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The Artinianness of Formal Local Cohomology Modules

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Abstract. Let I be an ideal of a commutative Noetherian local ring $(R,\mathfrak{m})$, M a finitely generated R-module and $\varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$ the i-th formal local cohomology module of M with respect to I. We prove some results concerning artinianness of $\varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$. We discuss the maximum and minimum integers such that $\varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$ is artinian.

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1. Introduction

Throughout this paper, we assume that $(R,\mathfrak{m})$ is a commutative Noetherian local ring with non-zero identity, I is an ideal of R and M a finitely generated R-module. Schenzel [9] has called $\mathfrak{F}^i_I(M) := \varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$ the i-th formal local cohomology module of M with respect to I and investigated their structure extensively. Let t be an integer. It is shown that the local cohomology module $H^i_I(M)$ is finitely generated for all $i < t$ if and only if there is some integer $r > 0$ such that $I^rH^i_I(M) = 0$ for all $i < t$. Recently, in [7, Theorem 2.8], it is proved that a similar result, that is, $\mathfrak{F}^i_I(M)$ is artinian for all $i < t$ if and only if there is some integer $r > 0$ such that $I^r\mathfrak{F}^i_I(M) = 0$ for all $i < t$. In this paper, we get the following result.

Theorem 1.1. *Let $t \geq 0$ be an integer. Then the following statements are equivalent:*
(a) *$\mathfrak{F}^i_I(M)$ is artinian for all $i > t$;*
(b) *$I \subseteq \mathrm{Rad}(0 : \mathfrak{F}^i_I(M))$ for all $i > t$.*

Set $q(I,M) := \sup\{i \mid \mathfrak{F}^i_I(M) \text{ is not artinian}\} = \sup\{i \mid I \nsubseteq \mathrm{Rad}(0 : \mathfrak{F}^i_I(M))\}$. We prove that if $\mathrm{Supp}\,L \subseteq \mathrm{Supp}\,M$, then $q(I,L) \leq q(I,M)$. In particular, if $\mathrm{Supp}\,L = \mathrm{Supp}\,M$, then $q(I,L) = q(I,M)$. In [3] and [8], the artinianness of local cohomology modules is considered. In [7, Theorem 2.9], it is shown that if $\mathfrak{F}^i_I(M)$ is artinian for all $i < t$, then $\mathfrak{F}^t_I(M)/I\mathfrak{F}^t_I(M)$ is artinian. As the dual case of the above result, we get another main result of this paper.

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Theorem 1.2. *Let t be an integer such that $\mathfrak{F}_I^i(M)$ is artinian for all $i < t$. Then* $\operatorname{Hom}_R(R/I, \mathfrak{F}_I^t(M))$ *is artinian.*

2. Main results

First, we give the following definition.

Definition 2.1. *For an ideal I of R, we define the formal filter depth,* ff-depth(I,M)*, by* ff-depth$(I,M) := \inf\{i \mid \mathfrak{F}_I^i(M)$ *is not artinian*$\}$.

Proposition 2.1. *Let I and J be ideals of R and* $\operatorname{Rad}(I) = \operatorname{Rad}(J)$*. Then we have that* ff-depth$(I,M) =$ ff-depth(J,M).

Proof. By [9, Proposition 3.3], we have $\mathfrak{F}_I^i(M) \cong \mathfrak{F}_{I\widehat{R}}^i(\widehat{M})$ for all $i \geq 0$. Therefore, we may assume that R is complete. Then, by Cohen's Structure Theorem, R is a homomorphic image of a regular complete local ring $(T, \mathfrak{n})$ such that $R = T/J$ for some ideal J of T. Set $b_1 := I \cap T$ and $b_2 := J \cap T$. In view of [1, Lemma 2.1], we have that

$$\mathfrak{F}_I^i(M) \cong \mathfrak{F}_{b_1}^i(M) \cong \operatorname{Hom}_T(H_{b_1}^{\dim T - i}(M,T), E_T(T/\mathfrak{n}))$$

