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FP-Gorenstein Cotorsion Modules

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Abstract. Let R be a ring. In this paper, FP-Gorenstein cotorsion modules are introduced and studied. An R-module N is said to be FP-Gorenstein cotorsion if $\mathrm{Ext}_R^1(F,N) = 0$ for any finitely presented Gorenstein flat R-module F. We prove that the class of FP-Gorenstein cotorsion modules is covering and preenveloping over coherent rings. FP-Gorenstein cotorsion dimension of modules and rings are also studied. Some properties of FP-Gorenstein cotorsion modules are given.

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1. Introduction and preliminaries

Throughout this paper, R will denote an associative ring with identity and all modules will be unitary. Unless otherwise stated, R-modules always denote left R-modules. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+; $\mathrm{fd}(M)$, $\mathrm{id}(M)$, $\mathrm{pd}(M)$ and FP-$\mathrm{id}(M)$ stand for the flat, injective, projective and FP-injective dimensions of M respectively. As usual, we use ${}_R\mathfrak{M}$ to denote the class of left R-modules, $\mathrm{wD}(R)$ the weakly global dimension of R and $\mathrm{D}(R)$ the left global dimension of R. For unexplained concepts, notions and facts, we refer the reader to [3, 7–9, 17, 18, 20, 21].

We first recall some notions and facts which we need in the later sections.

(1) Let M be an R-module and $\mathcal{X}$ a class of R-modules. A homomorphism $\phi : M \to X$ with $X \in \mathcal{X}$ is called an $\mathcal{X}$-*preenvelope* [7, 16, 18, 20] of M if for any homomorphism $f : M \to X'$ with $X' \in \mathcal{X}$, there is a homomorphism $g : X \to X'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of X when $X = X'$ and $f = \phi$, the $\mathcal{X}$-preenvelope ϕ is called an $\mathcal{X}$-*envelope* of M. $\mathcal{X}$ is a (pre)enveloping class provided that each module has an $\mathcal{X}$-(pre)envelope. Dually, $\mathcal{X}$-precovers, $\mathcal{X}$-covers, and covering classes of modules can be defined.

(2) Let $\mathcal{X}, \mathcal{Y}$ be two classes of R-modules. $\mathcal{X}^\perp = \{N \in {}_R\mathfrak{M} \mid \mathrm{Ext}_R^1(X,N) = 0 \text{ for all } X \in \mathcal{X}\}$ and ${}^\perp\mathcal{Y} = \{M \in {}_R\mathfrak{M} \mid \mathrm{Ext}_R^1(M,Y) = 0 \text{ for all } Y \in \mathcal{Y}\}$. A module M is said to have a *special*

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$\mathcal{X}$*-precover* [7] if there is an exact sequence $0 \to K \to X \to M \to 0$ with $X \in \mathcal{X}$ and $K \in \mathcal{X}^{\perp}$. Dually, M is said to be have a *special* $\mathcal{Y}$*-preenvelope* if there is an exact sequence $0 \to M \to Y \to L \to 0$ with $Y \in \mathcal{Y}$ and $L \in {}^{\perp}\mathcal{Y}$.

(3) Let $\mathcal{X},\mathcal{Y}$ be two classes of R-modules. The pair $(\mathcal{X},\mathcal{Y})$ is called a *cotorsion pair* (or *cotorsion theory*) [7–9] if $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. Let $\mathcal{S}$ be a class of R-modules. $({}^{\perp}(\mathcal{S}^{\perp}),\mathcal{S}^{\perp})$ is called the cotorsion pair *cogenerated* by $\mathcal{S}$. A cotorsion pair $(\mathcal{X},\mathcal{Y})$ is called *complete* if each module has a special $\mathcal{Y}$-preenvelope and *hereditary* if $\operatorname{Ext}^i_R(X,Y) = 0$ for all $i \geq 1, X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. $(\mathcal{X},\mathcal{Y})$ is called *perfect* provided that $\mathcal{X}$ is a covering class and $\mathcal{Y}$ is an enveloping class. We know that a cotorsion pair $(\mathcal{X},\mathcal{Y})$ is a complete cotorsion pair if it is cogenerated by a set [7, Theorem 7.4.1].

