BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37**(2) (2014), 465–485

Poisson Difference Integer Valued Autoregressive Model of Order One

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Abstract. This paper aims to model integer valued time series with possible negative values and either positive or negative correlations by introducing the Poisson difference integer valued autoregressive model of order one. This model has Poisson difference marginal distribution and is defined by a new operator called the extended binomial thinning operator. It includes previous integer valued autoregressive of order one model as special cases. The model can be used as a tool to model non-stationary count data. The model is applied to data from the Saudi stock exchange.

2010 Mathematics Subject Classification: 37M10, 62M10

Keywords and phrases: Integer valued time series, nonstationary, Poisson difference distribution, extended binomial distribution.

1. Introduction

Various models have been proposed for stationary discrete time series. Jacobs and Lewis [10–12] introduced what they had called the discrete autoregressive-moving average (DARMA) models, which were obtained by a probabilistic mixture of a sequence of independent identically distributed discrete random variables. Al-Osh and Alzaid [1], Alzaid and Al-Osh [2] and McKenzie [15] introduced the integer-valued autoregressive-moving average (INARMA) models. The INARMA models are defined on the basis of binomial thinning operator. In these models, the Poisson distribution plays the same role as the Normal distribution in Box-Jenkins models in terms of time reversibility and linear backward regression properties.

The nonstationary integer-valued time series are frequently encountered in the real life problems. Waiter *et al.* [17], Anderson and Grenfell [5] and Zaidi *et al.* [18] used the real valued ARIMA to model such kind of data. However, when the time series consists of small counts, this model may be inappropriate. Kim and Park [14] introduced an integer-valued autoregressive process of order p with signed binomial thinning operator (INARS (p)). Karlis and Anderson [13] defined the ZINAR process, as an extension of the INAR model using the signed binomial thinning operator and studied the case where the innovation has Skellam distribution. Freeland [8] defined the true integer-valued autoregressive process

Communicated by Anton Abdulbasah Kamil.
Received: November 23, 2011; *Revised:* February 29, 2012.

of order one (TINAR(1)) as the difference of two INAR processes which requires observing the two processes.

The aims of this paper are to define a model that can handle nonstationary integer valued time series, to model integer valued time series with possible negative values and to model integer valued time series with either positive or negative correlations. The paper is organized as follows. In Section 2, we define the extended binomial thinning operator. The Poisson difference integer valued autoregressive model of order one is introduced in Section 3. In Section 4, we study the properties of the model and the question of time reversibility. The estimation of the model parameters is discussed in Section 5. Section 6 includes applications from the Saudi stock exchange.

In the rest of this section we recall some definitions that are needed in the sequel.

Definition 1.1. *Let X be a non-negative integer valued random variable, then for any $\alpha \in (0,1)$ the "$\circ$" binomial thinning operator which is due to Steutel and Van Harn* [16] *is defined by*

$$\alpha \circ X = \sum_{i=1}^{X} Y_i \tag{1.1}$$

where $\{Y_i\}$ is a sequence of i.i.d. random variables, independent of X, such that

$$P(Y_i = 1) = 1 - P(Y_i = 0) = \alpha.$$

Al-Osh and Alzaid [1] introduced the integer valued autoregressive process of order one (INAR (1)).

Definition 1.2. *The INAR (1) process $\{X_t;\ t \in Z\}$ is defined by*

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t \tag{1.2}$$

where $\alpha \in (0,1)$ and $\{\varepsilon_t\}$ is a sequence of i.i.d. non-negative integer valued random variables having mean μ and variance σ^2.

Note that for any fixed parameter $\alpha \in (0,1)$ and any non-negative integer valued random variable X_t, the random variables $\alpha \circ X_t | X_t = x_t \sim Binomial(x_t, \alpha)$ are assumed to be independent of the history of the process and from the sequence$\{\varepsilon_t\}$.

The INAR (1) process has many properties similar to the AR (1) process. For example, any discrete self decomposable distribution can serve as a marginal distribution for the INAR (1). The Poisson distribution almost plays the same role of the normal distribution. Assuming that $\{\varepsilon_t\}$ is a sequence of i.i.d. Poisson (λ) then the process $\{X_t\}$ has Poisson marginal distribution with mean $\mu = \lambda/(1-\alpha)$.

Kim and Park [14] introduced an operator called the signed binomial thinning to develop the INARS (p).

Definition 1.3. *Let α be a real number on $(-1,1)$ and $\{w_{tj}(\alpha)\}$ be i.i.d. Bernoulli random variables with $P(w_{tj}(\alpha) = 1) = |\alpha|$ for each given t. Define $\operatorname{sgn}(x) = 1$ if $x \geqslant 0$ and $\operatorname{sgn}(x) = -1$ if $x < 0$. Using this notation, the signed binomial thinning is formally defined as*

$$\alpha \bullet y_t \equiv \operatorname{sgn}(\alpha)\operatorname{sgn}(y_t) \sum_{j=1}^{|y_t|} w_{tj}(\alpha) \tag{1.3}$$

where the subscript t in $w_{tj}(\alpha)$ describes the observed time of process y_t. When $y_t \geqslant 0$ and $\alpha \geqslant 0$, the signed binomial thinning is reduced to the binomial thinning operator.

Definition 1.4. *The integer-valued autoregressive process of order p with signed binomial thinning by Kim and Park* [14] *is defined by*

$$y_t = \sum_{i=1}^{p} \alpha_i \bullet y_{t-i} + \varepsilon_t, \quad t = 0, \pm 1, \pm 2, \dots \tag{1.4}$$

where the signed binomial thinning operator is given in (1.3), $\{\varepsilon_t\}$ *is a sequence of i.i.d. integer-valued random variables with mean* μ_ε *and variance* $\sigma_\varepsilon^2, 0 \leqslant |\alpha_i| \leqslant 1$ *for* $i = 1, \dots, p$. *The* $\{\varepsilon_t\}$ *are uncorrelated with* y_{t-i} *for* $i \geqslant 1$ *and the counting series* $w_{tj}(\alpha)$ *in the signed binomial thinning are i.i.d and independent of* y_t.

Under the condition that all roots of the polynomial $\lambda^p - \alpha_1 \lambda^{p-1} - \dots - \alpha_{p-1}\lambda - \alpha_p = 0$ are inside the unit circle, the process y_t is stationary and ergodic.

Karlis and Anderson [13] studied (1.4) for $p = 1$ and $\{\varepsilon_t\}$ has Skellam distribution. However the marginal distribution of the process does not has Skellam distribution. They computed the moment and conditional maximum likelihood estimates.

2. The extended binomial operator

It is well known that given two independent Poisson random variables the conditional distribution of one of them given their sum has binomial distribution. This idea was the basis for defining the INAR models. Recently, Alzaid and Omair [3] extended this result to the case where the two independent random variables are Poisson difference random variables and called the conditional distribution as the conditional Poisson difference distribution. A special case of this distribution was considered and named as the extended binomial distribution. In an analogy to the INAR models we will use the result of Alzaid and Omair [3] to introduce INAR model with Poisson difference marginal distributions.

For ease of reference, the definition of the Poisson difference distribution and the extended binomial distribution are given.

Definition 2.1. *A random variable Z is said to have Poisson difference (Skellam) distribution with parameters* $\theta_1 \geqslant 0$ *and* $\theta_2 \geqslant 0$ *if its probability mass function (p.m.f.) is given by:*

$$P(Z = z) = e^{-\theta_1 - \theta_2} \left(\frac{\theta_1}{\theta_2}\right)^{\frac{z}{2}} I_z\left(2\sqrt{\theta_1\theta_2}\right), \quad z = \dots, -1, 0, 1, \dots \tag{2.1}$$

where $I_y(x) = (x/2)^y \sum_{k=0}^{\infty} ((x^2/4)^k)/(k!(y+k)!)$ *is the modified Bessel function of the first kind.*

The Poisson difference distribution is denoted by $\mathrm{PD}(\theta_1, \theta_2)$.

Let X_1 and X_2 be two independent Poisson random variables with means $\theta_1 \geqslant 0$ and $\theta_2 \geqslant 0$, respectively. Let $Y_i = X_i + W, i = 1, 2$ where W is a random variable independent of X_1 and X_2. Then $Z = Y_1 - Y_2 = X_1 - X_2$ is $\mathrm{PD}(\theta_1, \theta_2)$.

