_p(pM)$. Hence $\cap_{i\in I} S_{p_i}(p_i M) \subseteq S_p(pM)$. Therefore $M$ is an s-top $R$-module by Theorem 3.1 (c).

(b) By part (a), it is enough to consider the case that $\mathrm{Ann}(M) = (0)$. Now let $N$ be a submodule of $M$. By Theorem 3.1 (b), it is sufficient to prove that $V^*(N) = V(\mathrm{rad}(N))$. Since $M$ is a top $R$-module and $\mathrm{Ann}(M) = (0)$, there exists a family $\{p_i\}_{i\in I}$ of elements of $\mathrm{Spec}(R)$ such that $\mathrm{rad}(N) = \cap_{i\in I} S_{p_i}(p_i M)$ by Remark 2.3. Since $M$ is a torsion (or torsion-free) $R$-module, we have $\mathrm{rad}(N) = (\mathbf{0})$ or $\{p_i : i \in I\} \subseteq \mathrm{Max}(R)$. If $\mathrm{rad}(N) = (\mathbf{0})$, then $N = (\mathbf{0})$ and we are done. Thus we can assume that $\mathrm{rad}(N) \neq (\mathbf{0})$ and $\{p_i : i \in I\} \subseteq \mathrm{Max}(R)$. Since $M$ is $X$-injective, $S_{p_i}(p_i M) = p_i M$ for every $i \in I$. As $M$ is a content module, we have $\mathrm{rad}(N) \subseteq \cap_{i\in I}(p_i M) \subseteq (\cap_{i\in I} p_i)M$. If $I$ is an infinite set, then $\mathrm{rad}(N) \subseteq (\cap_{i\in I} p_i)M = (\mathbf{0})$, a contradiction. Hence we assume that $I$ is finite and $Q \in V(\mathrm{rad}(N))$. It follows that $\cap_{i\in I} p_i \subseteq (Q : M)$. This implies that $p_j = (Q : M)$ for some $j \in I$, so $\mathrm{rad}(N) \subseteq \cap_{i\in I} p_i M \subseteq p_j M \subseteq Q$. Therefore $Q \in V^*(N)$ so that $V^*(N) = V(\mathrm{rad}(N))$.

(c) and (d) follow by part (b) and Remark 2.4. This completes the proof. ∎

**Remark 3.4.** The converse of part (a) of Theorem 3.2 is not true in general, for example $\mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module is an s-top $\mathbb{Z}$-module while $\mathrm{Ann}(\mathbb{Z}(p^\infty)) = 0$

The following proposition shows the behavior of s-top modules over localizations.

**Proposition 3.2.** *Let M be an s-top R-module. Then $M_p$ is an s-top $R_p$-module for every prime ideal p of R.*

*Proof.* Let $p$ be a prime ideal of $R$ and $N_p$ a submodule of $M_p$ for some submodule $N$ of $M$. By Theorem 3.1 (b), it is enough to prove that $V^*(N_p) = V(\mathrm{rad}(N_p))$. It is clear that $V^*(N_p) \subseteq V(\mathrm{rad}(N_p))$. Conversely, assume that $W \in V(\mathrm{rad}(N_p))$. Then there exists $Q \in \mathrm{Spec}_R(M)$ such that $W = Q_p$ and $p \supseteq (Q :_R M)$ by Remark 2.1 (c). It follows that

$$(Q_p :_{R_p} M_p) \supseteq (\mathrm{rad}(N_p) :_{R_p} M_p) \supseteq ((\mathrm{rad}(N))_p :_{R_p} M_p) \supseteq (\mathrm{rad}(N) :_R M)_p.$$

But $(Q_p :_{R_p} M_p) = (Q :_R M)_p$ by Remark 2.1 (d). Therefore $Q \in V(\mathrm{rad}(N))$ so that $Q \in V^*(N)$ by Theorem 3.1 (b). This implies that $W \in V^*(N_p)$. Hence the proof is completed. ∎

We need the following lemma.

**Lemma 3.1.** *Let $(R,m)$ be a quasi local ring and let $\{p_\alpha\}_{\alpha\in\Lambda}$ be a collection of prime ideals of R. Then $(\bigcap_{\alpha\in\Lambda} p_\alpha)_m = \bigcap_{\alpha\in\Lambda}(p_\alpha)_m$.*

*Proof.* Straightforward. ∎

Let $X$, $Y$ be two sets and let $g : X \longrightarrow Y$ be a map from $X$ into $Y$. Suppose $\tau$ is an arbitrary topology on $X$. Set

$$U = \{v \subseteq Y : g^{-1}(v) \in \tau\}.$$

Then $U$ is a topology in $X$, called the induced topology by $g$ in $Y$. We denote this topology by $g(\tau)$. In fact $U$ is the coarser topology in $Y$ that $g : (X,\tau) \longrightarrow (Y,U)$ is continuous.