(4) An R-module M is called *Gorenstein flat* [7, 9, 20] if there exists an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat R-modules such that $M = \ker(F^0 \to F^1)$ and that remains exact whenever $E \otimes_R -$ is applied for any injective right R-module E. The class of Gorenstein flat modules is denoted by $\mathcal{GF}$. An R-module N is called *Gorenstein cotorsion* [9] if $\operatorname{Ext}^1_R(M,N) = 0$ for any Gorenstein flat R-module M. The class of Gorenstein cotorsion modules is denoted by $\mathcal{GC}$. Over right coherent rings, $(\mathcal{GF},\mathcal{GC})$ is a hereditary and perfect cotorsion pair [9, Theorem 3.1.9]. So we can define the Gorenstein cotorsion dimension $\operatorname{Gcd}(M)$ of an R-module M as the least nonnegative integer n such that there is an exact sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to 0$ with $C^i \in \mathcal{GC}$ for $0 \leq i \leq n$.

In Section 2, we introduce the concept of FP-Gorenstein cotorsion modules. We show that the class of FP-Gorenstein cotorsion modules is closed under extensions, pure submodules, pure quotients, direct products and direct limits (and so direct sums) over coherent rings. Some basic properties of FP-Gorenstein cotorsion modules are given. In Section 3, we prove that over coherent rings, every R-module M has a surjective FP-Gorenstein cotorsion cover and an injective FP-Gorenstein cotorsion preenvelope. In Section 4, we introduce and investigate the FP-Goresntein cotorsion dimension of modules and rings. We characterize some rings through FP-Gorenstein cotorsion dimensions.

2. Some properties of FP-Gorenstein cotorsion modules

We begin with the following definition.

Definition 2.1. *An R-module N is called* FP-Gorenstein cotorsion *if* $\operatorname{Ext}^1_R(F,N) = 0$ *for all finitely presented Gorenstein flat R-modules F.*

Proposition 2.1. *The following hold:*

(1) *Injective modules, FP-injective modules and Gorenstein cotorsion modules are FP-Goresntein cotorsion.*
(2) *Every direct product of FP-Gorenstein cotorsion modules is FP-Gorenstein cotorsion.*
(3) *Every finite direct sum of FP-Gorenstein cotorsion modules is FP-Gorenstein cotorsion.*
(4) *Suppose* $N = N_1 \oplus N_2$, *then N is FP-Gorenstein cotorsion if and only if* N_1 *and* N_2 *are both FP-Gorenstein cotorsion.*

Proof. By Definition 2.1. ∎

Recall that a ring R is called *left coherent* (resp. *right coherent*) if every finitely generated left (resp. right) ideal is finitely presented. A ring R is coherent if it is both left and right coherent. A ring R is left coherent if and only if every finitely generated submodule of a finitely presented R-module is also finitely presented.

Proposition 2.2. *Suppose R is a coherent ring and N an FP-Gorenstein cotorsion R-module. Then* $\mathrm{Ext}_R^i(F,N)=0$ *for any finitely presented Gorenstein flat R-module F and for all $i\geq 1$.*

Proof. Let F be a finitely presented Gorenstein flat R-module. By Definition 2.1, we need only to prove that $\mathrm{Ext}_R^i(F,M)=0$ for $i\geq 2$. Since R is coherent, we have a finitely generated free resolution of F

$$\cdots \to F_n \overset{f_n}{\to} F_{n-1} \overset{f_{n-1}}{\to} \cdots \to F_1 \overset{f_1}{\to} F_0 \overset{f_0}{\to} F \to 0.$$

Then every $\ker f_i$ (for $i\geq 0$) is also finitely presented and Gorenstein flat by [9, Corollary 2.1.8]. Hence $\mathrm{Ext}_R^{i+1}(F,N)\cong \mathrm{Ext}_R^1(\ker f_{i-1},M)=0$ for all $i\geq 1$. ∎

