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Notes on Periodic Solutions for a Nonlinear Discrete System Involving the p-Laplacian

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Abstract. In this paper, we first improve two inequalities, then by using critical point theory, improve an existence theorem of periodic solutions for a nonlinear discrete system involving the p-Laplacian, and present some estimates of periodic solutions.

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1. Introduction and main results

Let $\mathbb{R}$ denote the real number, $\mathbb{Z}$ the integers. Given $a<b$ in $\mathbb{Z}$, let $\mathbb{Z}[a,b]=\{a,a+1,\ldots,b\}$. Let $T>1$ and N be fixed positive integers. Consider the following nonlinear discrete system involving the p-Laplacian

$$\Delta[\Phi_p(\Delta x(t-1))]+\nabla F(t,x(t))=0,\quad t\in\mathbb{Z},\tag{1.1}$$

where $p>1, q>1$, $1/p+1/q=1$, $\Phi_p(u)=|u|^{p-2}u=\left(\sqrt{\sum_{i=1}^N u_i^2}\right)^{p-2}(u_1,u_2,\ldots,u_N)^\tau$, $u\in\mathbb{R}^N$, $\cdot^\tau$ stands for the transpose of a vector or a matrix, $F:\mathbb{Z}\times\mathbb{R}^N\to\mathbb{R}$, $(t,x)\to F(t,x)$ is T-periodic in t for all $x\in\mathbb{R}^N$ and continuously differentiable and convex in x for every $t\in\mathbb{Z}$, $\nabla F(t,x)$ denotes the gradient of $F(t,x)$ in x, and $\Delta x(t)=x(t+1)-x(t)$, $\Delta^2 x(t)=\Delta(\Delta x(t))$.

When $p=2$, problem (1.1) becomes the second order discrete nonlinear system. By using the variational methods, some existence results for periodic solutions are obtained, such as [1, 5, 6, 10–12]. When $p>1$, recently, there are also some results, see [2–4, 7, 8]. Especially, in [7], by using the dual least principle, the authors obtained the following result:

Theorem 1.1. *Suppose F satisfies the following conditions:*

(A_1) *there exists $\beta:\mathbb{Z}\to\mathbb{R}^N$ such that for all $(t,y)\in\mathbb{Z}\times\mathbb{R}^N$,*

$$F(t,y)\geq\left(\beta(t),|y|^{\frac{p-2}{2}}y\right)\quad \textit{and}\quad \beta(t+T)=\beta(t);$$

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(A_2) *there are constants* $\alpha \in (0, T^{-1})$, *and* $\gamma : \mathbb{Z} \to \mathbb{R}$ *such that for all* $(t,y) \in \mathbb{Z} \times \mathbb{R}^N$,

$$F(t,y) \le \frac{\alpha^p}{p}|y|^p + \gamma(t) \quad \text{and} \quad \gamma(t+T) = \gamma(t);$$

(A_3) $\sum_{t=1}^{T} F(t,y) \to +\infty$, *as* $|y| \to \infty$, $y \in \mathbb{R}^N$.
Then, system (1.1) has at least one T-periodic solution.

In our paper, we improve two discrete inequalities in [7,8]. Furthermore, we improve the condition (A_2) and also obtain some estimates of periodic solution for system (1.1).

2. Preliminaries

In the following, we use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^N$. Let

$$S = \{u = (u_1, u_2)^\tau = \{u(t)\} | u(t) = (u_1(t), u_2(t))^\tau \in \mathbb{R}^{2N},$$
$$u_i = \{u_i(t)\}, u_i(t) \in \mathbb{R}^N, i = 1,2, t \in \mathbb{Z}\}.$$

E is defined as a subspace of S by

$$E = \{u = \{u(t)\} \in S | u(t+T) = u(t), t \in \mathbb{Z}\}.$$

For $u = (u_1, u_2)^\tau \in E$, set

$$\|u_i\|_r = \left(\sum_{t=1}^{T} |u_i(t)|^r \right)^{1/r},$$

where $i = 1,2, r > 1$. Then E can be equipped with the norm as follows:

$$\|u\| = \|u_1\|_p + \|u_2\|_q$$

for $u = (u_1, u_2)^\tau \in E$. It is obvious that E is a reflexive Banach space with dimension $2NT$, which can be identified with $\mathbb{R}^{2NT}$. Let

$$W = \left\{ u = (u_1, u_2)^\tau \in E | u_i(1) = u_i(2) = \cdots = u_i(T) = \frac{1}{T} \sum_{t=1}^{T} u_i(t), i = 1,2 \right\}$$

and

$$Y = \left\{ u = (u_1, u_2)^\tau \in E | \sum_{t=1}^{T} u_i(t) = 0, i = 1,2 \right\}.$$

Then E can be decomposed into the direct sum $E = Y \oplus W$. So, for any $u \in E$, u can be expressed in the form $u = \tilde{u} + \bar{u}$, where $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)^\tau \in Y$ and $\bar{u} = (\bar{u}_1, \bar{u}_2)^\tau \in W$. Obviously, $u_i = \tilde{u}_i + \bar{u}_i, i = 1,2$.