Moreover if $g$ is bijective, then

$$g(\tau)=\{g(w): w\in\tau\}.$$

**Theorem 3.3.** *Let $M$ be an $R$-module and $p\in \mathrm{Spec}(R)$. Let $f:\bar{R}\longrightarrow \bar{R_p}$ be the canonical homomorphism and let $f^*:\mathrm{Spec}\,(\bar{R_p})\longrightarrow \mathrm{Spec}\,(\bar{R})$ be the associated mapping. Consider the following diagram*

$$\begin{array}{ccc} (\mathrm{Spec}_{R_p}(M_p),\tau^*_{M_p}) & \overset{f_p}{\longrightarrow} & (\mathrm{Spec}_R(M),\tau^*_M) \\ \psi_p\downarrow & & \downarrow\psi \\ \mathrm{Spec}\,(\bar{R_p}) & \overset{f^*}{\longrightarrow} & \mathrm{Spec}\,(\bar{R}) \end{array}$$

*with natural maps, where $f_p$ is the natural map as in the Remark 2.1 (c). Then we have the following.*

(a) *The above diagram is commutative.*

(b) *If $(R,p)$ is quasi local ring, then*

(i) *$f_p$ is bijective;*

(ii) *If $M_p$ is a primeful (X-injective) $R_p$-module, then $\psi$ is surjective (injective) so that $M$ is a primeful (X-injective) $R$-module;*

(iii) *If $M_p$ is a top $R_p$-module, then we have*

$$f_p(\tau_{M_p})\subseteq\tau_M\subseteq f_p(\tau^*_{M_p})=\tau^*_M.$$

*Consequently, $M$ is top $R$-module and all maps in the above diagram are continuous.*

*Proof.* (a) Use Remark 2.1 parts (c) and (d).

(b)(i) By Remark 2.1 (c).

(b)(ii) Consider the map $f^*:\mathrm{Spec}\,(\bar{R_p})\longrightarrow\mathrm{Spec}\,(\bar{R})$ given by $f^*(q)=f^{-1}(q)$, where $f:\bar{R}\longrightarrow\bar{R_p}$ is the canonical homomorphism and $q\in\mathrm{Spec}\,(\bar{R_p})$. Then by [3, p.46, Exercise 21], $f^*$ is a homeomorphism of $\mathrm{Spec}\,(\bar{R_p})$ onto its image in $\mathrm{Spec}\,(\bar{R})$. Since $(R,p)$ is quasi local ring, $f^*$ is a surjective map so that it is a homeomorphism. Now the claim follows from part (a).

(b)(iii) First we show that $f_p(\tau_{M_p})\subseteq\tau_M$. It is enough to prove

$$f_p(\{V(W)|W\leq M_p\})\subseteq\{V(N)|N\leq M\}.$$

To see this, let $A\in f_p(\{V(W)|W\leq M_p\})$ so that $A=f_p(V(N_p))$ for some submodule $N$ of $M$. We show that $A$ is closed in $(X,\tau_M)$ or equivalently, $V(\bigcap_{Q\in A}Q)=A$ by [9, Proposition 5.1]. Clearly, $A\subseteq V(\bigcap_{Q\in A}Q)$. Now let $Q^*\in V(\bigcap_{Q\in A}Q)$. It follows that $(Q^*:M)\supseteq\bigcap_{Q\in A}(Q:M)$ so that $(Q^*:M)_p\supseteq\bigcap_{Q\in A}(Q:M)_p$ by Lemma 3.1. On the other hand, $(Q^*:M)_p=(Q^*_p:M_p)$ by Remark 2.1 (d). Hence $(Q^*_p:M_p)\supseteq\bigcap_{Q\in A}(Q_p:M_p)$. But $Q\in A$ implies that $(Q_p:M_p)\supseteq(N_p:M_p)$. Thus $(Q^*_p:M_p)\supseteq(N_p:M_p)$. It follows that $Q^*\in f_p(V(N_p))=A$. To complete the first assertion, since $\tau_M\subseteq\tau^*_M$, it is enough to show that $f_p(\tau^*_{M_p})=\tau^*_M$. As $f_p$ is bijective, it suffices to show that $H:=f_p(\{V^*(W)|W\leq M_p\})=\{V^*(N)|N\leq M\}$. If $K\in H$, there exists a submodule $W$ of $M_p$ such that $f_p(V^*(W))=K$. Also there exists a submodule $N$ of $M$ such that $W=N_p$. It is easy to check that $K=f_p(V^*(W))=V^*(N)$. This implies that $K\in\{V^*(N)|N\leq M\}$. We have similar arguments for the reverse inclusion.