and

$$\mathfrak{F}_J^i(M) \cong \mathfrak{F}_{b_2}^i(M) \cong \operatorname{Hom}_T(H_{b_2}^{\dim T - i}(M,T), E_T(T/\mathfrak{n}))$$

for all $i \geq 0$. Since $\operatorname{Rad}(I) = \operatorname{Rad}(J)$, then $\operatorname{Rad}(b_1) = \operatorname{Rad}(b_2)$. Let $E^\bullet$ be a minimal injective resolution of T. We know that $H_{b_1}^{\dim T - i}(M,T) = H^{\dim T - i}(\operatorname{Hom}_T(M, \Gamma_{b_1}(E^\bullet)))$ and $H_{b_2}^{\dim T - i}(M,T) = H^{\dim T - i}(\operatorname{Hom}_T(M, \Gamma_{b_2}(E^\bullet)))$. Now the result follows by $\operatorname{Rad}(b_1) = \operatorname{Rad}(b_2)$. ∎

Proposition 2.2. ff-depth$(I,M) =$ ff-depth$(I\widehat{R}, \widehat{M})$.

Proof. Since $\mathfrak{F}_I^i(M) \cong \mathfrak{F}_{I\widehat{R}}^i(\widehat{M})$ for all $i \geq 0$. The result is clear. ∎

Proposition 2.3. *Let $I \subseteq J$ be ideals of R. Then we have that* ff-depth$(I,M) \leq$ ff-depth$(J,M) +$ ara(J/I).

Proof. By Proposition 2.1, we may assume that there are $x_1, x_2, \ldots, x_n \in R$ such that $J = I + (x_1, x_2, \ldots, x_n)$. By induction on n, it suffices to treat only the case $n = 1$. So, let $J = I + (x)$ for some $x \in R$. By [9, Theorem 3.15], there is the following long exact sequence

$$\cdots \to \operatorname{Hom}(R_x, \mathfrak{F}_I^i(M)) \to \mathfrak{F}_I^i(M) \to \mathfrak{F}_J^i(M) \to \operatorname{Hom}(R_x, \mathfrak{F}_I^{i+1}(M)) \to \cdots.$$

For all $i <$ ff-depth$(I,M) - 1$, $\mathfrak{F}_I^i(M)$ and $\mathfrak{F}_I^{i+1}(M)$ are artinian, then $\mathfrak{F}_J^i(M)$ is artinian by the above exact sequence, and so ff-depth$(I,M) \leq$ ff-depth$(J,M) + 1$. ∎

In [1, Proposition 4.4], it is proved that if L is a pure submodule of M. Then $\inf\{i \mid \mathfrak{F}_I^i(L) \neq 0\} \geq \inf\{i \mid \mathfrak{F}_I^i(M) \neq 0\}$. Next, we give a similar result.

Proposition 2.4. *Let L be a pure submodule of M. Then* ff-depth$(I,M) \leq$ ff-depth(I,L).

Proof. Since L is a pure submodule of M, we have that the natural map $L/I^nL \to M/I^nM$ is pure for all $n > 0$. [6, Corollary 3.2(a)] implies the exact sequence

$$0 \to H_{\mathfrak{m}}^i(L/I^nL) \to H_{\mathfrak{m}}^i(M/I^nM)$$

for all $i \geq 0$ and $n \geq 0$. This induces the exact sequence $0 \to \mathfrak{F}_I^i(L) \to \mathfrak{F}_I^i(M)$ and so ff-depth$(I,M) \leq$ ff-depth(I,L). ∎

Lemma 2.1. *Let $(R,\mathfrak{m})$ is a local ring possessing a dualizing complex $D_R^{\cdot}$ and let p denote a prime ideal and i be an integer such that $\mathfrak{F}^i_{IR_p}(M_p)$ is not artinian. Then $\mathfrak{F}_I^{i+\dim R/p}(M)$ is not artinian.*

Proof. The proof is similar to the one of [9, Corollary 3.7], here we omit it. ∎

Proposition 2.5.