For $u = (u_1, u_2)^\tau \in Y$, let

$$\|\Delta u_i\|_r = \left(\sum_{t=1}^{T} |\Delta u_i(t)|^r \right)^{1/r},$$

where $i = 1,2, r > 1$. It is easy to verify that

$$\|\Delta u\| = \|\Delta u_1\|_q + \|\Delta u_2\|_p$$

is also a norm on Y. Since Y is finite-dimensional, the norm $\|\Delta u\|$ is equivalent to the norm $\|u\|$ in E if $u \in Y$.

$\Gamma_0(\mathbb{R}^N)$ denotes the set of all convex lower semi-continuous(l.s.c.) functions $F : \mathbb{R}^N \to (-\infty, +\infty]$ whose effective domain $D(F) = \{u \in \mathbb{R}^N : F(u) < +\infty\}$ is nonempty. Let $H : \mathbb{Z} \times$

$\mathbb{R}^{2N} \to \mathbb{R}$, $(t,u) \to H(t,u)$ be a smooth Hamiltonian such that for each $t \in \mathbb{Z}[1,T]$, $H(t,\cdot) \in \Gamma_0(\mathbb{R}^{2N})$ is strictly convex and $H(t,u)/|u| \to +\infty$, if $|u| \to \infty$. The Fenchel transform $H^*(t,\cdot)$ of $H(t,\cdot)$ is defined by

$$H^*(t,v) = \sup_{u\in\mathbb{R}^{2N}} \{(v,u) - H(t,u)\} \tag{2.1}$$

or

$$\begin{cases} H^*(t,v) = (v,u) - H(t,u), \\ v = \nabla H(t,u), \quad \text{or} \quad u = \nabla H^*(t,v). \end{cases} \tag{2.2}$$

If for $u = (u_1,u_2), u_1,u_2 \in \mathbb{R}^N$, $H(t,u)$ can be split into parts $H(t,u) = H_1(t,u_1) + H_2(t,u_2)$, then by (2.2), $H^*(t,v) = H_1^*(t,v_1) + H_2^*(t,v_2), v = (v_1,v_2), v_1,v_2 \in \mathbb{R}^N$. We denote by J the symplectic matrix. Then $J^2 = -I$ and $(Ju,v) = -(u,Jv)$ for all $u,v \in \mathbb{R}^{2N}$. It is clear that $(J\dot{v},v) = (\dot{v}_2,v_1) - (\dot{v}_1,v_2)$, where $v = (v_1,v_2)^\tau \in \mathbb{R}^N \times \mathbb{R}^N, i = 1,2$.

Let $u_1(t) = x(t)$, $u_2(t) = \alpha^{-1}\Phi_p(\Delta u_1(t-1)), t \in \mathbb{Z}$. Then problem (1.1) is equivalent to the non-autonomous system

$$\begin{cases} \Delta u_2(t) + \alpha^{-1}\nabla F(t,u_1(t)) = 0, & t \in \mathbb{Z}, \\ -\Delta u_1(t-1) + \alpha^{q-1}\Phi_q(u_2(t)) = 0, \end{cases} \tag{2.3}$$

that is

$$J\Delta u(t) + \nabla H(t,Lu(t)) = 0, \quad t \in \mathbb{Z}, \tag{2.4}$$

where $Lu(t) = (u_1(t),u_2(t+1))^\tau$, $L^{-1}u(t) = (u_1(t),u_2(t-1))^\tau, u = (u_1,u_2)^\tau, H(t,u) = H_1(t,u_1) + H_2(t,u_2)$ and

$$H_1(t,u_1) = \frac{1}{\alpha}F(t,u_1), \quad H_2(t,u_2) = \frac{\alpha^{q-1}}{q}|u_2|^q.$$

The dual action is defined on E by

$$I(v) = \frac{1}{2}\sum_{t=1}^{T}(L(J\Delta v(t-1)),v(t)) + \sum_{t=1}^{T}\left[H_1^*(t,\Delta v_1(t)) + H_2^*(t,\Delta v_2(t))\right],$$

where $v = (v_1,v_2)^\tau \in E$. Since $I(v) = I(\tilde{v} + \bar{v}) = I(\tilde{v})$ for $v = \tilde{v} + \bar{v} \in E$, in order to find the T-periodic solution of (1.1), it suffices to find the critical point of I on subspace Y of E. The above knowledge and statement come from [7,9,12].