Alzaid and Omair [4] introduced the following alternative formulas for the probability mass function of the Poisson difference distribution

$$P(Z = z) = e^{-\theta_1 - \theta_2} \theta_1^z {}_0\tilde{F}_1(;z+1; \theta_1\theta_2), \quad z = \dots, -1, 0, 1, \dots$$

using the regularized hypergeometric function ${}_0\tilde{F}_1$, which is defined by

$${}_0\tilde{F}_1(;y;\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{k!\Gamma(y+k)} \tag{2.2}$$

This function is linked with the modified Bessel function of the first kind through the identity

$$I_y(\theta) = \left(\frac{\theta}{2}\right)^y {}_0\tilde{F}_1\left(;y+1;\frac{\theta^2}{4}\right). \tag{2.3}$$

Definition 2.2. *A random variable X in Z has extended binomial distribution with parameters $0 < p < 1(q = 1-p), \theta > 0$ and $z \in Z$, denoted by $X \sim \mathrm{EB}(z,p,\theta)$ if*

$$P(X=x) = \frac{p^x q^{z-x} {}_0\tilde{F}_1\left(;x+1;p^2\theta\right) {}_0\tilde{F}_1\left(;z-x+1;q^2\theta\right)}{{}_0\tilde{F}_1\left(;z+1;\theta\right)}, \quad x = \ldots,-1,0,1,\ldots. \tag{2.4}$$

For $X \sim \mathrm{EB}\left(z,p,\theta\right)$:

I. The characteristic function:

$$\Phi_X(t) = \left(pe^{it}+q\right)^z \frac{{}_0\tilde{F}_1\left(;z+1;\theta\left(pqe^{it}+pqe^{-it}+1-2pq\right)\right)}{{}_0\tilde{F}_1\left(;z+1;\theta\right)}. \tag{2.5}$$

II. The mean:

$$E(X) = pz. \tag{2.6}$$

III. The variance:

$$V(X) = zpq + 2pq\theta \frac{{}_0\tilde{F}_1(;z+2;\theta)}{{}_0\tilde{F}_1(;z+1;\theta)} \tag{2.7}$$

Next, we will introduce a new operator which will be used in defining the PDINAR (1) model.

Definition 2.3. *Let Z be an integer-valued random variable (which can take negative integers); then for any $\alpha \in [0,1]$ and $\theta \geqslant 0$ the extended binomial thinning operator denoted by "$S_{\alpha,\theta}(Z)$" is defined such that $S_{\alpha,\theta}(Z)|Z \sim \mathrm{EB}(Z,\alpha,\theta)$.*

The extended binomial thinning operator has the following representation

$$S_{\alpha,\theta}(Z) = (\operatorname{sgn} Z)\sum_{i=1}^{|Z|} Y_i + \sum_{i=1}^{W(Z)} B_i, \tag{2.8}$$

where Y_i is a sequence of i.i.d. random variables, independent of B_i, Z and $W(Z)$, such that $P(Y_i = 1) = 1 - P(Y_i = 0) = \alpha, \{B_i\}$ is a sequence of i.i.d. random variables independent of $\{Y_i\}, Z$ and $W(Z)$ such that

$$P(B_i = 1) = P(B_i = -1) = \alpha(1-\alpha) \quad \text{and} \quad P(B_i = 0) = 1 - 2\alpha(1-\alpha)$$

and $W(Z)|Z = z$ is a random variable having Bessel distribution with parameters $(|z|,\theta)$, (see, for example, Devroye [7]).

Since $\sum_{i=1}^{|Z|} Y_i|Z = z \sim binomial(|z|,\alpha), \sum_{i=1}^{W(Z)} B_i|Z = z$ has the distribution with characteristic function given by

$$\Phi(t) = \frac{{}_0\tilde{F}_1\left(;|z|+1;\theta\left(\alpha\bar{\alpha}e^{it}+\alpha\bar{\alpha}e^{-it}+1-2\alpha\bar{\alpha}\right)\right)}{{}_0\tilde{F}_1\left(;|z|+1;\theta\right)},$$

where $\bar{\alpha} = 1-\alpha$ and since they are independent it is clear that $S_{\alpha,\theta}(Z)|Z = z \sim \mathrm{EB}(z,\alpha,\theta)$. See Alzaid and Omair [3].

Remark 2.1.

I. The extended binomial thinning includes the binomial thinning as a special case when Z is non-negative integer valued random variable and $\theta = 0$.

II. The extended binomial thinning operator multiplied by a sign yields the binomial signed operator of Kim and Park [14] as a special case when $\theta = 0$ as we will see in the next section.

3. The Poisson difference integer valued autoregressive model of order one

In this section, we will define a new integer-valued autoregressive process of order one that can handle negative integer-valued time series and allow for both positive and negative autocorrelation. This process is called Poisson difference integer-valued process (PDINAR(1)). Unlike the PINAR (1) where only positive correlation is obtained, in the PDINAR(1) process we can model processes with positive and negative correlation. We will use the notation PDINAR$^+$(1) for the process with positive correlation and PDINAR$^-$(1) for the process with negative correlation.

Definition 3.1. *Let $\{\varepsilon_t\}$ be a sequence of i.i.d. random variables with the Poisson difference distribution* $\mathrm{PD}(\theta_1, \theta_2)$. *The PDINAR (1) process $\{Z_t\}$ is defined by*

$$Z_t = \delta S_{\alpha,\theta}(Z_{t-1}) + \varepsilon_t \quad t = 0, 1, 2, \ldots, \tag{3.1}$$

where $\{S_{\alpha,\theta}(Z_t)\}$ is a sequence of i.i.d. integer-valued random variables that arise from Z_t by the extended binomial thinning and are assumed to be independent of the history of the process and from the sequence $\{\varepsilon_t\}$. δ is a parameter with possible values 1 *and* -1 *describing the sign of the correlation.*

Throughout we denote by PDINAR$^+$(1) or PDINAR$^-$(1) to the PDINAR(1) according to the value of δ is 1 or -1 respectively.

Let $S_{\alpha,\theta}(Z_t)|Z_t \sim \mathrm{EB}(Z_t, \alpha, \theta)$ such that

$$\alpha \in (0,1), \quad \theta = \frac{1}{4}\left(\left(\frac{\theta_1+\theta_2}{1-\alpha}\right)^2 - \left(\frac{\theta_1-\theta_2}{1-\delta\alpha}\right)^2\right).$$

It is also assumed that

$$Z_0 \sim \mathrm{PD}\left(\frac{1}{2}\left(\frac{\theta_1+\theta_2}{1-\alpha} + \frac{\theta_1-\theta_2}{1-\delta\alpha}\right), \frac{1}{2}\left(\frac{\theta_1+\theta_2}{1-\alpha} - \frac{\theta_1-\theta_2}{1-\delta\alpha}\right)\right).$$

According to the above definition, the process is Markovian.

The following proposition is proved in Alzaid and Omair [3].

Proposition 3.1. *If $Z \sim \mathrm{PD}(\theta_1, \theta_2)$ and $X|Z = z \sim \mathrm{EB}(z, p, \theta)$, where $\theta = \theta_1\theta_2$ then X and $Z - X$ are independent and $X \sim \mathrm{PD}(p\theta_1, p\theta_2)$ and $Z - X \sim \mathrm{PD}(q\theta_1, q\theta_2)$.*

Proposition 3.2. *Under all the above conditions, the PDINAR(1) process is a stationary Markov process with the Poisson difference marginal distribution having parameters*

$$\left(\frac{1}{2}\left(\frac{\theta_1+\theta_2}{1-\alpha} + \frac{\theta_1-\theta_2}{1-\delta\alpha}\right), \frac{1}{2}\left(\frac{\theta_1+\theta_2}{1-\alpha} - \frac{\theta_1-\theta_2}{1-\delta\alpha}\right)\right).$$

Proof.

Case 1. When $\delta = 1$, the process is PDINAR$^+(1)$.
According to Proposition 3.1, if

$$Z_0 \sim \mathrm{PD}\left(\frac{1}{2}\left(\frac{\theta_1+\theta_2}{1-\alpha}+\frac{\theta_1-\theta_2}{1-\alpha}\right)=\frac{\theta_1}{1-\alpha},\frac{1}{2}\left(\frac{\theta_1+\theta_2}{1-\alpha}-\frac{\theta_1-\theta_2}{1-\alpha}\right)=\frac{\theta_2}{1-\alpha}\right)$$

and

$$S_{\alpha,\theta}(Z_0)|Z_0 \sim \mathrm{EB}\left(Z_0,\alpha,\frac{\theta_1\theta_2}{(1-\alpha)^2}\right),$$

then

$$S_{\alpha,\theta}(Z_0) \sim \mathrm{PD}\left(\frac{\alpha\theta_1}{1-\alpha},\frac{\alpha\theta_2}{1-\alpha}\right).$$

Since $S_{\alpha,\theta}(Z_0) \sim \mathrm{PD}\left(\alpha\theta_1/(1-\alpha),\alpha\theta_2/(1-\alpha)\right)$ is independent of $\varepsilon_1 \sim \mathrm{PD}(\theta_1,\theta_2)$ and the sum of two independent Poisson difference random variables is Poisson difference random variable, we conclude that Z_1 has the Poisson difference distribution with parameters $\theta_1/(1-\alpha)$ and $\theta_2/(1-\alpha)$. That is Z_1 has the same distribution as Z_0. Since the process is Markovian, $\{Z_t\}$ is stationary $\mathrm{PD}\left(\theta_1/(1-\alpha),\theta_2/(1-\alpha)\right)$.