Corollary 2.1. *Let R be a coherent ring and $0\to N'\to N\to N''\to 0$ a short exact sequence. If N' is FP-Goresntein cotorsion, then N is FP-Goresntein cotorsion if and only if N'' is FP-Gorenstein cotorsion.*

Proof. Let F be any finitely presented Gorenstein flat R-module, we get the following exact sequence

$$0=\mathrm{Ext}_R^1(F,N')\to \mathrm{Ext}_R^1(F,N)\to \mathrm{Ext}_R^1(F,N'')\to \mathrm{Ext}_R^2(F,N').$$

By Proposition 2.2, $\mathrm{Ext}_R^2(F,N')=0$. Hence the result follows. ∎

Lemma 2.1. *Let R be a coherent ring. Then $\varinjlim N_i$ is FP-Gorenstein cotorsion, where $((N_i),(f_{ji}))$ is a direct system of FP-Gorenstein cotorsion R-modules. In particular, the class $\mathcal{FGC}$ of FP-Gorenstein cotorsion R-modules is closed under direct sums.*

Proof. Let F be a finitely presented Gorenstein flat R-module. By [19, Theorem 3.2], we get

$$\mathrm{Ext}_R^1(F,\varinjlim N_i)\cong \varinjlim \mathrm{Ext}_R^1(F,N_i)=0.$$

Then the result follows. ∎

It is not hard to see that the condition "R is commutative" can be dropped in [2, Proposition 1.3]. Then we have the next lemma.

Lemma 2.2. *If R is coherent, then a finitely presented R-module is Gorenstein flat if and only if it is Gorenstein projective.*

Remark 2.1.

(1) Let R be a coherent ring. Then each R-module with finite projective dimension is FP-Gorenstein cotorsion since finitely presented Gorenstein projective R-modules coincide with finitely presented Gorenstein flat R-modules by Lemma 2.2. Hence any R-module with finite injective dimension is also FP-Gorenstein cotorsion by [4, Lemma 2.1].

(2) Let $R=\mathbb{Z}$. Then $\mathrm{D}(R)=1$, so every Goresntein flat R-module is flat. Since finitely presented flat R-modules are finitely generated projective, every R-module is FP-Gorenstein cotorsion by Definition 2.1. Note that the quotient field $\mathbb{Q}$ of R is a flat R-module, but it is not a projective R-module. So there is an R-module L such

that $\operatorname{Ext}_R^1(\mathbb{Q}, L) \neq 0$, i.e., L is neither cotorsion nor Gorenstein cotorsion. This example shows that FP-Gorenstein cotorsion modules need not to be cotorsion or Gorenstein cotorsion. Then we get the following implications:

injective modules ⇒ Gorenstein cotorsion modules ⇒ cotorsion modules,
injective modules ⇒ FP-injective modules ⇒ FP-Gorenstein cotorsion modules.

Proposition 2.3. *Let R be a coherent ring.*

(1) *If an R-module N has finite FP-injective dimension, then N is FP-Gorenstein cotorsion.*
(2) *If a right R-module N has finite FP-injective dimension, then N^+ is FP-Gorenstein cotorsion.*
(3) *If an R-module M has finite flat dimension, then M is FP-Gorenstein cotorsion.*

Proof. (1). Suppose that $FP\text{-id}(N) = n < \infty$. Let F be a finitely presented Gorenstein flat R-module. Then there exists an exact sequence

$$0 \to F \to P^0 \to P^1 \to \cdots \to P^{n-1} \to L \to 0$$

such that P^i is finitely generated projective for $0 \le i \le n-1$ and L is a finitely presented Gorenstein flat R-module. Thus $\operatorname{Ext}_R^1(F, N) \cong \operatorname{Ext}_R^{n+1}(L, N) = 0$ and hence N is FP-Gorenstein cotorsion.