(1) *Let $x \in \mathfrak{m}$ be an M-filter regular element. Then we have that* ff-depth$(I, M/xM) \geq$ ff-depth$(I,M) - 1$.

(2) *Suppose that* f-depth$M < \infty$. *Then* ff-depth$(I,M) \leq \min\{$f-depth$M, \dim M/IM\}$.

(3) *Suppose that R possesses a dualizing complex. Then*

$$\text{ff-depth}(I,M) \leq \text{ff-depth}(IR_p, M_p) + \dim R/p$$

for all $p \in \operatorname{Supp} M \cap V(I)$.

Proof. (1) It is easy to prove by [9, Theorem 3.14].

(2) Since f-depth$M =$ f-depth$\widehat{M}$ and $\dim M/IM = \dim \widehat{M}/I\widehat{M}$, we can assume that R is complete by Proposition 2.2. Note that

$$\text{ff-depth}(I,M) \leq \sup\{i \mid \mathfrak{F}_I^i(M) \text{ is not artinian }\} \leq \sup\{i \mid \mathfrak{F}_I^i(M) \neq 0\} = \dim M/IM.$$

Now we prove ff-depth$(I,M) \leq$ f-depth$M$ by induction on $t =$ ff-depth(I,M). When $t = 0$, the claim holds. Let $t \geq 1$. Then $\mathfrak{F}_I^0(M)$ is artinian. It follows that $\dim R/(I+p) > 0$ for all $p \in \operatorname{Ass} M \backslash \{\mathfrak{m}\}$ by [5, Proposition 2.2]. Then we can choose $x \in \mathfrak{m}$ which forms a parameter of $R/(I,p)$ for all $p \in \operatorname{Ass} M \backslash \{\mathfrak{m}\}$, so $x \in \mathfrak{m}$ be an M-filter regular element. Thus

$$t - 1 \leq \text{ff-depth}(I, M/xM) \leq \text{f-depth}(M/xM) = \text{f-depth}M - 1$$

by (1) and the inductive hypothesis. So $t \leq$ f-depth$M$.

(3) We get the result by Lemma 2.1. ∎

Theorem 2.1. *Let M be a non-zero finitely generated R-module and let $t \geq 1$ be an integer. Then the following four conditions are equivalent:*

(1) $\mathfrak{F}_I^i(M) = 0$ *for all $i \geq t$;*

(2) $\mathfrak{F}_I^i(M)$ *is finitely generated for all $i \geq t$;*

(3) $\mathfrak{F}_I^i(R/p) = 0$ *for all $i \geq t$, $p \in$* Supp$M$;

(4) $\mathfrak{F}_I^i(R/p)$ *is finitely generated for all $i \geq t$, $p \in$* Supp$M$.

Proof. $(1) \Rightarrow (2)$. It is clear.

$(2) \Rightarrow (1)$. We use induction on $d = \dim M$. For $d = 0$, then $\mathfrak{F}_I^i(M) = 0$ for all $i \geq 1$.

Now let $d > 0$ and $\mathfrak{F}_I^i(M) = 0$ for all $i > t$. Now we will prove that $\mathfrak{F}_I^t(M) = 0$. First, we assume that depth$M > 0$, then there is an element $x \in \mathfrak{m}$ which is M-regular. From the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we can get the long exact sequence

$$\cdots \to \mathfrak{F}_I^i(M) \xrightarrow{x} \mathfrak{F}_I^i(M) \to \mathfrak{F}_I^i(M/xM) \to \mathfrak{F}_I^{i+1}(M) \to \cdots,$$

then $\mathfrak{F}_I^i(M/xM) = 0$ for all $i \geq t$. By the inductive hypothesis, we get that $\mathfrak{F}_I^t(M/xM) = 0$, then $x\mathfrak{F}_I^t(M) = \mathfrak{F}_I^t(M)$. Since $\mathfrak{F}_I^t(M)$ is finitely generated, then $\mathfrak{F}_I^t(M) = 0$.