Case 2. When $\delta = -1$, the process is PDINAR$^-(1)$.
According to Proposition 3.1, if

$$Z_0 \sim \mathrm{PD}\left(\frac{1}{2}\left(\frac{\theta_1+\theta_2}{1-\alpha}+\frac{\theta_1-\theta_2}{1+\alpha}\right)=\frac{\theta_1+\alpha\theta_2}{1-\alpha^2},\frac{1}{2}\left(\frac{\theta_1+\theta_2}{1-\alpha}-\frac{\theta_1-\theta_2}{1+\alpha}\right)=\frac{\theta_2+\alpha\theta_1}{1-\alpha^2}\right)$$

and

$$S_{\alpha,\theta}(Z_0)|Z_0 \sim \mathrm{EB}\left(Z_0,\alpha,\left(\frac{\theta_1+\alpha\theta_2}{1-\alpha^2}\right)\left(\frac{\theta_2+\alpha\theta_1}{1-\alpha^2}\right)\right)$$

then

$$S_{\alpha,\theta}(Z_0) \sim \mathrm{PD}\left(\alpha\frac{\theta_1+\alpha\theta_2}{1-\alpha^2},\alpha\frac{\theta_2+\alpha\theta_1}{1-\alpha^2}\right).$$

Since $S_{\alpha,\theta}(Z_0) \sim \mathrm{PD}\left(\alpha(\theta_1+\alpha\theta_2)/(1-\alpha^2),\alpha(\theta_2+\alpha\theta_1)/(1-\alpha^2)\right)$ is independent of $\varepsilon_1 \sim \mathrm{PD}(\theta_1,\theta_2)$, Z_1 has the Poisson difference distribution as the difference of two independent Poisson difference distributions with parameters $(\theta_1+\alpha\theta_2)/(1-\alpha^2)$ and $(\theta_2+\alpha\theta_1)/(1-\alpha^2)$. That is Z_1 has the same distribution as Z_0. Since the process is Markovian, $\{Z_t\}$ is stationary $\mathrm{PD}\left((\theta_1+\alpha\theta_2)/(1-\alpha^2),(\theta_2+\alpha\theta_1)/(1-\alpha^2)\right)$. ∎

4. Properties of the PDINAR (1) model

In this section, we discuss some distributional properties of the PDINAR (1) model.

1. The conditional mean is

$$E(Z_t|Z_{t-1}) = \alpha\delta Z_{t-1}+\theta_1-\theta_2 \tag{4.1}$$

and hence it is linear in Z_{t-1}. This implies that the PDINAR (1) model can be viewed as a new member of the conditional linear model of Grunwald [9].

2. The conditional variance is given by

$$V(Z_t|Z_{t-1}) = \alpha(1-\alpha)Z_{t-1} + 2\alpha(1-\alpha)\theta \frac{{}_0\tilde{F}_1\left(;Z_{t-1}+2;\theta\right)}{{}_0\tilde{F}_1\left(;Z_{t-1}+1;\theta\right)} + \theta_1 + \theta_2, \tag{4.2}$$

where

$$\theta = \frac{1}{4}\left(\left(\frac{\theta_1+\theta_2}{1-\alpha}\right)^2 - \left(\frac{\theta_1-\theta_2}{1-\delta\alpha}\right)^2\right) = \begin{cases} \frac{\theta_1\theta_2}{(1-\alpha)^2} & \delta = 1 \\ \left(\frac{\theta_1+\alpha\theta_2}{1-\alpha^2}\right)\left(\frac{\theta_2+\alpha\theta_1}{1-\alpha^2}\right) & \delta = -1 \end{cases}$$

which is clearly not linear.

3. The unconditional mean is given by

$$E(Z_t) = \frac{\theta_1-\theta_2}{1-\delta\alpha}. \tag{4.3}$$

4. The unconditional variance is given by

$$V(Z_t) = \frac{\theta_1+\theta_2}{1-\alpha}. \tag{4.4}$$

5. For any $k = 0, \pm 1, \pm 2, \ldots$, the covariance is given by

$$\begin{aligned} \gamma_k &= \mathrm{Cov}(Z_{t+k}, Z_t) = \mathrm{Cov}\left(\delta S_{\alpha,\theta}\left(Z_{t+k-1}\right) + \varepsilon_{t+k}, Z_t\right) \\ &= E\left(\mathrm{Cov}\left(\delta S_{\alpha,\theta}\left(Z_{t+k-1}\right) + \varepsilon_{t+k}, Z_t|Z_{t+k-1}\right)\right) \\ &\quad + \mathrm{Cov}\left(E\left(\delta S_{\alpha,\theta}\left(Z_{t+k-1} + \varepsilon_{t+k}|Z_{t+k-1}\right), E\left(Z_t|Z_{t+k-1}\right)\right)\right) \\ &= \mathrm{Cov}\left(E\left(S_{\alpha,\theta}\left(Z_{t+k-1} + \varepsilon_{t+k}|Z_{t+k-1}\right), E\left(Z_t|Z_{t+k-1}\right)\right)\right) \\ &= \delta\alpha\,\mathrm{Cov}\left(Z_{t+k-1}, Z_t\right) = (\delta\alpha)^k\,\mathrm{Cov}(Z_t, Z_t) = (\alpha\delta)^k\left(\frac{\theta_1+\theta_2}{1-\alpha}\right). \end{aligned} \tag{4.5}$$

6. The autocorrelation is

$$\rho_k = \mathrm{Corr}(Z_{t+k}, Z_t) = (\alpha\delta)^k. \tag{4.6}$$

We can see that the autocorrelation function decays exponentially.

7. The one step ahead predictive distribution is

$$P(Z_t = z_t \mid Z_{t-1} = z_{t-1}) = \sum_{i=-\infty}^{\infty} P(S_{\alpha,\theta}\left(z_{t-1}\right) = i \mid z_{t-1})\, P(\varepsilon_t = z_t - \delta i)$$

$$= \sum_{i=-\infty}^{\infty} \frac{\alpha^i(1-\alpha)^{z_{t-1}-i} e^{-\theta_1-\theta_2}\theta_1^{z_t-\delta i}\,{}_0\widetilde{F}_1(;i+1;\theta)\,{}_0\widetilde{F}_1(;z_{t-1}-i+1;\theta)\,{}_0\widetilde{F}_1(;z_t-\delta i+1;\theta_1\theta_2)}{{}_0\widetilde{F}_1(;z_{t-1}+1;\theta)}$$

where

$$\theta = \frac{1}{4}\left(\left(\frac{\theta_1+\theta_2}{1-\alpha}\right)^2 - \left(\frac{\theta_1-\theta_2}{1-\delta\alpha}\right)^2\right).$$

Remark 4.1.

(1) The PDINAR$^+$(1) has the PINAR(1) as a special case when $\theta_2 = 0$. In the PDINAR$^+$(1), if $\theta_2 = 0$, then $\{\varepsilon_t\}$ is a sequence of i.i.d. random variables with Poisson distribution *Poisson* (θ_1) and the extended binomial thinning operator $S_{\alpha,\theta}\left(Z\right)$ reduces to the binomial thinning operator since $\theta_2 = 0$ implies that $\theta = 0$.

(2) If one defines a stationary Poisson difference INARS(1) process with positive correlation, then one of the parameters of the Poisson difference distribution must be zero, i.e it has either a Poisson distribution or the negative of a Poisson distribution. It cannot be a difference of two Poisson distributions with nonzero parameters. Since in the INARS(1) process $\theta = \theta_1\theta_2/((1-\alpha)^2) = 0$ this implies that either $\theta_1 = 0$ or $\theta_2 = 0$. It is impossible to define a stationary Poisson difference INARS (1) process with negative correlation since $\theta = \left((\theta_1+\alpha\theta_2)/(1-\alpha^2)\right)\left((\theta_2+\alpha\theta_1)/(1-\alpha^2)\right) = 0$ implies that both $\theta_1 = 0$ and $\theta_2 = 0$. We mention that Kim and Park [14] did not discuss the marginal distribution of their process.

(3) If $\alpha = 0$, the random variable $S_{\alpha,\theta}(Z_t)|Z_t$ has a degenerate distribution. In this case, the PDINAR(1) reduces to a sequence of i.i.d. random variables PD(θ_1,θ_2).