(2). Let F be a finitely presented Gorenstein flat R-module and E an injective right R-module. Then $\operatorname{Tor}_1^R(E, F) = 0$ and [7, Theorem 3.2.1] shows

$$\operatorname{Ext}_R^1(F, E^+) \cong \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}_1^R(E, F), \mathbb{Q}/\mathbb{Z}) = 0,$$

which implies that E^+ is FP-Gorenstein cotorsion for every injective right R-module E.

Next, we assume that $FP\text{-id}(N) = n < \infty$. Then there exists an exact sequence

$$0 \to N \to E^0 \to E^1 \cdots \to E^{n-1} \to L \to 0$$

such that each E^i is injective for $0 \le i \le n-1$ and L is FP-injective by [19, Lemma 3.1]. This exact sequence induces the following exact sequence

$$0 \to L^+ \to (E^{n-1})^+ \to \cdots \to (E^1)^+ \to (E^0)^+ \to N^+ \to 0.$$

By Corollary 2.1, it is sufficient to prove that L^+ is FP-Gorenstein cotorsion. Since L is FP-injective, L is a pure submodule of any right R-module which contains L. Then we get a pure exact sequence

$$0 \to L \to E \to K \to 0$$

with E injective. Note that

$$0 \to K^+ \to E^+ \to L^+ \to 0$$

splits, so L^+ is FP-Gorenstein cotorsion since E^+ is FP-Gorenstein cotorsion by the proof above. This completes the proof.

(3). Let F be a finitely presented Gorenstein flat R-module and F' a flat R-module. Then $F' = \varinjlim P_i$ for some direct system $((P_i), (f_{ji}))$, where each P_i is projective. By [10, Lemma 3.1.6], we have

$$\operatorname{Ext}_R^1(F, F') \cong \operatorname{Ext}_R^1(F, \varinjlim P_i) \cong \varinjlim \operatorname{Ext}_R^1(F, P_i) = 0.$$

Hence any flat R-module is FP-Gorenstein cotorsion. Assume that $\mathrm{fd}(M) = n$, then we have the exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where F_i is flat for $0 \le i \le n$. By the proof above, each F_i is FP-Gorenstein cotorsion and hence M is also FP-Gorenstein cotorsion by Corollary 2.1. ∎

Recall that a submodule T of an R-module N is said to be a *pure submodule* of N if $0 \to A \otimes_R T \to A \otimes_R N$ is exact for all right R-modules A, or equivalently, if $\mathrm{Hom}_R(A,N) \to \mathrm{Hom}_R(A,N/T) \to 0$ is exact for all finitely presented R-modules A. An exact sequence $0 \to T \xrightarrow{\lambda} N$ is said to be *pure exact* if $\lambda(T)$ is a pure submodule of N.

Proposition 2.4. *Let R be a ring and N an FP-Gorenstein cotorsion R-module. If the exact sequence $0 \to N' \to N \xrightarrow{\pi} N'' \to 0$ is pure, then N' is FP-Gorenstein cotorsion. In addition, if R is coherent, then N'' is also FP-Gorenstein cotorsion.*

Proof. Let F be a finitely presented Gorenstein flat R-module. Then we have an exact sequence

$$\begin{aligned} \mathrm{Hom}_R(F,N) \xrightarrow{\pi_*} \mathrm{Hom}_R(F,N'') &\to \mathrm{Ext}^1_R(F,N') \to \mathrm{Ext}^1_R(F,N)\ (=0) \\ &\to \mathrm{Ext}^1_R(F,N'') \to \mathrm{Ext}^2_R(F,N'). \end{aligned}$$

Since F is finitely presented and $0 \to N' \to N \xrightarrow{\pi} N'' \to 0$ is pure exact, π_* is epimorphic. So $\mathrm{Ext}^1_R(F,N') = 0$ and hence N' is FP-Gorenstein cotorsion. If R is coherent, then $\mathrm{Ext}^2_R(F,N') = 0$ by Proposition 2.3. So $\mathrm{Ext}^1_R(F,N'') = 0$ and then N'' is also FP-Gorenstein cotorsion. ∎

Corollary 2.2. *Suppose R is coherent. Then M is FP-Gorenstein cotorsion if and only if M^{++} is FP-Gorenstein cotorsion.*

Proof. Note that $0 \to M \to M^{++}$ is a pure exact sequence, then M is FP-Gorenstein cotorsion whenever M^{++} is by Proposition 2.4.