Now let depth$M = 0$ and $N = H_{\mathfrak{m}}^0(M)$, then $\mathfrak{F}_I^0(N) = \varprojlim_n H_{\mathfrak{m}}^0(N/I^nN) = N$ and $\mathfrak{F}_I^i(N) = 0$ for all $i \geq 1$. From the short exact sequence $0 \to N \to M \to M/N \to 0$, we get that $\mathfrak{F}_I^i(M) = \mathfrak{F}_I^i(M/N)$ for all $i \geq 1$. Since depth$M/N > 0$, the desired result follows the above argument.

(1) $\Rightarrow$ (3). Note that $\dim M/IM = \sup\{i \mid \mathfrak{F}_I^i(M) \neq 0\}$. For all $p \in \operatorname{Supp} M$, $\dim R/(I+p) \leq \dim M/IM$, hence $\mathfrak{F}_I^i(R/p) = 0$ for all $i \geq t$.

(3) $\Rightarrow$ (1). It is enough for us to prove that $\mathfrak{F}_I^t(M) = 0$. There is a prime filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ of submodules of M such that $M_j/M_{j-1} \cong R/p_j$, where $p_j \in \operatorname{Supp} M$, $1 \leq j \leq s$. From the exact sequence $\mathfrak{F}_I^t(M_{j-1}) \to \mathfrak{F}_I^t(M_j) \to \mathfrak{F}_I^t(R/p_j)$, we obtain that $\mathfrak{F}_I^t(M) = 0$ by the assumption and induction on j.

The proof of (3) $\Leftrightarrow$ (4) is similar to the proof of (1) $\Leftrightarrow$ (2). ∎

Next corollary is proved in [1, Theorem 2.6 (ii)]. Here we provide an easy method.

Corollary 2.1. *Assume that* $\dim M/IM = c > 0$. *Then* $\mathfrak{F}_I^c(M)$ *is not finitely generated.*

Proof. If $\mathfrak{F}_I^c(M)$ is finitely generated, then $\mathfrak{F}_I^i(M)$ is finitely generated for all $i \geq c$. Hence $\mathfrak{F}_I^i(M) = 0$ for all $i \geq c$ by Theorem 2.1. In fact, $\mathfrak{F}_I^c(M) \neq 0$. It is a contradiction. ∎

Now, we will present one of the main results in this paper.

Theorem 2.2. *Let t be a non-negative integer such that* $\mathfrak{F}_I^i(M)$ *is artinian for all* $i < t$. *Then* $\operatorname{Hom}_R(R/I, \mathfrak{F}_I^t(M))$ *is artinian.*

Proof. Since $\mathfrak{F}_I^i(M) \cong \mathfrak{F}_{I\widehat{R}}^i(\widehat{M})$ and

$$\begin{aligned}\operatorname{Hom}_{\widehat{R}}(\widehat{R}/I\widehat{R}, \mathfrak{F}_{I\widehat{R}}^t(\widehat{M})) &\cong \operatorname{Hom}_{\widehat{R}}(R/I \otimes \widehat{R}, \mathfrak{F}_I^t(M)) \\ &= \operatorname{Hom}_R(R/I, \operatorname{Hom}_{\widehat{R}}(\widehat{R}, \mathfrak{F}_I^t(M))) = \operatorname{Hom}_R(R/I, \mathfrak{F}_I^t(M)).\end{aligned}$$

Hence, we can assume that R is complete. Next, we use induction on t. When $t = 0$, we get that $\operatorname{Ass}_R(\mathfrak{F}_I^0(M)) = \{p \in \operatorname{Ass} M : \dim R/(I+p) = 0\}$ by [9, Lemma 4.1], then $V(I) \cap \operatorname{Supp}(\mathfrak{F}_I^0(M)) \subseteq \{\mathfrak{m}\}$, it turns out that $\operatorname{Hom}_R(R/I, \mathfrak{F}_I^0(M))$ is artinian.