Lemma 2.1. *Let $u = (u_1,u_2) \in Y$. Then*

$$\max_{t\in\mathbb{Z}[1,T]}|u_i(t)| \le \min\left\{\frac{(T-1)^{(p+1)/p}}{T}, \left(\frac{(T+1)^{p+1}-2}{T^p(p+1)}\right)^{1/p}\right\}\left(\sum_{s=1}^{T}|\Delta u_i(s)|^q\right)^{1/q}, \quad i = 1,2, \tag{2.5}$$

$$\max_{t\in\mathbb{Z}[1,T]}|u_i(t)| \le \min\left\{\frac{(T-1)^{(q+1)/q}}{T}, \left(\frac{(T+1)^{q+1}-2}{T^q(q+1)}\right)^{1/q}\right\}\left(\sum_{s=1}^{T}|\Delta u_i(s)|^p\right)^{1/p}, \quad i = 1,2, \tag{2.6}$$

and

$$\sum_{t=1}^{T}|u_i(t)|^q \le \min\left\{\frac{(T-1)^{2q-1}}{T^{q-1}}, \frac{T^{q-1}\Theta(p,q)}{(p+1)^{q/p}}\right\}\sum_{s=1}^{T}|\Delta u_i(s)|^q, \quad i = 1,2, \tag{2.7}$$

$$\sum_{t=1}^{T}|u_i(t)|^p \le \min\left\{\frac{(T-1)^{2p-1}}{T^{p-1}}, \frac{T^{p-1}\Theta(q,p)}{(q+1)^{p/q}}\right\}\sum_{s=1}^{T}|\Delta u_i(s)|^p, \quad i=1,2. \tag{2.8}$$

where

$$\Theta(p,q)=\sum_{t=1}^{T}\left[\left(\frac{t}{T}\right)^{p+1}+\left(1-\frac{t}{T}+\frac{1}{T}\right)^{p+1}-\frac{2}{T^{p+1}}\right]^{q/p},$$

$$\Theta(q,p)=\sum_{t=1}^{T}\left[\left(\frac{t}{T}\right)^{q+1}+\left(1-\frac{t}{T}+\frac{1}{T}\right)^{q+1}-\frac{2}{T^{q+1}}\right]^{p/q}.$$

Proof. Fix $t\in\mathbb{Z}[1,T]$. For every $\tau\in\mathbb{Z}[1,t-1]$, we have

$$u_1(t)=u_1(\tau)+\sum_{s=\tau}^{t-1}\Delta u_1(s) \tag{2.9}$$

and for every $\tau\in\mathbb{Z}[t,T]$,

$$u_1(t)=u_1(\tau)-\sum_{s=t}^{\tau-1}\Delta u_1(s). \tag{2.10}$$

Summing (2.9) over $\mathbb{Z}[1,t-1]$ and (2.10) over $\mathbb{Z}[t,T]$, we have

$$(t-1)u_1(t)=\sum_{\tau=1}^{t-1}u_1(\tau)+\sum_{\tau=1}^{t-1}\sum_{s=\tau}^{t-1}\Delta u_1(s)=\sum_{\tau=1}^{t-1}u_1(\tau)+\sum_{s=1}^{t-1}s\Delta u_1(s) \tag{2.11}$$

and

$$(T-t+1)u_1(t)=\sum_{\tau=t}^{T}u_1(\tau)-\sum_{\tau=t}^{T}\sum_{s=t}^{\tau-1}\Delta u_1(s)=\sum_{\tau=t}^{T}u_1(\tau)-\sum_{s=t}^{T-1}(T-s)\Delta u_1(s). \tag{2.12}$$

Set

$$\phi(s)=\begin{cases} s, & 1\le s\le t-1,\\ T-s, & t\le s\le T.\end{cases}$$

Since $\sum_{\tau=1}^{T}u_1(\tau)=0$, combining (2.11) with (2.12) and using the Hölder inequality, we obtain

$$\begin{aligned} T|u_1(t)| &= \left|\sum_{s=1}^{t-1}s\Delta u_1(s)-\sum_{s=t}^{T-1}(T-s)\Delta u_1(s)\right| \le \sum_{s=1}^{t-1}s|\Delta u_1(s)|+\sum_{s=t}^{T-1}(T-s)|\Delta u_1(s)| \\ &= \sum_{s=1}^{T-1}\phi(s)|\Delta u_1(s)| = \sum_{s=1}^{T}\phi(s)|\Delta u_1(s)| \le \left(\sum_{s=1}^{T}[\phi(s)]^p\right)^{1/p}\left(\sum_{s=1}^{T}|\Delta u_1(s)|^q\right)^{1/q} \\ &= \left(\sum_{s=1}^{t-1}s^p+\sum_{s=t}^{T-1}(T-s)^p\right)^{1/p}\left(\sum_{s=1}^{T}|\Delta u_1(s)|^q\right)^{1/q}. \end{aligned} \tag{2.13}$$