Therefore $f_p(\tau^*_{M_p}) = \tau^*_M$, so $M$ is a top $R$-module. To see the last assertion, we note that $f_p : (\text{Spec}_{R_p}(M_p), \tau^*_{M_p}) \longrightarrow (\text{Spec}_R(M), f_p(\tau^*_{M_p}))$ is continuous and $f_p(\tau^*_{M_p}) = \tau^*_M$ by the above arguments. Hence $f_p : (\text{Spec}_{R_p}(M_p), \tau^*_{M_p}) \longrightarrow (\text{Spec}_R(M), \tau^*_M)$ is a continuous map. Also $f^*$ is continuous map as we saw in the proof of part (b)(ii). Moreover, if $L$ is a top $R$-module, then the natural map $\psi : (\text{Spec}_R(L), \tau^*_L) \longrightarrow \text{Spec}(\bar{R})$ is always a continuous map (for, if $\bar{I}$ is an ideal of $\bar{R}$, $\psi^{-1}(V(\bar{I})) = V(IL) = V^*(IL)$ by [9, Proposition 3.1] and [9, Result 3]). As, $M_p$ and $M$ are top module, it follows that $\psi$ and $\psi_p$ are continuous as desired. This completes the proof. ∎

**Corollary 3.2.** *Let $(R, p)$ be a quasi local ring and $M$ an $R$-module. Then $M$ is an s-top module if and only if $M_p$ is an s-top $R_p$-module.*

*Proof.* ($\Rightarrow$) This follows by Proposition 3.2. To see the reverse implication, since $M_p$ is an s-top $R_p$-module, $\tau_{M_p} = \tau^*_{M_p}$. Thus $f_p(\tau_{M_p}) = f_p(\tau^*_{M_p})$. By Theorem 3.3, part (b)(iii), $\tau_M = \tau^*_M$. This completes the proof. ∎

The following example shows that if $M$ is a top module over a local ring, then it is not necessarily an s-top module.

**Example 3.2.** Let $p$ be a prime integer and let $\mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$, where $S = \mathbb{Z} \setminus (p)$. Consider the $\mathbb{Z}_{(p)}$-module $M = (\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}) \oplus \mathbb{Q}$. Then $M$ is a top module, and $\tau_M \neq \tau^*_M$ by [1, Example 4.12].

**Theorem 3.4.** *Let $M$ be an $R$-module. Then $(X, \tau_M)$ and $(X, \tau^*_M)$ are spectral spaces in each of the following cases:*

(a) *$M$ is an s-top $R$-module and $Im(\psi)$ is a closed subspace of* Spec$(\bar{R})$*, where $\psi$ is a natural map of $X$ (for example, when $M$ is primeful);*
(b) *$(R, p)$ be a quasi local ring and $M_p$ be a primeful s-top $R_p$-module;*
(c) *$M$ is an s-top $R$-module and $R$ is a PID;*
(d) *$R$ is a PID and $M$ has the property listed in (a), (b), (c), and (d) of Theorem 3.2.*

*Proof.* (a) This follows from [9, Theorem 6.7].

(b) By Theorem 3.3 (b)(ii) and Corollary 3.2, $M$ is a primeful s-top module. Hence by part (a), $(X, \tau_M)$ and $(X, \tau^*_M)$ are both spectral spaces.

(c) Since $R$ is a PID and $M$ is s-top, $(X, \tau_M)$ and $(X, \tau^*_M)$ are Noetherian by [2, Proposition 3.3 (a)]. Therefore, $(X, \tau^*_M)$ is spectral by [2, Theorem 3.8 (b)] and [2, Remark 1.2] as desired.

(d) This is an immediate consequence of part (c) and Theorem 3.2. This completes the proof. ∎

**Remark 3.5.** The module $M$ in Example 3.2 shows that the converse of Theorem 3.4 (a) is not true in general. Note that $M$ is primeful and $(X, \tau_M)$ and $(X, \tau^*_M)$ are spectral spaces by [1, Example 4.12].

We have not found any example of an s-top $R$-module $M$ for which $X = \text{Spec}_R(M)$ is not spectral with both topologies $\tau_M$ and $\tau^*_M$. The lack of such counterexamples together with Theorem 3.4 which explains $(X, \tau_M)$ and $(X, \tau^*_M)$ are both spectral spaces for certain classes of s-top modules, motivates the following conjecture.

**Conjecture 3.1.** *Let $M$ be an s-top $R$-module. Then $(X, \tau_M)$ and $(X, \tau^*_M)$ are spectral spaces.*

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