Proposition 4.1. *The Poisson difference integer-valued autoregressive process of order one with positive correlation PDINAR$^+$(1) is time reversible.*

Proof. Since the PDINAR$^+$(1) is a Markov process, it is enough to compute the bivariate probability characteristic function $\Phi_{Z_{t-1},Z_t}(u,v)$, which is of the form

$$\Phi_{Z_{t-1},Z_t}(u,v) = E\left(e^{iuZ_{t-1}+ivZ_t}\right) = E\left(e^{iuZ_{t-1}}E\left(e^{ivS_{\alpha,\theta}(Z_{t-1})}|Z_{t-1}\right)\right)\Phi_{\in_t}(v)$$

$$= E\left(e^{iuZ_{t-1}}\left(\alpha e^{iv}+\overline{\alpha}\right)^{Z_{t-1}}\frac{{}_0\tilde{F}_1\left(;Z_{t-1}+1;\frac{\theta_1\theta_2}{(1-\alpha)^2}\left(\alpha\overline{\alpha}e^{iv}+\alpha\overline{\alpha}e^{-iv}+1-2\alpha\overline{\alpha}\right)\right)}{{}_0\tilde{F}_1\left(Z_{t-1}+1;\frac{\theta_1\theta_2}{(1-\alpha)^2}\right)}\right)\Phi_{\in_t}(v)$$

$$= e^{-\frac{\theta_1+\theta_2}{1-\alpha}}\sum_{z_{t-1}=-\infty}^{\infty}\left(\left(\frac{\theta_1 e^{iu}\left(\alpha e^{iv}+\overline{\alpha}\right)}{1-\alpha}\right)^{z_{t-1}}\right.$$

$$\left.\times\, {}_0\tilde{F}_1\left(;z_{t-1}+1;\left(\frac{\theta_1 e^{iu}\left(\alpha e^{iv}+\overline{\alpha}\right)}{1-\alpha}\right)\left(\frac{\theta_2 e^{-iu}\left(\alpha e^{-iv}+\overline{\alpha}\right)}{1-\alpha}\right)\right)\right)\Phi_{\in_t}(v)$$

(4.7)

$$= e^{-\frac{\theta_1+\theta_2}{1-\alpha}}e^{\frac{\theta_1}{1-\alpha}e^{iu}\left(\alpha e^{iv}+\overline{\alpha}\right)+\frac{\theta_2}{1-\alpha}e^{-iu}\left(\alpha e^{-iv}+\overline{\alpha}\right)}\Phi_{\in_t}(v)$$

$$= e^{-\frac{\theta_1+\theta_2}{1-\alpha}-\theta_1-\theta_2}e^{\frac{\theta_1\alpha}{1-\alpha}e^{iu}e^{iv}+\theta_1 e^{iu}+\frac{\theta_2\alpha}{1-\alpha}e^{-iu}e^{-iv}+\theta_2 e^{-iu}+\theta_1 e^{iv}+\theta_2 e^{-iv}}$$

where in (4.7) we used the identity$\sum_{x=-\infty}^{\infty}\lambda_1^x {}_0\tilde{F}_1(;x+1;\lambda_1\lambda_2) = e^{\lambda_1+\lambda_2}$.

The bivariate characteristic function is a symmetric function in u and v, which implies $(Z_t,Z_{t-1})\overset{d}{=}(Z_{t-1},Z_t)$ and hence the process is time reversible. Moreover, since the PDINAR$^+$(1) has linear forward regression it will have linear backward regression, that is

$$E(Z_t|Z_{t-1}=z) = E(Z_{t-1}|Z_t=z) = \alpha z+\theta_1-\theta_2. \qquad ∎$$

5. Estimation

Let us assume that we have $n+1$ observations $z_0,z_1,\ldots,z_n$ from PDINAR(1) process. In the PDINAR(1) model we have four parameters to be estimated δ,α,θ_1 and θ_2. Three methods will be considered in this section, Yule-Walker method, conditional maximum likelihood method and conditional least squares method. In all methods δ is estimated by

$$\hat{\delta} = \begin{cases} 1 & \text{if } r_1 > 0 \\ -1 & \text{if } r_1 < 0 \end{cases}$$

where r_1 is the sample autocorrelation function.

1. Yule-Walker Method:

   The simplest way to get an estimator for α is to replace ρ_1 with the sample autocorrelation function r_1 in the Yule-Walker equation and solve for α to obtain

$$\hat{\alpha}_{YW} = \widehat{\delta} r_1 = \widehat{\delta} \frac{\sum_{t=0}^{n-1} (z_t - \bar{z})(z_{t+1} - \bar{z})}{\sum_{t=0}^{n} (z_t - \bar{z})^2},$$

   where $\bar{z}$ is the sample mean.

   The following set of equations are used for estimating θ_1 and θ_2:

$$E(Z_t) = \frac{\theta_1 - \theta_2}{(1 - \delta\alpha)} \quad \text{and} \quad V(Z_t) = \frac{\theta_1 + \theta_2}{(1 - \alpha)}.$$

   For the PDINAR$^+$(1), the Yule-Walker estimators are given by:

$$\begin{aligned} \hat{\alpha}^+_{YW} &= \frac{\sum_{t=0}^{n-1} (z_t - \bar{z})(z_{t+1} - \bar{z})}{\sum_{t=0}^{n} (z_t - \bar{z})^2}, \\ \hat{\theta}^+_{1YW} &= \frac{\left(1 - \hat{\alpha}^+_{YW}\right)}{2} \left(\bar{z} + s_z^2\right), \\ \hat{\theta}^+_{2YW} &= \hat{\theta}^+_{1YW} - \left(1 - \hat{\alpha}^+_{YW}\right) \bar{z}. \end{aligned}$$

   For the PDINAR$^-$(1), the Yule-Walker estimators are given by:

$$\begin{aligned} \hat{\alpha}^-_{YW} &= -\frac{\sum_{t=0}^{n-1} (z_t - \bar{z})(z_{t+1} - \bar{z})}{\sum_{t=0}^{n} (z_t - \bar{z})^2}, \\ \hat{\theta}^-_{1YW} &= \frac{1}{2} \left(\left(1 + \hat{\alpha}^-_{YW}\right) \bar{z} + \left(1 - \hat{\alpha}^-_{YW}\right) s_z^2 \right), \\ \hat{\theta}^-_{2YW} &= \hat{\theta}^-_{1YW} - \left(1 + \hat{\alpha}^-_{YW}\right) \bar{z}. \end{aligned}$$

2. The Conditional Maximum Likelihood CML Method:

   The conditional likelihood for the PDINAR(1) model is defined by

$$\begin{aligned} CL(\alpha, \theta_1, \theta_2) &= \prod_{j=1}^{n} P(Z_t = z_t \mid Z_{t-1} = z_{t-1}) \\ &= \prod_{t=1}^{n} \sum_{i=-\infty}^{\infty} P(S_{\alpha,\theta}(z_t - 1) = i \mid (z_{t-1}) = P(\varepsilon_t = z_t - \delta i) \end{aligned}$$

   For the PDINAR$^+$(1), the CML estimators for $\hat{\alpha}^+_{CML}, \hat{\alpha}^+_{1CML}$ and $\hat{\alpha}^+_{2CML}$ are obtained by maximizing the following conditional likelihood numerically

$$CL^+(\alpha, \theta_1, \theta_2)$$
$$= \prod_{t=1}^{n} \sum_{i=-\infty}^{\infty} \frac{\alpha^i (1-\alpha)^{z_{t-1}-i} e^{-\theta_1 - \theta_2} \theta_1^{z_t - i} {}_0\widetilde{F}_1\left(; i+1; \alpha^2 \theta_1 \theta_2 / (1-\alpha)^2\right) {}_0\widetilde{F}_1(; z_{t-1} - i + 1; \theta_1 \theta_2) {}_0\widetilde{F}_1(; z_t - i + 1; \theta_1 \theta_2)}{{}_0\widetilde{F}_1\left(; z_{t-1} + 1; \theta_1 \theta_2 / (1-\alpha)^2\right)}$$

For the PDINAR^{-}(1), the CML estimators for $\hat{\alpha}^{+}_{CML}, \hat{\alpha}^{+}_{1CML}$ and $\hat{\alpha}^{+}_{2CML}$ are obtained by maximizing the following conditional likelihood numerically

$$CL^{-}(\alpha,\theta_1,\theta_2) = \prod_{t=1}^{n}\sum_{i=-\infty}^{\infty} \frac{\alpha^{i}(1-\alpha)^{z_{t-1}-i}e^{-\theta_1-\theta_2}\theta_1^{z_t+i}{}_0\widetilde{F}_1(;i+1;\alpha^2K)_0\widetilde{F}_1(;z_{t-1}-i+1;(1-\alpha)^2K)_0\widetilde{F}_1(;z_t+i+1;\theta_1\theta_2)}{{}_0\widetilde{F}_1(;z_{t-1}+1;K)}$$

where

$$K=\left(\frac{\theta_1+\alpha\theta_2}{1-\alpha^2}\right)\left(\frac{\theta_2+\alpha\theta_1}{1-\alpha^2}\right).$$

3. Conditional Least Squares Method:

The estimation procedure that we are going to apply was developed by Klimko and Nelson (1978) with some modifications in order to be able to estimate all the parameters. The Conditional least squares method is based on minimization of the sum of squared deviations about the conditional expectation. The CLS estimator minimizes the criterion function S_{1CLS} given by

$$S_{1CLS}=\sum_{t=1}^{n}e_{1t}^2=\sum_{t=1}^{n}\left(Z_t-E\left(Z_t|Z_{t-1}\right)\right)^2=\sum_{t=1}^{n}\left(Z_t-\alpha\delta Z_{t-1}-\theta_1+\theta_2\right)^2.$$

It is clear that differentiating S_{1CLS} with respect to θ_1 and θ_2 and equating the resulting expressions to zero give the same equation. Therefore, θ_1 and θ_2 are not estimable directly. In order to estimate these parameters using conditional least squares method, we will use the following reparametrization

$$\mu=\theta_1-\theta_2,\quad \sigma^2=\theta_1+\theta_2$$

and estimate all the three parameters α,μ and σ^2 as follows.