Since

$$\sum_{s=1}^{t-1}s^p<\frac{t^{p+1}-1}{p+1}, \quad \sum_{s=t}^{T-1}(T-s)^p=\sum_{k=1}^{T-t}k^p<\frac{(T-t+1)^{p+1}-1}{p+1}, \tag{2.14}$$

and

$$\sum_{s=1}^{t-1} s^p + \sum_{s=t}^{T-1} (T-s)^p \le \sum_{s=1}^{T-1} (T-1)^p = (T-1)^{p+1}, \tag{2.15}$$

it follows from (2.13) that (2.5) with $i = 1$ holds. On the other hand, from (2.13),(2.14) and (2.15), we have

$$\begin{aligned}
&T^q \sum_{t=1}^{T} |u_1(t)|^q \\
&\le \left(\sum_{s=1}^{T} |\Delta u_1(s)|^q\right) \sum_{t=1}^{T} \left(\sum_{s=1}^{t-1} s^p + \sum_{s=t}^{T-1} (T-s)^p\right)^{q/p} \\
&\le \left(\sum_{s=1}^{T} |\Delta u_1(s)|^q\right) \min\left\{\sum_{t=1}^{T} \left(\frac{t^{p+1}-1}{p+1} + \frac{(T-t+1)^{p+1}-1}{p+1}\right)^{q/p}, T(T-1)^{2q-1}\right\} \\
&= \left(\sum_{s=1}^{T} |\Delta u_1(s)|^q\right) \\
&\quad \cdot \min\left\{\frac{T^{2q-1}}{(p+1)^{q/p}} \sum_{t=1}^{T} \left[\left(\frac{t}{T}\right)^{p+1} + \left(1 - \frac{t}{T} + \frac{1}{T}\right)^{p+1} - \frac{2}{T^{p+1}}\right]^{q/p}, T(T-1)^{2q-1}\right\} \\
&= \min\left\{\frac{T^{2q-1}\Theta(p,q)}{(p+1)^{q/p}}, T(T-1)^{2q-1}\right\} \left(\sum_{s=1}^{T} |\Delta u_1(s)|^q\right).
\end{aligned}$$

It follows that (2.7) with $i = 1$ holds. Similarly, we can prove other inequalities also hold. Thus the proof is complete. ∎

Remark 2.1. Since

$$\min\left\{\frac{(T-1)^{(p+1)/p}}{T}, \left(\frac{(T+1)^{p+1}-2}{T^p(p+1)}\right)^{1/p}\right\} \le \frac{(T-1)^{(p+1)/p}}{T} < \frac{T^{(p+1)/p}}{T} = T^{1/p}$$

and

$$\min\left\{\frac{(T-1)^{(q+1)/q}}{T}, \left(\frac{(T+1)^{q+1}-2}{T^q(q+1)}\right)^{1/q}\right\} \le \frac{(T-1)^{(q+1)/q}}{T} < \frac{T^{(q+1)/q}}{T} = T^{1/q},$$

(2.5) and (2.6) improve (2.8) and (2.9) in [7] which shows that for $u = (u_1, u_2) \in Y$ and $t \in \mathbb{Z}[1, T]$,

$$|u_i(t)| \le T^{1/p} \|\Delta u_i\|_{L^q}, \quad |u_i(t)| \le T^{1/q} \|\Delta u_i\|_{L^p}, \quad i = 1, 2,$$

respectively. Moreover, Lemma 2.1 also improves [8, Lemma 2.2].

Lemma 2.2. *For every* $u = (u_1, u_2)^\tau \in E$,

$$\sum_{t=1}^{T} (L(J\Delta u(t-1)), u(t)) \ge -\frac{C}{q} \|\Delta u_1\|_q^q - \frac{C}{p} \|\Delta u_2\|_p^p$$

and

$$\sum_{t=1}^{T} (L^{-1}(J\Delta u(t)), u(t)) \ge -\frac{C}{p} \|\Delta u_1\|_p^p - \frac{C}{q} \|\Delta u_2\|_q^q, \tag{2.16}$$

where

$$C = C(p,q) + C(q,p), \quad C^q(p,q) = \min\left\{\frac{(T-1)^{2q-1}}{T^{q-1}}, \frac{T^{q-1}\Theta(p,q)}{(p+1)^{q/p}}\right\},$$

and

$$C^p(q,p) = \min\left\{\frac{(T-1)^{2p-1}}{T^{p-1}}, \frac{T^{p-1}\Theta(q,p)}{(q+1)^{p/q}}\right\}.$$