Now we suppose that $t > 0$, and the result holds for all values less than t. From the short exact sequence $0 \to H_I^0(M) \to M \to M/H_I^0(M) \to 0$, one has the following long exact sequence

$$\cdots \to H_{\mathfrak{m}}^i(H_I^0(M)) \to \mathfrak{F}_I^i(M) \to \mathfrak{F}_I^i(M/H_I^0(M)) \to H_{\mathfrak{m}}^{i+1}(H_I^0(M)) \to \cdots$$

by [1, Lemma 2.3], so $\mathfrak{F}_I^i(M/H_I^0(M))$ is artinian for all $i < t$. We split the exact sequence

$$H_{\mathfrak{m}}^t(H_I^0(M)) \to \mathfrak{F}_I^t(M) \xrightarrow{f} \mathfrak{F}_I^t(M/H_I^0(M)) \xrightarrow{g} H_{\mathfrak{m}}^{t+1}(H_I^0(M))$$

to the following exact sequences

$$0 \to \ker f \to \mathfrak{F}_I^t(M) \to \operatorname{im} f \to 0$$

and

$$0 \to \operatorname{im} f \to \mathfrak{F}_I^t(M/H_I^0(M)) \to \operatorname{im} g \to 0.$$

Then we have the following exact sequences

$$\begin{aligned}0 \to \operatorname{Hom}_R(R/I, \ker f) &\to \operatorname{Hom}_R(R/I, \mathfrak{F}_I^t(M)) \\ &\to \operatorname{Hom}_R(R/I, \operatorname{im} f) \to \operatorname{Ext}_R^1(R/I, \ker f) \to \cdots,\end{aligned}$$

$$\begin{aligned}0 \to \operatorname{Hom}_R(R/I, \operatorname{im} f) &\to \operatorname{Hom}_R(R/I, \mathfrak{F}_I^t(M/H_I^0(M))) \\ &\to \operatorname{Hom}_R(R/I, \operatorname{im} g) \to \cdots.\end{aligned}$$

Note that $\ker f$ and $\operatorname{im} g$ are artinian, it is enough to show that $\operatorname{Hom}_R(R/I,\mathfrak{F}_I^t(M/H_I^0(M)))$ is artinian. So, we may assume that $H_I^0(M)=0$. Then there is an M-regular element $x\in I$. The short exact sequence $0\to M\xrightarrow{x} M\to M/xM\to 0$ provides the long exact sequence

$$\cdots\to\mathfrak{F}_I^i(M)\xrightarrow{x}\mathfrak{F}_I^i(M)\to\mathfrak{F}_I^i(M/xM)\to\mathfrak{F}_I^{i+1}(M)\xrightarrow{x}\mathfrak{F}_I^{i+1}(M)\to\cdots. \tag{2.1}$$

This induces that $\mathfrak{F}_I^i(M/xM)$ is artinian for all $i<t-1$. So $\operatorname{Hom}_R(R/I,\mathfrak{F}_I^{t-1}(M/xM))$ is artinian by the inductive hypothesis. From (2.1) we get the exact sequence

$$0\to\mathfrak{F}_I^{t-1}(M)/x\mathfrak{F}_I^{t-1}(M)\to\mathfrak{F}_I^{t-1}(M/xM)\to(0:_{\mathfrak{F}_I^t(M)}x)\to 0,$$

which induces the exact sequence

$$\operatorname{Hom}_R(R/I,\mathfrak{F}_I^{t-1}(M/xM))\to\operatorname{Hom}_R(R/I,(0:_{\mathfrak{F}_I^t(M)}x))\to\operatorname{Ext}_R^1(R/I,\mathfrak{F}_I^{t-1}(M)/x\mathfrak{F}_I^{t-1}(M)).$$