For the first step, the conditional mean prediction error is considered

$$e_{1t}=Z_t-E\left(Z_t|Z_{t-1}\right)=Z_t-\alpha\delta Z_{t-1}-\mu$$

The CLS estimators of α and μ minimizes the criterion function $S_{1CLS}=\sum_{t=1}^{n}e_{1t}^2$. From the first step we obtain

$$\hat{\alpha}_{CLS}=\hat{\delta}\frac{\sum_{t=1}^{n}Z_tZ_{t-1}-n\bar{Z}\bar{Z}_0}{\sum_{t=1}^{n}Z_{t-1}^2-n\bar{Z}_0^2},\quad \text{where}\quad \bar{Z}=\frac{1}{n}\sum_{t=1}^{n}Z_t\quad\text{and}\quad \bar{Z}_0=(1/n)\sum_{t=1}^{n}Z_{t-1}$$

$$\hat{\mu}_{CLS}=\bar{Z}-\hat{\alpha}_{CLS}\hat{\delta}\bar{Z}_0.$$

Note that in the case of PDINAR^{+}(1), $\hat{\alpha}^{+}_{CLS}$ and $\hat{\mu}^{+}_{CLS}$ (when $\hat{\delta}$=1) are similar to the CLS estimators of the PINAR(1) process.

To obtain an estimate of σ^2 a second step is needed. The normal equations based on the conditional variance prediction error (e_{2t}) are used. Brannas and Quoreshi [6] used the conditional variance prediction error as a second step to obtain feasible generalized least square estimator for a long-lag integer-valued moving average model. The conditional variance prediction error is defined by

$$e_{2t}=\left(Z_t-E\left(Z_t|Z_{t-1}\right)\right)^2-V\left(Z_t|Z_{t-1}\right)$$

The two proposed methods for estimating σ^2 are:

1. Method 1 :

   From the fact that $\sum_{t=1}^{n} e_{2t} = 0$, one can obtain a direct estimator of σ^2 by solving the nonlinear equation in the case of PDINAR$^+$(1)

$$\sum_{t=1}^{n}\left(\hat{e}_{1t}^2 - \hat{\alpha}_{CLS}^{+}\left(1-\hat{\alpha}_{CLS}^{+}\right)Z_{t-1} - 2\hat{\alpha}_{CLS}^{+}\left(1-\hat{\alpha}_{CLS}^{+}\right)A\frac{{}_0\tilde{F}_1\left(;Z_{t-1}+2;A\right)}{{}_0\tilde{F}_1\left(;Z_{t-1}+1;A\right)} - \sigma^2\right) = 0,$$

   where

$$A = \frac{1}{4}\left(\left(\frac{\theta_1+\theta_2}{1-\alpha}\right)^2 - \left(\frac{\theta_1-\theta_2}{1-\alpha}\right)^2\right) = \frac{\sigma^4 - \left(\hat{\mu}_{CLS}^{+}\right)^2}{4\left(1-\hat{\alpha}_{CLS}^{+}\right)^2}.$$

   $\hat{\alpha}_{CLS}^{+}$ and $\hat{\mu}_{CLS}^{+}$ are those estimators obtained from the first step, and $\hat{e}_{1t} = Z_t - \hat{\alpha}_{CLS}^{+}Z_{t-1} - \hat{\mu}_{CLS}^{+}$.

   In the PDINAR$^-$(1) process a direct estimator of σ^2 is obtained by solving the following nonlinear equation

$$\sum_{t=1}^{n}\left(\hat{e}_{1t}^2 - \hat{\alpha}_{CLS}^{-}\left(1-\hat{\alpha}_{CLS}^{-}\right)Z_{t-1} - 2\hat{\alpha}_{CLS}^{-}\left(1-\hat{\alpha}_{CLS}^{-}\right)B\frac{{}_0\tilde{F}_1\left(;Z_{t-1}+2;B\right)}{{}_0\tilde{F}_1\left(;Z_{t-1}+1;B\right)} - \sigma^2\right) = 0,$$

   where $\hat{\alpha}_{CLS}^{-}$ and $\hat{\mu}_{CLS}^{-}$ are those estimators obtained from the first step,

$$B = \frac{1}{4}\left(\left(\frac{\theta_1+\theta_2}{1-\alpha}\right)^2 - \left(\frac{\theta_1-\theta_2}{1+\alpha}\right)^2\right) = \frac{\left(1+\hat{\alpha}_{CLS}^{-}\right)^2\sigma^4 - \left(1-\hat{\alpha}_{CLS}^{-}\right)^2\left(\hat{\mu}_{CLS}^{-}\right)^2}{4\left(1-\hat{\alpha}_{CLS}^{-}\right)^2}$$

   and $\hat{e}_{1t} = Z_t + \hat{\alpha}_{CLS}^{-}Z_{t-1} - \hat{\mu}_{CLS}^{-}$.

2. Method 2:

   Minimize the criterion $S_{2CLS} = \sum_{t=1}^{n} e_{2t}^2$ with respect to σ^2 by differentiating S_{2CLS} with respect to σ^2 and equating the result to zero, with $\hat{\alpha}_{CLS}$ and $\hat{\mu}_{CLS}$ as those estimators obtained from the first step.

   In the case of PDINAR$^+$(1)

$$\frac{\partial S_{2CLS}}{\partial\sigma^2} = 2\sum_{t=1}^{n}\left\{\left(\hat{e}_{1t}^2 - \hat{\alpha}_{CLS}\left(1-\hat{\alpha}_{CLS}\right)Z_{t-1} - \hat{\alpha}_{CLS}\frac{\left(\sigma^4-\hat{\mu}_{CLS}^2\right)}{2\left(1-\hat{\alpha}_{CLS}\right)}\frac{{}_0\tilde{F}_1\left(;Z_{t-1}+2;A\right)}{{}_0\tilde{F}_1\left(;Z_{t-1}+1;A\right)} - \sigma^2\right) * \right.$$
$$\left(-\sigma^2\frac{\hat{\alpha}_{CLS}}{\left(1-\hat{\alpha}_{CLS}\right)}\frac{{}_0\tilde{F}_1\left(;Z_{t-1}+2;A\right)}{{}_0\tilde{F}_1\left(;Z_{t-1}+1;A\right)} - \hat{\alpha}_{CLS}\frac{\left(\sigma^4-\hat{\mu}_{CLS}^2\right)}{\left(1-\hat{\alpha}_{CLS}\right)}\sigma^2/\left(4\left(1-\hat{\alpha}_{CLS}\right)^2\right) * \right.$$
$$\left.\left.\frac{{}_0\tilde{F}_1\left(;Z_{t-1}+1;A\right){}_0\tilde{F}_1\left(;Z_{t-1}+3;A\right) - \left({}_0\tilde{F}_1\left(;Z_{t-1}+2;A\right)\right)^2}{\left({}_0\tilde{F}_1\left(;Z_{t-1}+1;A\right)\right)^2} - 1\right)\right\} = 0$$

   By solving the last nonlinear equation, another estimate of σ^2 is obtained.

   After estimating σ^2 either using the first or the second method, the CLS estimates of θ_1 and θ_2 are obtained from the following set of equations:

$$\hat{\theta}_{1CLS}^{+} = 1/2\left(\hat{\sigma}_{CLS}^{2+} + \hat{\mu}_{CLS}^{+}\right)$$
$$\hat{\theta}_{2CLS}^{+} = 1/2\left(\hat{\sigma}_{CLS}^{2+} - \hat{\mu}_{CLS}^{+}\right).$$

   In the case of PDINAR$^-$(1)

$$\frac{\partial S_{2CLS}}{\partial\sigma^2} = 2\sum_{t=1}^{n}\left\{\left(\hat{e}_{1t}^2 - \hat{\alpha}_{CLS}\left(1-\hat{\alpha}_{CLS}\right)Z_{t-1} - 2\hat{\alpha}_{CLS}\left(1-\hat{\alpha}_{CLS}\right)B * \frac{{}_0\tilde{F}_1\left(;Z_{t-1}+2;B\right)}{{}_0\tilde{F}_1\left(;Z_{t-1}+1;B\right)} - \sigma^2\right) * \right.$$

$$\left(-\hat{\alpha}_{CLS}\frac{\sigma^2}{1-\hat{\alpha}_{CLS}} * \frac{{}_0\tilde{F}_1\left(;Z_{t-1}+2;B\right)}{{}_0\tilde{F}_1\left(;Z_{t-1}+1;B\right)} \right.$$
$$\left. -\hat{\alpha}_{CLS}B\frac{\sigma^2}{1-\hat{\alpha}_{CLS}}\frac{{}_0\tilde{F}_1\left(;Z_{t-1}+1;B\right){}_0\tilde{F}_1\left(;Z_{t-1}+3;B\right)-{}_0\tilde{F}_1\left(;Z_{t-1}+2;B\right)^2}{{}_0\tilde{F}_1\left(;Z_{t-1}+1;B\right)^2}-1\right)\Bigg\}$$
$$=0$$

By solving the last nonlinear equation, another estimate of σ^2 is obtained.