Proof. For $u = (u_1, u_2) \in E$, we write $u_i = \tilde{u}_i + \bar{u}_i$, where $\bar{u}_i = 1/T \sum_{t=1}^{T} u_i(t), i = 1,2$. Since $\sum_{t=1}^{T} \tilde{u}_i(t) = 0$ and $\Delta u_i(t) = \Delta\tilde{u}_i(t), i = 1,2$, then by (2.7), (2.8), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\sum_{t=1}^{T}(L(J\Delta u(t-1)), u(t)) &= \sum_{t=1}^{T}[(\Delta u_2(t-1), u_1(t)) - (\Delta u_1(t), u_2(t))] \\ &= \sum_{t=1}^{T}[(\Delta\tilde{u}_2(t-1), \tilde{u}_1(t)) - (\Delta\tilde{u}_1(t), \tilde{u}_2(t))] \\ &\geq -C(p,q)\|\Delta\tilde{u}_2\|_p\|\Delta\tilde{u}_1\|_q - C(q,p)\|\Delta\tilde{u}_2\|_p\|\Delta\tilde{u}_1\|_q \\ &= -C\|\Delta u_2\|_p\|\Delta u_1\|_q \geq -\frac{C}{q}\|\Delta u_1\|_q^q - \frac{C}{p}\|\Delta u_2\|_p^p.\end{aligned}$$

Similarly to the above process, (2.16) also holds for $u = (u_1, u_2) \in E$. ∎

Remark 2.2. Note that

$$C = C(p,q) + C(q,p) \leq \left(\frac{(T-1)^{2q-1}}{T^{q-1}}\right)^{1/q} + \left(\frac{(T-1)^{2p-1}}{T^{p-1}}\right)^{1/p} < 2T. \tag{2.17}$$

So our Lemma 2.2 improves [7, Lemma 2.3].

Lemma 2.3. [9, Proposition 1.4] *Let $G \in C^1(\mathbb{R}^N, \mathbb{R})$ be a convex function. Then, for all $x, y \in \mathbb{R}^N$, we have*

$$G(x) \geq G(y) + (\nabla G(y), x - y).$$

3. Main results and proofs

Theorem 3.1. *Suppose F satisfies (A_1), (A_3) and the following conditions:*

$(A_2)'$ *there are constants $\alpha \in (0, 2/C)$, and $\gamma : \mathbb{Z} \to \mathbb{R}$ such that for all $(t,y) \in \mathbb{Z} \times \mathbb{R}^N$,*

$$F(t,y) \leq \frac{\alpha^p}{p}|y|^p + \gamma(t) \quad \text{and} \quad \gamma(t+T) = \gamma(t).$$

Then, system (2.3) has at least one solution $u \in E$ such that

$$v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = -J\left[u(t) - \frac{1}{T}\sum_{s=1}^{T} u(s)\right] = \begin{pmatrix} -u_2(t) + \frac{1}{T}\sum_{s=1}^{T} u_2(s) \\ u_1(t) - \frac{1}{T}\sum_{s=1}^{T} u_1(s) \end{pmatrix}$$

minimizes the dual action I, that is to say, system (1.1) has at least one solution $x = u_1$.

Proof. The proof is the same as in [7]. We only need to replace [7, Lemma 2.3] with our Lemma 2.2 in the proof. In order to make the paper self-contained, we present a brief outline of the proof. More details can be seen in [7].

Step 1. We consider the existence of one T-periodic solution for a perturbed problem. Note that $\alpha < 2/C$. So there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\alpha(1+\varepsilon)^{p-1} < 2/C, \quad \alpha(1+\varepsilon)^{q-1} < 2/C.$$

Consider the following perturbed problem:

$$\begin{cases} \Delta u_2(t) + \varepsilon\alpha^{p-1}\phi_p(u_1(t)) + \nabla H_1(t, u_1(t)) = 0, & t \in \mathbb{Z}, \\ -\Delta u_1(t-1) + \varepsilon\alpha^{q-1}\phi_q(u_2(t)) + \nabla H_2(t, u_2(t)) = 0, \\ u_1(t+T) = u_1(t), \quad u_2(t+T) = u_2(t). \end{cases} \tag{3.1}$$

In order to obtain the solution of the perturbed problem, consider the following perturbed dual action functional

$$I_\varepsilon(v) = \frac{1}{2}\sum_{t=1}^{T}(L(J\Delta v(t-1)), v(t)) + \sum_{t=1}^{T} H_\varepsilon^*(t, \Delta v(t)),$$

where

$$H_\varepsilon(t, \Delta v) = \varepsilon\alpha^{p-1}\frac{|u_1|^p}{p} + H_1(t, u_1) + \varepsilon\alpha^{q-1}\frac{|u_2|^q}{q} + H_2(t, u_2).$$

By (A1), (A2)$'$, [7, Lemma 2.1] and Lemma 2.2, one can obtain that

$$\begin{aligned} I_\varepsilon(v) \geq & -\frac{C}{2q}\|\Delta v_1\|_q^q - \frac{C}{2p}\|\Delta v_2\|_p^p + \frac{(1+\varepsilon)^{-(q-1)}\alpha^{-1}}{q}\|\Delta v_1\|_q^q \\ & + \frac{(1+\varepsilon)^{-(p-1)}\alpha^{-1}}{p}\|\Delta v_2\|_p^p - \frac{1}{\alpha}\sum_{t=1}^{T}\gamma(t). \end{aligned} \tag{3.2}$$