It follows that $\operatorname{Hom}_R(R/I,(0:_{\mathfrak{F}_I^t(M)}x))$ is artinian. Since $x\in I$, we have that

$$\operatorname{Hom}_R(R/I,(0:_{\mathfrak{F}_I^t(M)}x))\cong\operatorname{Hom}_R(R/I\otimes R/xR,\mathfrak{F}_I^t(M))\cong\operatorname{Hom}_R(R/I,\mathfrak{F}_I^t(M)),$$

and so $\operatorname{Hom}_R(R/I,\mathfrak{F}_I^t(M))$ is artinian. ∎

Theorem 2.3. *Let M be a non-zero finitely generated R-module and let t be a non-negative integer. Then the following statements are equivalent:*

(a) $\mathfrak{F}_I^i(M)$ *is artinian for all $i>t$;*
(b) $I\subseteq\operatorname{Rad}(0:\mathfrak{F}_I^i(M))$ *for all $i>t$.*

Proof. $(a)\Rightarrow(b)$. Let $i>t$. Since $\mathfrak{F}_I^i(M)$ is artinian, we get that $I^s\mathfrak{F}_I^i(M)=0$ for some positive integer s by [7, Proposition 2.1]. So $I\subseteq\operatorname{Rad}(0:\mathfrak{F}_I^i(M))$ for all $i>t$.

$(b)\Rightarrow(a)$. We use induction on $d=\dim M$. For $d=0$, $\mathfrak{F}_I^i(M)=0$ for all $i>0$. So, in this case the claim holds. Now, let $d>0$ and assume that the claim holds for all values less than d. One has the following long exact sequence

$$\cdots\to H_{\mathfrak{m}}^i(H_I^0(M))\to\mathfrak{F}_I^i(M)\to\mathfrak{F}_I^i(M/H_I^0(M))\to H_{\mathfrak{m}}^{i+1}(H_I^0(M))\to\cdots \tag{2.2}$$

by [1, Lemma 2.3]. So, it is enough to prove that $\mathfrak{F}_I^i(M/H_I^0(M))$ is artinian for all $i>t$. From (2.2) we can see that $I\subseteq\operatorname{Rad}(0:\mathfrak{F}_I^i(M/H_I^0(M)))$ for all $i>t$. Thus, we may assume that $H_I^0(M)=0$. Then there is an M-regular element $x\in I$. For all $i>t$, there exists a positive integer s_i such that $x^{s_i}\mathfrak{F}_I^i(M)=0$ by hypothesis. The short exact sequence $0\to M\xrightarrow{x^{s_i}} M\to M/x^{s_i}M\to 0$ provides the exact sequence

$$0\to\mathfrak{F}_I^i(M)\to\mathfrak{F}_I^i(M/x^{s_i}M)\to\mathfrak{F}_I^{i+1}(M)$$

for all $i>t$. This induces that $I\subseteq\operatorname{Rad}(0:\mathfrak{F}_I^i(M/x^{s_i}M))$ is artinian and by the inductive hypothesis $\mathfrak{F}_I^i(M/x^{s_i}M)$ is artinian for all $i>t$. Hence $\mathfrak{F}_I^i(M)$ is artinian for all $i>t$. ∎

Assume that M and N are finitely generated R-modules. Set $q(I,M):=\sup\{i\mid\mathfrak{F}_I^i(M)$ is not artinian$\}=\sup\{i\mid I\nsubseteq\operatorname{Rad}(0:\mathfrak{F}_I^i(M))\}$ and $f_I(M,N)=\inf\{i\mid H_I^i(M,N)$ is not finitely generated$\}$.