After estimating σ^2 either using the first or the second method, the CLS estimates of θ_1 and θ_2 are obtained from the following set of equations:

$$\hat{\theta}^-_{1CLS} = 1/2\left(\hat{\sigma}^{2-}_{CLS}+\hat{\mu}^-_{CLS}\right)$$
$$\hat{\theta}^-_{2CLS} = 1/2\left(\hat{\sigma}^{2-}_{CLS}-\hat{\mu}^-_{CLS}\right).$$

The CLS estimates obtained using the first method and the second method will be denoted by CLS (1) and CLS (2), respectively.

To provide an idea about the relative merits of each of the methods of estimation, a Monte Carlo study is conducted. The estimation of α,θ_1 and θ_2 in the stationary PDINAR (1) by the Yule-Walker method, the conditional maximum likelihood and the CLS using the two proposed method are compared. The conditional maximum likelihood has an infinite sum, it was approximated to a finite one i ranges from -50 to 50 since increasing this bound to larger values made no difference. To generate the data, the parameters selected are $(\theta_1,\theta_2) = (1,1),(5,5),(2,4),(4,2)$, $\alpha = 0.2,0.5,0.7$ and $n = 50,100,200$. 1000 replications were made on each sample, and the average bias and the mean square error of the parameters' estimates are calculated. All simulation and estimation procedures are done using the Mathematica 8 software.

Steps of generation of PDINAR(1):

1. Generate n independent observations from $PD(\theta_1,\theta_2)$ which serves as $\{\varepsilon_t\}_{t=1}^n$.
2. Generate one observation from
$$PD\left(\left(\frac{1}{2}\right)\left(\frac{\theta_1+\theta_2}{1-\alpha}+\frac{\theta_1-\theta_2}{1-\delta\alpha}\right),\left(\frac{1}{2}\right)\left(\frac{\theta_1+\theta_2}{1-\alpha}-\frac{\theta_1-\theta_2}{1-\delta\alpha}\right)\right)$$
and call it z_0.
3. Generate one observation from $EB(z_0,\alpha,\theta)$ and call it $S_{\alpha,\theta}(z_0)$ where
$$\theta = \frac{1}{4}\left(\left(\frac{\theta_1+\theta_2}{1-\alpha}\right)^2-\left(\frac{\theta_1-\theta_2}{1-\delta\alpha}\right)^2\right).$$
4. Let $z_1 = \delta S_{\alpha,\theta}(z_0)+\varepsilon_1$.
5. Generate one observation from $EB(z_1,\alpha,\theta)$ and call it $S_{\alpha,\theta}(z_1)$.
6. Let $z_2 = \delta S_{\alpha,\theta}(z_1)+\varepsilon_2$.
7. Repeat steps 5 and 6 until we obtain the nth observation z_n.

Findings:

Regarding the MSE results we found that:

(I) In the three methods of estimation, for both the models (PDINAR$^+$(1) and PDINAR$^-$(1)), the MSE of each parameter is reciprocally related to the sample size. The pace of decrease in the MSE as a result of increase in the sample size is similar for all methods (the MSE would be reduced by about 50% if the sample size is doubled).

(II) The MSE of $\hat{\alpha}$ for both models in the Yule-Walker, CML and CLS estimates decreases with the increase in α.

(III) The MSE of all estimates in the three methods are very close for both models. If a ranking were to be made the CML would be first, followed by CLS (2), followed by the CLS (1), and then the Yule-Walker method. The magnitude of gain, in terms of MSE does not make it worth the extra calculations in the CML and CLS methods.

Regarding the bias for the PDINAR(1) we can summarize as follows:

(I) For the PDINAR$^+$(1), in all cases, $\hat{\alpha}$ is biased negatively using all methods. While for PDINAR$^-$(1), when $\alpha = 0.5$ and 0.7, $\hat{\alpha}$ is always biased negatively using all methods and when $\alpha = 0.2$, $\hat{\alpha}$ is biased positively.

(II) For the PDINAR$^+$(1), the magnitude of the biases of $\hat{\alpha}$ increase with the increase in α using all methods except for some few cases of $\hat{\theta}_1$ and $\hat{\theta}_2$. This is only true for the PDINAR$^-$(1) using the Yule-Walker method.

(III) For the PDINAR$^+$(1), the amount of biases of all estimates of $\hat{\alpha}$ is reciprocally related to the sample size. This is not true for the PDINAR$^-$(1).

(IV) For both models, the proposed methods of CLS estimation give close biases of $\hat{\theta}_1$ and $\hat{\theta}_2$. It has been noticed that for the PDINAR$^+$(1)

$$\left|\left|bias\left(\hat{\theta}_{1CLS(1)}\right)\right| - \left|bias\left(\hat{\theta}_{1CLS(2)}\right)\right|\right| = \left|\left|bias\left(\hat{\theta}_{2CLS(1)}\right)\right| - \left|bias\left(\hat{\theta}_{2CLS(2)}\right)\right|\right|$$

in all cases.

Table 1. MSE (α) results of different estimation method for PDINAR$^+$(1) model ($\theta_1 = 1$, $\theta_2 = 1$)

n	α	**MSE (α)**		
		CML	**Y-W**	**CLS**
50	0.2	0.01621	0.02002	0.02058
	0.5	0.01909	0.01998	0.01939
	0.7	0.01641	0.01701	0.01880
100	0.2	0.009464	0.01002	0.01018
	0.5	0.008547	0.00887	0.00874
	0.7	0.006410	0.00707	0.00670
200	0.2	0.005225	0.00538	0.00540
	0.5	0.004302	0.00450	0.00444
	0.7	0.002987	0.00320	0.00309

Table 2. MSE (θ_1) and MSE (θ_2) results of different estimation method for PDINAR$^+$(1) model ($\theta_1 = 1$, $\theta_2 = 1$)

			MSE ($\theta 1$)					MSE ($\theta 2$)		
n	α	**CML**	**Y-W**	**CLS(1)**	**CLS(2)**		**CML**	**Y-W**	**CLS(1)**	**CLS(2)**
50	0.2	0.07091	0.07770	0.07670	0.07652		0.08331	0.08838	0.08722	0.08699
	0.5	0.07375	0.07778	0.07479	0.07435		0.08339	0.08753	0.083995	0.08369
	0.7	0.07823	0.08965	0.08018	0.07986		0.0857	0.0965	0.0863	0.0862
100	0.2	0.03784	0.04012	0.03950	0.03926		0.03447	0.03651	0.036276	0.03608
	0.5	0.03594	0.03873	0.03736	0.037006		0.03256	0.034488	0.033948	0.03359
	0.7	0.03596	0.04038	0.03721	0.036984		0.03344	0.036897	0.034758	0.03459
200	0.2	0.01941	0.02005	0.01994	0.019883		0.01801	0.0185826	0.018626	0.01855
	0.5	0.01738	0.01807	0.01786	0.01775		0.016265	0.017001	0.016936	0.016786
	0.7	0.01685	0.01788	0.01723	0.017177		0.015699	0.016758	0.016272	0.016184

6. Applications

In this section we present an application of the PDINAR(1) model from the Saudi stock market. In 2007, the minimum amount of change was SR 0.25 for all stocks. The daily close price as number of ticks (ticks=close price*4) in 2007 for Saudi Telecommunication Company (STC) stock and the Electricity stock are considered. The time series plots of the two stocks are illustrated in Figures 1 and 2 respectively. The autocorrelation function (ACF) and the partial autocorrelation function (PACF) for STC and Electricity are shown in Figures 3–6.

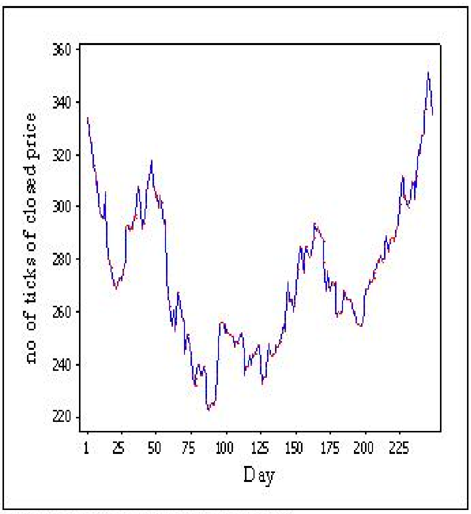

Figure 1. Time series plot of STC

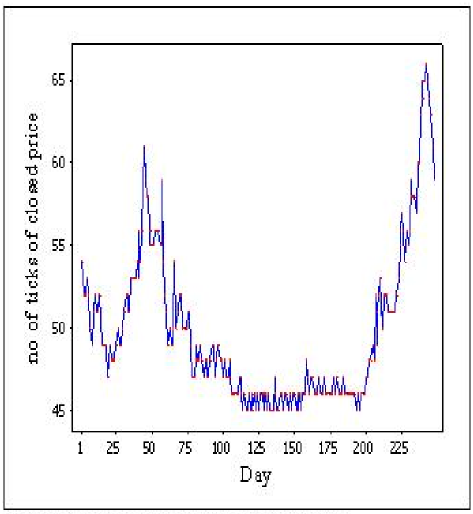

Figure 2. Time series plot of Electricity

Figures 1 and 2 show that both series are nonstationary in the mean. Figures 3–6 shows the same phenomenon of the sustained large ACF and exceptionally large first lag PACF indicating that differencing is needed.