Since $(1+\varepsilon)^{-(q-1)}\alpha^{-1} > C/2$ and $(1+\varepsilon)^{-(p-1)}\alpha^{-1} > C/2$, I_ε is bounded from below and coercive in subspace Y. By [7, Lemma 2.2], we know that I_ε is continuously differentiable in Y. Then by [9, Theorem 1.1], I_ε attains its minimum at some point $v_\varepsilon \in Y$. Then by [7, Lemma 2.2],

$$u_\varepsilon(t) = L^{-1}(\nabla H_\varepsilon^*(t, \Delta v_\varepsilon(t))), \quad u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})^\tau, \quad v_\varepsilon = (v_{1\varepsilon}, v_{2\varepsilon})^\tau$$

is a solution of the perturbed problem (3.1).

Step 2. We prove that u_ε is bounded in E. By (A3), we can get a $y_0 \in E$ such that $\sum_{t=1}^{T} y_0(t) = 0$. Then

$$I_\varepsilon(v_\varepsilon) \leq I_\varepsilon(y_0) \leq \frac{1}{2}\sum_{t=1}^{T}(L(J\Delta y_0(t-1)), y_0(t)) + \sum_{t=1}^{T} H^*(t, \Delta y_0(t)) < +\infty.$$

Note that $\Delta u_\varepsilon(t) = J\Delta v_\varepsilon(t)$. So (3.2), (2.7) and (2.8) imply that there exists a constant K_1 such that

$$\|\tilde{u}_{1\varepsilon}\|_p \leq K_1 \quad \text{and} \quad \|\tilde{u}_{2\varepsilon}\|_q \leq K_1. \tag{3.3}$$

By virtue of the convexity of $H_i(t, \cdot)(i = 1, 2)$, (3.3), (A2)$'$ and (A3), we can obtain that there exists a constant K_2 such that

$$|\bar{u}_{1\varepsilon}| \leq K_2 \quad \text{and} \quad |\bar{u}_{2\varepsilon}| \leq K_2.$$

So

$$\|u_\varepsilon\| = \|u_{1\varepsilon}\|_p + \|u_{2\varepsilon}\|_q \leq \|\tilde{u}_{1\varepsilon}\|_p + |\bar{u}_{1\varepsilon}|_p + \|\tilde{u}_{2\varepsilon}\|_q + |\bar{u}_{2\varepsilon}|_q \leq 2K_1 + K_2(T^{1/p} + T^{1/q}),$$

which shows that u_ε is bounded in E.

Step 3. We prove the existence of a T-periodic solution for system (1.1). Note that u_ε is bounded in E and E is dimensional. Then there exists a sequence $\{\varepsilon_n\} \subset (0,\varepsilon_0)$ and some point $u = (u_1,u_2)^\tau \in E$ such that

$$\varepsilon_n \to 0, \quad u_{\varepsilon_n} \to u \quad \text{as} \quad n \to \infty.$$

Let $n \to \infty$ in (3.1). Then it is easy to obtain that u_1 is a T-periodic solution of system (1.1). Moreover, since $\Delta v_{\varepsilon_n}(t) = -J\Delta u_{\varepsilon_n}(t)$, we have $v_{\varepsilon_n}(t) = -J(u_{\varepsilon_n}(t) - \bar{u}_{\varepsilon_n})$. Let $n \to \infty$. Then

$$v_{\varepsilon_n}(t) \to -J(u(t) - \bar{u}) := v(t). \tag{3.4}$$

Step 4. We prove that $v = (v_1,v_2)^\tau \in E$ minimizes the dual action I. Since $\Delta v_{\varepsilon_n}(t) = \nabla H_{\varepsilon_n}(t, Lu_{\varepsilon_n}(t))$,

$$\Delta v_{1\varepsilon_n}(t) = \nabla H_{1\varepsilon_n}(t, u_{1\varepsilon_n}(t)), \quad \Delta v_{2\varepsilon_n}(t-1) = \nabla H_{2\varepsilon_n}(t, u_{2\varepsilon_n}(t)).$$

Let $n \to \infty$. Then (3.4) and (2.4) imply that

$$\Delta v_1(t) = \nabla H_1(t, u_1(t)), \quad \Delta v_2(t-1) = \nabla H_2(t, u_2(t)). \tag{3.5}$$

As $H_\varepsilon^*(t,v) \le H^*(t,v)$, we obtain that

$$I_{\varepsilon_n}(v_{\varepsilon_n}) \le I_{\varepsilon_n}(h) \le I(h).$$

for all $h \in E$. Let $n \to \infty$. By (3.5) and [7, Lemma 2.1], we can get $I(v) \le I(h)$ for all $h \in E$. Thus the proof is complete. ∎

Remark 3.1. By (2.17), it is easy to obtain that $2/C > 2/(2T) = 1/T$. So Theorem 3.1 improves Theorem 1.1 since the range of α is larger.