Conversely, suppose that M is FP-Gorenstein cotorsion. Let F be a finitely presented Gorenstein flat R-module and $\mathbf{P}$ a finitely generated projective resolution of F. Then we have

$$\begin{aligned} \mathrm{Ext}^1_R(F,M^{++}) &= H_{-1}(\mathrm{Hom}_R(\mathbf{P},M^{++})) \\ &\cong H_{-1}(\mathrm{Hom}_{\mathbb{Z}}(M^+ \otimes_R \mathbf{P},\mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(M^+ \otimes_R \mathbf{P}),\mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(H_{-1}(\mathrm{Hom}_R(\mathbf{P},M)),\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ext}^1_R(F,M),\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) = 0. \end{aligned}$$

The second step is Hom-tensor adjointness. The fourth step follows from the proof of [18, Theorem 9.51] and [18, Remark, p.257]. Hence M^{++} is FP-Gorenstein cotorsion. ∎

3. Existences of FP-Gorenstein cotorsion covers and preenvelopes

In the rest of this article, $\mathcal{GF}_{fp}$ always denotes the class of finitely presented Gorenstein flat R-modules.

Theorem 3.1. *Let R be a coherent ring.*

(1) *Every R-module M has a surjective FP-Gorenstein cotorsion cover* $f : C \to M$.
(2) *The pair* $({}^{\perp}\mathcal{FGC}, \mathcal{FGC})$ *is a complete and hereditary cotorsion pair. In particular, every R-module M has a special* ${}^{\perp}\mathcal{FGC}$*-precover and a special FP-Gorenstein cotorsion preenvelope.*

Proof. (1). Since the class of FP-Gorenstein cotorsion modules is closed under pure quotient modules by Proposition 2.4 and closed under direct sums by Lemma 2.1, every R-module M has an FP-Gorenstein cotorsion cover $f : C \to M$ by [12, Theorem 2.5]. Note that each projective R-module is FP-Gorenstein cotorsion by Remark 2.1, then f is surjective.

(2). Firstly, it is easy to see that $({}^{\perp}\mathcal{FGC}, \mathcal{FGC}) = ({}^{\perp}(\mathcal{GF}_{fp}{}^{\perp}), \mathcal{GF}_{fp}^{\perp})$ is a cotorsion pair.

Secondly. For any finitely presented Gorenstein flat R-module F, $\mathrm{Card}(F) \leq \aleph_0 \cdot \mathrm{Card}(R)$. Let Y be the set of all finitely presented Gorenstein flat R-modules F such that $\mathrm{Card}(F) \leq \aleph_0 \cdot \mathrm{Card}(R)$. Then C is in $\mathcal{FGC}$ if and only if $\mathrm{Ext}^1_R(F, C) = 0$ for all $F \in Y$. This just says that the cotorsion pair $({}^{\perp}\mathcal{FGC}, \mathcal{FGC})$ is cogenerated by the set Y and hence $({}^{\perp}\mathcal{FGC}, \mathcal{FGC})$ is a complete cotorsion pair by [10, Theorem 3.2.1]. In particular, every R-module M has a special ${}^{\perp}\mathcal{FGC}$-precover and a special $\mathcal{FGC}$-preenvelope.

Thirdly. $\mathcal{FGC}$ is coresolving by Proposition 2.1 and Corollary 2.1, so $({}^{\perp}\mathcal{FGC}, \mathcal{FGC})$ is a hereditary cotorsion pair by [8, Theorem 2.1.4]. ∎

Remark 3.1.