Remark 2.1. [1, Example 4.3(i)] In general, $\operatorname{Supp}M=\operatorname{Supp}N$ not necessarily lead to $\operatorname{fgrade}(I,M)=\operatorname{fgrade}(I,N)$ for any finitely generated R-modules M and N. For example, let $(R,\mathfrak{m})$ be a 2-dimensional regular local ring and I an ideal with $\dim R/I=1$. The Hartshorne-Lichtenbaum Vanishing Theorem yields that $\operatorname{cd}(I,R)=1$, $\operatorname{cd}(I,R/\mathfrak{m})=0$, fgrade$(I,R)=1$ and fgrade$(I,R/\mathfrak{m})=0$. Set $M=:R\oplus R/\mathfrak{m}$. Then M is a 2-dimensional sequentially Cohen-Macaulay R-module and $\operatorname{Supp}M=\operatorname{Supp}R$, but fgrade$(I,M)=\inf\{$fgrade (I,R), fgrade$(I,R/\mathfrak{m})\}=0$. However, we have the following result.

Proposition 2.6. *Let M and L be finitely generated R-modules and* $\operatorname{Supp}L\subseteq\operatorname{Supp}M$. *Then* $q(I,L)\le q(I,M)$. *In particular, if* $\operatorname{Supp}L=\operatorname{Supp}M$. *Then* $q(I,M)=q(I,L)$.

Proof. Since $\mathfrak{F}^i_I(K)\cong\mathfrak{F}^i_{I\widehat{R}}(\widehat{K})$ for any R-module K and all $i\ge 0$. Therefore, we may assume that R is complete. Then, by Cohen's Structure Theorem, R is a homomorphic image of a regular complete local ring $(T,\mathfrak{n})$ such that $R=T/J$ for some ideal J of T. Set $b:=I\cap T$. In view of [1, Lemma 2.1], we have that

$$\mathfrak{F}^i_I(M)\cong\mathfrak{F}^i_b(M)\cong\operatorname{Hom}_T(H_b^{\dim T-i}(M,T),E_T(T/\mathfrak{n}))$$

and

$$\mathfrak{F}^i_I(L)\cong\mathfrak{F}^i_b(L)\cong\operatorname{Hom}_T(H_b^{\dim T-i}(L,T),E_T(T/\mathfrak{n}))$$

for all $i\ge 0$. It induces that

$$\begin{aligned} q(I,M)&=\sup\{i\mid H_b^{\dim T-i}(M,T)\text{ is not finitely generated}\}\\ &=\dim T-\inf\{i\mid H_b^i(M,T)\text{ is not finitely generated}\}=\dim T-f_b(M,T)\end{aligned}$$

and $q(I,L)=\dim T-\inf\{i\mid H_b^i(L,T)$ is not finitely generated$\}=\dim T-f_b(L,T)$. The claim follows by [2, Theorem 2.1]. ∎

Next, we will give a proposition, before this, we give a lemma.

Lemma 2.2. *Let* $0\to M_1\to M_1\oplus M_2\to M_2\to 0$ *be an exact sequence of finitely generated R-modules. Then* $q(I,M_1\oplus M_2)=\sup\{q(I,M_1),q(I,M_2)\}$.

Proof. As formal local cohomology functor is additive, the result is clear. ∎

Proposition 2.7. $q(I,M)=\sup\{q(I,R/p)\mid p\in\operatorname{Supp}M\}$.

Proof. Set $K:=\oplus_{p\in\operatorname{Ass}M}R/p$. Then K is finitely generated and $\operatorname{Supp}K=\operatorname{Supp}M$. So we have that

$$q(I,M)=q(I,K)=\sup\{q(I,R/p)\mid p\in\operatorname{Ass}M\}=\sup\{q(I,R/p)\mid p\in\operatorname{Supp}M\},$$

where the first equality is by Proposition 2.6, the second equality follows by Lemma 2.2. ∎

Theorem 2.4. *Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring, I_1 and I_2 be two ideals of R such that $I_1\subseteq I_2$, and M a finitely generated R-module of dimension n. Then there is a surjective homomorphism:* $\mathfrak{F}^n_{I_1}(M)\to\mathfrak{F}^n_{I_2}(M)$.