Next, we consider the estimate of solutions for system (1.1).

Theorem 3.2. *Assume that there exists $\alpha \in (0, C^{-1})$, $\beta,\gamma \in [0,+\infty)$, $\delta \in (0,+\infty)$ such that*

$$\delta|y|^{p/2} - \beta \le F(t,y) \le \frac{\alpha^p}{p}|y|^p + \gamma, \tag{3.6}$$

for all $t \in \mathbb{Z}$ and $y \in \mathbb{R}^N$. Then each solution $x = u_1$ of system (1.1) satisfies

$$\sum_{t=1}^{T} |x(t)|^{p/2} \le \frac{(\gamma+\beta)T}{\delta} + \frac{\alpha^q C(q,p) B^{1/p} D^{1/q}}{\delta}, \tag{3.7}$$

$$\|\Delta x\|_p^p \le \frac{pT(\gamma+\beta)}{1 - C\alpha}, \tag{3.8}$$

where

$$B = \frac{pT(\gamma+\beta)}{\alpha^q - C\alpha^{q+1}}, \quad D = \frac{qT(\gamma+\beta)}{\alpha^{1-q/p} - C\alpha}.$$

Proof. By (3.6), for all $u = (u_1,u_2) \in \mathbb{R}^N \times \mathbb{R}^N$, we have

$$\begin{aligned}\frac{\delta}{\alpha}|u_1|^{p/2} - \frac{\beta}{\alpha} + \frac{\alpha^{q-1}}{q}|u_2|^q \le H(t,u) &= \frac{1}{\alpha}F(t,u_1) + \frac{\alpha^{q-1}}{q}|u_2|^q \\ &\le \frac{\alpha^{p-1}}{p}|u_1|^p + \frac{\gamma}{\alpha} + \frac{\alpha^{q-1}}{q}|u_2|^q. \end{aligned}\tag{3.9}$$

Then, we have

$$(u,v) - H(t,u) \ge (u,v) - \frac{\alpha^{p-1}}{p}|u_1|^p - \frac{\gamma}{\alpha} - \frac{\alpha^{q-1}}{q}|u_2|^q, \quad \forall\, u \in \mathbb{R}^N \times \mathbb{R}^N.$$

Since

$$
\begin{aligned}
&(u,v)-\frac{\alpha^{p-1}}{p}|u_1|^p-\frac{\gamma}{\alpha}-\frac{\alpha^{q-1}}{q}|u_2|^q\\
&=(u_1,v_1)+(u_2,v_2)-\frac{\alpha^{p-1}}{p}|u_1|^p-\frac{\gamma}{\alpha}-\frac{\alpha^{q-1}}{q}|u_2|^q\\
&\leq |u_1||v_1|-\frac{\alpha^{p-1}}{p}|u_1|^p-\frac{\gamma}{\alpha}+|u_2||v_2|-\frac{\alpha^{q-1}}{q}|u_2|^q\\
&\leq \sup_{u_1\in\mathbb{R}^N}\left\{|u_1||v_1|-\frac{\alpha^{p-1}}{p}|u_1|^p-\frac{\gamma}{\alpha}\right\}+\sup_{u_2\in\mathbb{R}^N}\left\{|u_2||v_2|-\frac{\alpha^{q-1}}{q}|u_2|^q\right\}\\
&=\frac{|v_1|^q}{q\alpha}-\frac{\gamma}{\alpha}+\frac{1}{p\alpha}|v_2|^p,\quad \forall\, u\in\mathbb{R}^N\times\mathbb{R}^N.
\end{aligned}
$$

Hence, by (2.1), we have

$$H^*(t,v)\geq \frac{|v_1|^q}{q\alpha}-\frac{\gamma}{\alpha}+\frac{1}{p\alpha}|v_2|^p.$$

When $v=\nabla H(t,u)$, by (2.2) and (3.9), we get

$$H^*(t,v)=(u,v)-H(t,u)\leq (u,v)+\frac{\beta}{\alpha}.$$

Then

$$\frac{|v_1|^q}{q\alpha}-\frac{\gamma}{\alpha}+\frac{1}{p\alpha}|v_2|^p\leq (u,v)+\frac{\beta}{\alpha}. \tag{3.10}$$

Note that

$$v=\nabla H(t,u)=\begin{pmatrix}\nabla H_1(t,u_1)\\ \nabla H_2(t,u_2)\end{pmatrix}=\begin{pmatrix}\frac{1}{\alpha}\nabla F(t,u_1)\\ \alpha^{q-1}|u_2|^{q-2}u_2\end{pmatrix}.$$