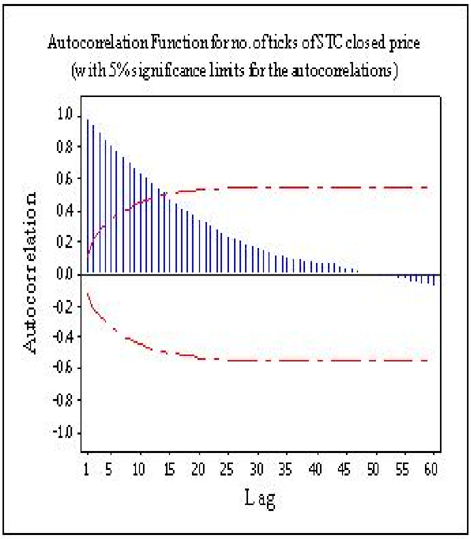

Figure 3. ACF of STC

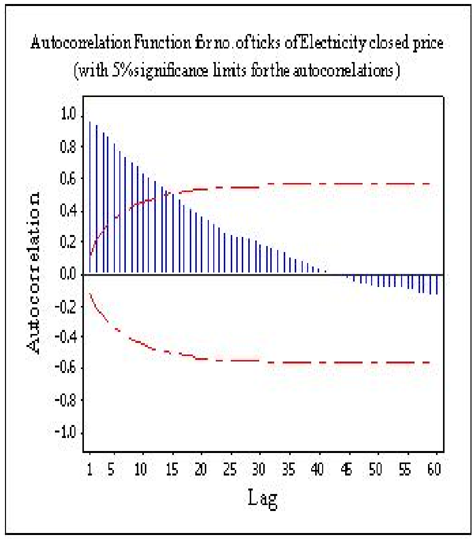

Figure 4. ACF of Electricity

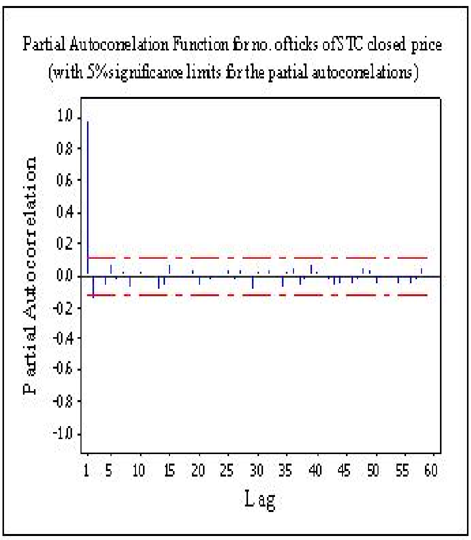

Figure 5. PACF of STC

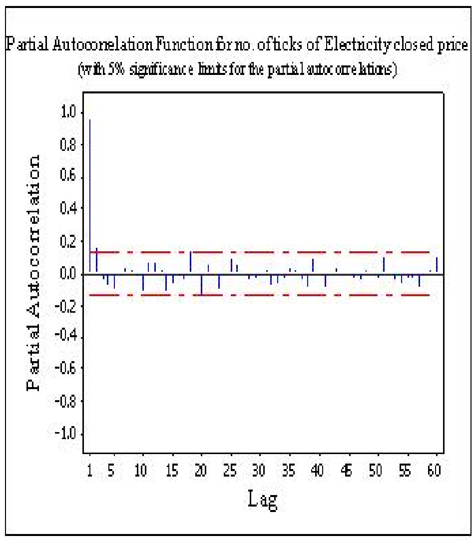

Figure 6. PACF of Electricity

In order to get more information about the data, Table 3 displays some descriptive statistics for the number of ticks of STC and Electricity and their lag one differences.

The time series plots of the two differenced stocks are illustrated in Figures 7 and 8 respectively. The autocorrelation function and the partial autocorrelation function for the differenced stocks are shown in Figures 9–12.

Table 3. Descriptive statistics of STC and Electricity with their lag one difference.

Variable	No. of observation	Mean	Variance	Minimum	Median	Maximum	Range
STC	248	273.6	766.54	222	270	351	129
STC difference	247	0.004	22.232	-19	0	13	32
Electricity	248	49.819	22.06	45	48	66	21
Electricity difference	247	0.0202	1.6215	-5	0	5	10

It is clear from Figures 7 and 8 that both differenced series are stationary in the mean. Figure 9 shows that the lag one correlation is positive and significant hence $\hat{\delta} = 1$ and PDINAR$^+$(1) is proposed to model the differenced series of STC price as number of ticks. Figure 10 shows that the lag one correlation is negative and significant hence $\hat{\delta} = -1$ and PDINAR$^-$(1) is proposed to model the differenced series of Electricity price as number of ticks. The estimates are obtained using the Yule-Walker method, the conditional maximum likelihood method and the conditional least squares method (using the two proposed methods) and are displayed in Table 4. Standard errors for the CML estimates were derived from the Hessian matrix and the standard errors for the YW and the CLS estimates were derived through simulation based methods.

Table 4. Estimation results of STC and Electricity as PDINAR(1) with (standard errors)

Stock	Parameter	CML	Y-W	CLS (1)	CLS (2)
STC	α	0.2187 (0.003)	0.2181 (0.062)	0.2201 (0.063)	0.2201 (0.063)
	θ_1	8.5586 (0.319)	8.69273 (0.921)	10.4088 (1.093)	10.397 (1.092)
	θ_2	8.5455 (0.154)	8.68956 (0.927)	10.3957 (1.101)	10.3839 (1.101)
Electricity	α	0.2798 (0.063)	0.2436 (0.063)	0.2462 (0.063)	0.2462 (0.064)
	θ_1	0.5574 (0.074)	0.6258 (0.082)	0.62467 (0.082)	0.616791 (0.081)
	θ_2	0.4995 (0.073)	0.6007 (0.079)	0.58921 (0.079)	0.58133 (0.078)

Using the Y-W estimates, the differenced price of the STC stock as number of ticks is PDINAR$^+$(1) with $\alpha = 0.22$ and the process has marginal PD$(11.1179,\ 11.1138)$. The differenced price of the Electricity stock as number of ticks is PDINAR$^-$(1) with $\alpha = 0.2436$ and the process has marginal PD$(0.821,\ 0.801)$. The relative frequency of the data and the fitted Poisson difference distribution are plotted in the following graphs.

In order to visualize the adequacy of the models, the ACF and the PACF of the estimated residuals are plotted in Figures 15–18 for both of the stocks. The estimated residuals are computed as $\hat{e}_t = Z_t - \delta\hat{\alpha}Z_{t-1} - \hat{\theta}_1 + \hat{\theta}_2$. Clearly, from Figures 15–18 of the ACF and the PACF of the residuals, both the stocks can be considered as white noise.

It is worth noting that even though we have fitted PDINAR (1) models to the data, based on Figures 9–12, it is possible to find alternative models which may give better approximations.