Proof. Let $\overline{R}=R/\operatorname{Ann}_R M$. Note that $\mathfrak{F}^i_{I_1}(M)\cong\mathfrak{F}^i_{I_1\overline{R}}(M)$ and $\mathfrak{F}^i_{I_2}(M)\cong\mathfrak{F}^i_{I_2\overline{R}}(M)$. So we can assume that $\operatorname{Ann}_R M=0$, and then $\dim R=n$. We may assume that R is complete by [9, Theorem 3.3]. Then, by Cohen's Structure Theorem, there exists a complete regular local ring $(T,\mathfrak{n})$ such that $R=T/J$ for some ideal J of T. Set $J_1=I_1\cap J$ and $J_2=I_2\cap J$. Since $\dim_R M=\dim_T M$, $\mathfrak{F}^n_{I_1}(M)\cong\mathfrak{F}^n_{J_1}(M)$ and $\mathfrak{F}^n_{I_2}(M)\cong\mathfrak{F}^n_{J_2}(M)$. Thus we may assume that $R=T$. Then by [1, Lemma 2.1], it follows that

$$\mathfrak{F}^n_{I_1}(M)\cong\operatorname{Hom}_T(H^0_{J_1}(M,T),E_T(T/\mathfrak{n}))$$

and

$$\mathfrak{F}^n_{I_2}(M)\cong\operatorname{Hom}_T(H^0_{J_2}(M,T),E_T(T/\mathfrak{n})).$$

Since $H^0_{J_2}(M,T)$ is a submodule of $H^0_{J_1}(M,T)$, the result is follows. ∎

Remark 2.2. In the above theorem, if $\mathfrak{F}^n_{I_1}(M) = \mathfrak{F}^n_{I_2}(M) = 0$, then the result always holds. Now, we construct an example such that $\mathfrak{F}^n_{I_1}(M) \neq 0$ and $\mathfrak{F}^n_{I_2}(M) \neq 0$. Let k be a field. Let $R = k[[x,y]]$ denote the formal power series ring in two variables over k. Put $I_1 = (x^2)R$, $I_2 = (x)R$ and $M = R/I_2$. Then $I_1 \subseteq I_2$ and $\dim M = 1$, $\mathfrak{F}^1_{I_1}(M) \neq 0$ and $\mathfrak{F}^1_{I_2}(M) \neq 0$.

Proposition 2.8. *Let* $(R,\mathfrak{m})$ *be a commutative Noetherian local ring of dimension n and M a finitely generated R-module. Then* $\operatorname{Coass}\mathfrak{F}^n_I(M) \subseteq \{p \in \operatorname{Spec} R \mid p \supseteq \operatorname{Ann} M, \dim R/p = n\}$.

Proof. Since $\operatorname{Coass}\mathfrak{F}^n_I(M) = \operatorname{Coass}(\mathfrak{F}^n_I(R) \otimes M) = \operatorname{Supp} M \cap \operatorname{Coass}\mathfrak{F}^n_I(R)$, let $p \in \operatorname{Coass}\mathfrak{F}^n_I$ (M), we have that $p \supseteq \operatorname{Ann} M$ and $p \in \operatorname{Coass}\mathfrak{F}^n_I(R/p)$, then $\dim R/p = n$. ∎

Remark 2.3.

(1) In Proposition 2.8, if $\mathfrak{F}^n_I(M) = 0$, then the result is clear. Here, we give an example such that $\mathfrak{F}^n_I(M) \neq 0$. To this end, let R be a local domain of dimension 3, $I = (0)$ and $M = R$. Then $\mathfrak{F}^3_{(0)}(R) \neq 0$.

(2) The inclusion in the above Proposition is not an equality in general. Let R be a local domain of dimension 3 and I an ideal of R of dimension 1. Then $\operatorname{Coass}\mathfrak{F}^3_I(R) = \emptyset$, but $(0) \in \{p \in \operatorname{Spec} R \mid p \supseteq \operatorname{Ann} R, \dim \operatorname{Ann} R, \dim R/p = 3\}$.

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