Then by (2.2) and (3.10), we have

$$\frac{\left|\frac{1}{\alpha}\nabla F(t,u_1)\right|^q}{q\alpha}-\frac{\gamma}{\alpha}+\frac{1}{p\alpha}\left|\alpha^{q-1}|u_2|^{q-2}u_2\right|^p\leq (u,\nabla H(t,u))+\frac{\beta}{\alpha},$$

that is

$$\frac{\alpha^{-(1+q)}}{q}|\nabla F(t,u_1)|^q-\frac{\gamma}{\alpha}+\frac{\alpha^{q-1}}{p}|u_2|^q\leq (u,\nabla H(t,u))+\frac{\beta}{\alpha}.$$

For each solution $u\in E$ of system (1.1), by (2.3) and (2.4), we know

$$\nabla F(t,u_1(t))=-\alpha\Delta u_2(t)$$

and

$$L\nabla H(t,u(t))=\nabla H(t,Lu(t))=-J\Delta u(t).$$

Hence

$$\frac{1}{q\alpha}|\Delta u_2(t)|^q-\frac{\gamma}{\alpha}+\frac{\alpha^{q-1}}{p}|u_2(t)|^q\leq (u(t),-L^{-1}(J\Delta u(t)))+\frac{\beta}{\alpha}.$$

Summing the above inequality over $\mathbb{Z}[1,T]$ and using Lemma 2.2 and (2.3), we obtain

$$\frac{1}{q\alpha}\|\Delta u_2\|_q^q-\frac{\gamma T}{\alpha}+\frac{\alpha^{q-1}}{p}\|u_2\|_q^q$$

$$\leq -\sum_{t=1}^{T}(u(t), L^{-1}(J\Delta u(t))) + \frac{\beta T}{\alpha} \leq \frac{C}{q}\|\Delta u_2\|_q^q + \frac{C}{p}\|\Delta u_1\|_p^p + \frac{\beta T}{\alpha}$$
$$= \frac{C}{q}\|\Delta u_2\|_q^q + \frac{C\alpha^q}{p}\|\Phi_q(u_2)\|_p^p + \frac{\beta T}{\alpha} = \frac{C}{q}\|\Delta u_2\|_q^q + \frac{C\alpha^q}{p}\|u_2\|_q^q + \frac{\beta T}{\alpha}.$$

So

$$\left(\frac{1}{q\alpha} - \frac{C}{q}\right)\|\Delta u_2\|_q^q + \left(\frac{\alpha^{q-1}}{p} - \frac{C\alpha^q}{p}\right)\|u_2\|_q^q \leq \frac{T(\beta+\gamma)}{\alpha}.$$

Since $\alpha \in \left(0, C^{-1}\right)$, we have

$$\|u_2\|_q^q \leq \frac{pT(\gamma+\beta)}{\alpha^q - C\alpha^{q+1}} = B, \quad \|\Delta u_2\|_q^q \leq \frac{qT(\gamma+\beta)}{1-C\alpha} = D. \tag{3.11}$$

Hence,

$$\|\Delta u_1\|_p^p = \alpha^q\|\Phi_q(u_2)\|_p^p = \alpha^q\|u_2\|_q^q \leq B\alpha^q. \tag{3.12}$$

It follows that (3.8) holds. Since F is continuously differentiable and convex in x, then by Lemma 2.3, (3.6), (2.3), Lemma 2.2, Hölder's inequality, (3.11) and (3.12), we have

$$\delta\sum_{t=1}^{T}|u_1(t)|^{p/2} - \beta T \leq \sum_{t=1}^{T}F(t,u_1(t)) \leq \sum_{t=1}^{T}[F(t,0) + (\nabla F(t,u_1(t)), u_1(t))]$$
$$\leq \gamma T - \sum_{t=1}^{T}(\alpha\Delta u_2(t), u_1(t)) = \gamma T - \sum_{t=1}^{T}(\alpha\Delta u_2(t), \tilde{u}_1(t))$$
$$\leq \gamma T + \alpha\left(\sum_{t=1}^{T}|\tilde{u}_1(t)|^p\right)^{1/p}\left(\sum_{t=1}^{T}|\Delta u_2(t)|^q\right)^{1/q}$$
$$\leq \gamma T + \alpha C(q,p)\|\Delta u_1\|_p\|\Delta u_2\|_q \leq \gamma T + \alpha^q C(q,p)B^{1/p}D^{1/q}.$$

So, we get

$$\sum_{t=1}^{T}|u_1(t)|^{p/2} \leq \frac{(\gamma+\beta)T}{\delta} + \frac{\alpha^q C(q,p)B^{1/p}D^{1/q}}{\delta}.$$

It follows that (3.7) holds. The proof is complete. ∎

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