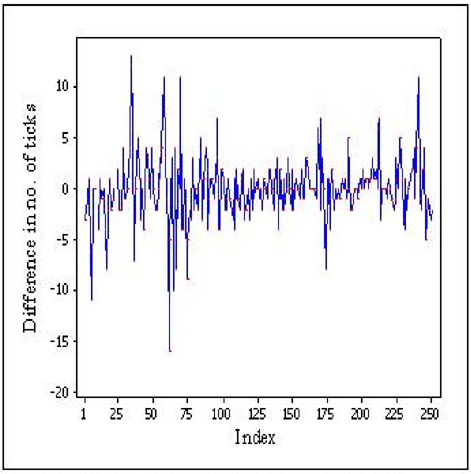

Figure 7. Time series plot of differenced STC

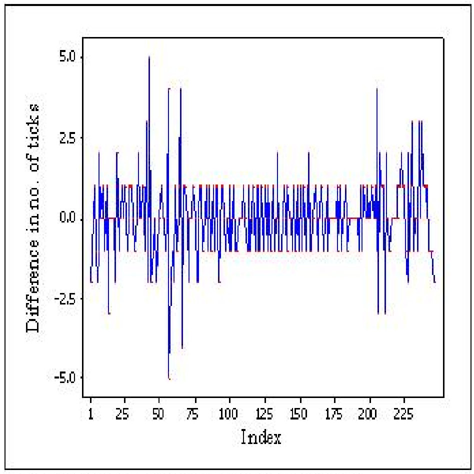

Figure 8. Time series plot of differenced Electricity

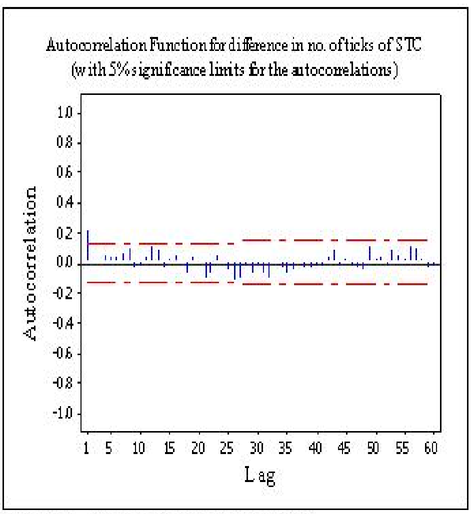

Figure 9. ACF for differenced STC

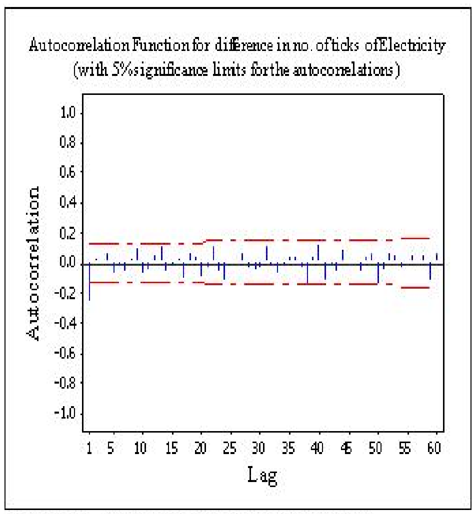

Figure 10. ACF for differenced Electricity

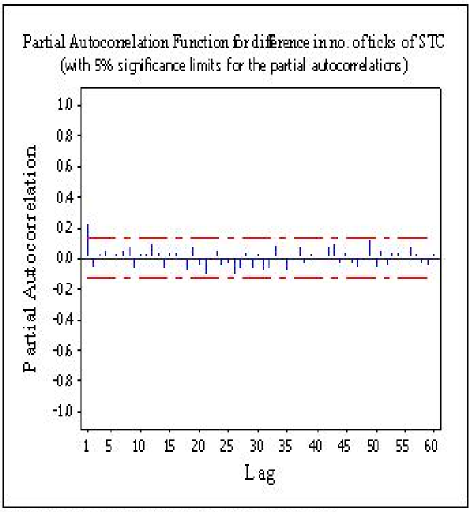

Figure 11. PACF for differenced STC

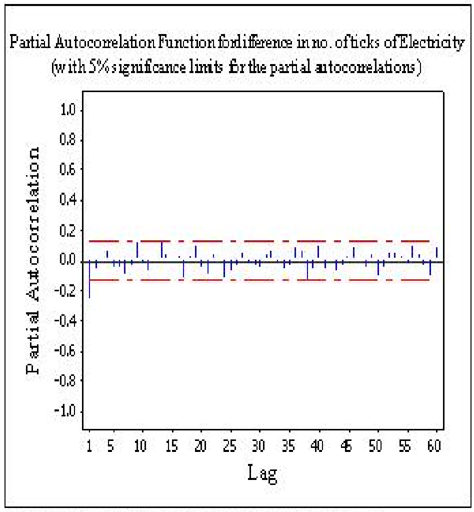

Figure 12. PACF for differenced Electricity

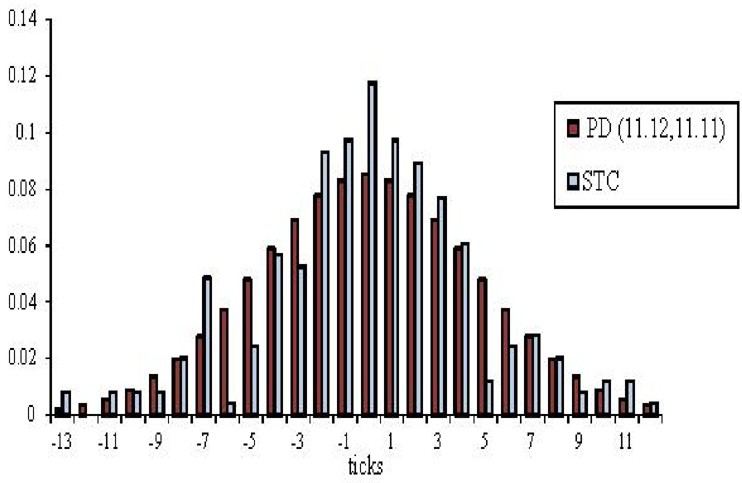

Figure 13. Relative frequency of STC and fitted Poisson difference

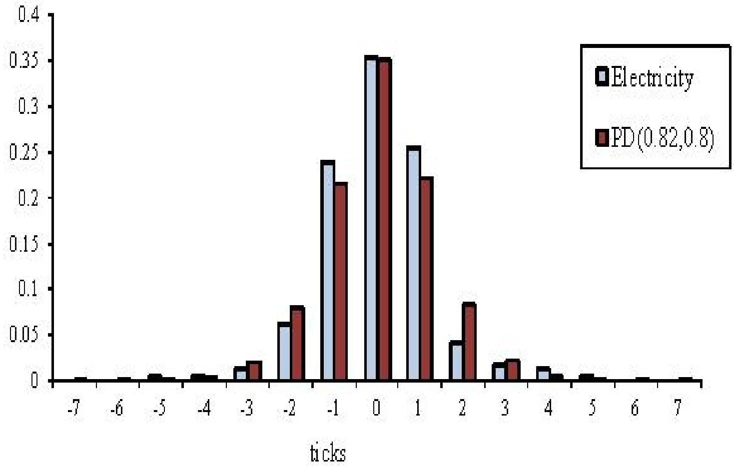

Figure 14. Relative frequency of Electricity and fitted Poisson difference

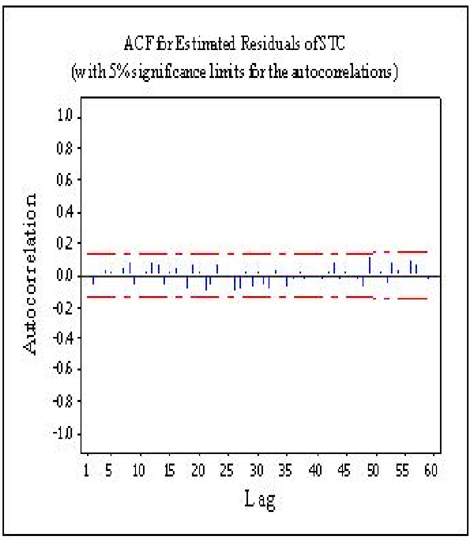

Figure 15. ACF for residuals of STC

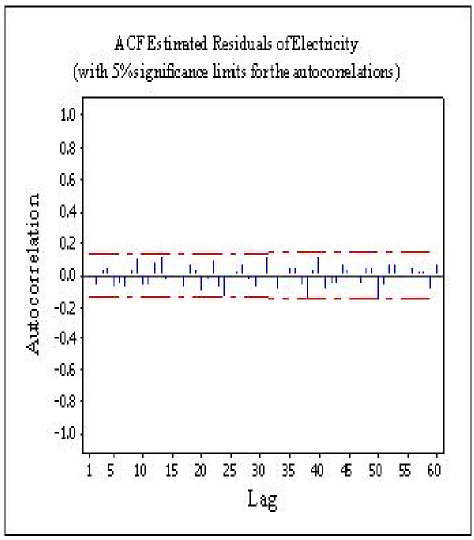

Figure 16. ACF for residuals of Electricity

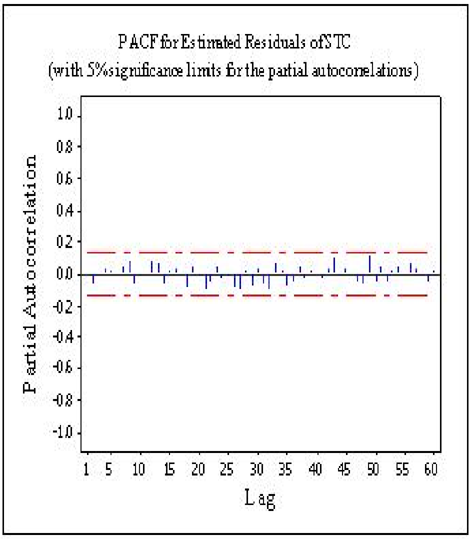

Figure 17. PACF for residuals of STC

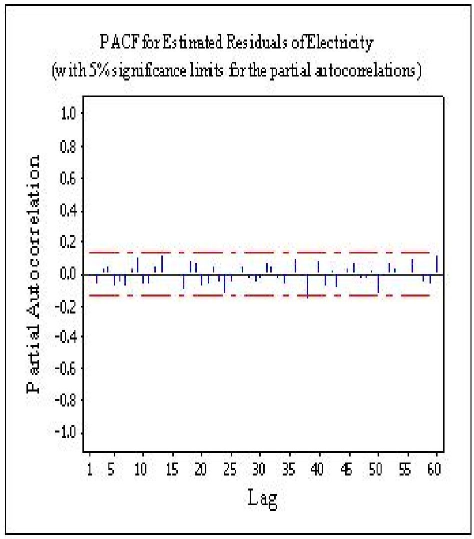

Figure 18. PACF for residuals of Electricity

Acknowledgement. The authors extend their appreciation to the Deanship of Scientific Research at King Saud University for funding the work through the research group project RGB - VPP - 053.

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