(1) Note that $\mathcal{FGC}$ contains all injective modules, then every $\mathcal{FGC}$-preenvelope $g : M \to C$ of an R-module M is a monomorphism. Clearly, ${}^{\perp}\mathcal{FGC}$ contains all projective R-modules, so each ${}^{\perp}\mathcal{FGC}$-precover $f : G \to N$ of an R-module N is an epimorphism.
(2) $\mathcal{GF} \supseteq {}^{\perp}\mathcal{FGC}$ since $\mathcal{GC} \subseteq \mathcal{FGC}$. So every R-module $M \in {}^{\perp}\mathcal{FGC}$ is Gorenstein flat. In general, ${}^{\perp}\mathcal{FGC}$ isn't closed under direct limits. If ${}^{\perp}\mathcal{FGC}$ is closed under direct limits, then ${}^{\perp}\mathcal{FGC}$ contains all flat R-modules since every flat module is a direct limit of finitely generated free R-modules. Even over the ring $\mathbb{Z}$, ${}^{\perp}\mathcal{FGC}$ doesn't contain all flat modules (see Remark 2.1(2)).

Corollary 3.1. *Let R be a coherent ring and* $f : M \to N$ *a monomorphism.*

(1) *If* $\mathrm{coker}(f) \in {}^{\perp}\mathcal{FGC}$, *then* $gf : M \to C$ *is also an* $\mathcal{FGC}$*-preenvelope of M whenever* $g : N \to C$ *is an* $\mathcal{FGC}$*-preenvelope of N.*
(2) *If* $g : N \to C$ *is a special* $\mathcal{FGC}$*-preenvelope of N, then* $\mathrm{coker}(f) \in {}^{\perp}\mathcal{FGC}$ *if and only if* $gf : M \to C$ *is a special* $\mathcal{FGC}$*-preenvelope of M.*

Proof. This is similar to the proof of [15, Proposition 2.6]. ∎

Proposition 3.1. *The following conditions are equivalent for a coherent ring R:*

(1) *Every R-module is FP-Gorenstein cotorsion.*
(2) *Every R-module* $M \in {}^{\perp}\mathcal{FGC}$ *is FP-Gorenstein cotorsion.*

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (1). Let M be an R-module. By Theorem 3.1, we have a short exact sequence:

$$0 \to C \to F \xrightarrow{f} M \to 0$$

such that $f : F \to M$ is a special ${}^{\perp}\mathcal{FGC}$-precover. So C is FP-Gorenstein cotorsion and hence M is FP-Gorenstein cotorsion by Corollary 2.1. ∎

4. FP-Gorenstein cotorsion dimension of modules and rings

Definition 4.1. *Let R be a ring. For an R-module M, the* FP-Gorenstein cotorsion dimension FP-Gcd(M) *of M is defined to be the smallest integer $n \geq 0$ such that* $\mathrm{Ext}_R^{n+1}(F, M) = 0$ *for any finitely presented Gorenstein flat R-module F. If there is no such n, set* FP-Gcd$(M) = \infty$. *The (left)* global FP-Gorenstein cotorsion dimension FP-G-cot.D(R) *of R is defined as the supremum of the FP-Gorenstein cotorsion dimensions of R-modules.*

Dually, we can define the ${}^{\perp}\mathcal{FGC}$ *dimension* of M, denoted by $\mathrm{Gfd}^*(M)$. Note that ${}^{\perp}\mathcal{FGC}$ contains all projective R-modules, then $\mathrm{Gfd}(M) \leq \mathrm{Gfd}^*(M) \leq \mathrm{pd}(M)$ for all R-modules M. The (left) *global* ${}^{\perp}\mathcal{FGC}$ *dimension* of R is defined by G-wD$^*(R) = \sup\{\mathrm{Gfd}^*(M) | M \in {}_R\mathfrak{M}\}$.

Proposition 4.1. *Let R be coherent and N an R-module.*

(1) *Consider the following two exact sequences*

$$0 \to N \to G^0 \to G^1 \to \cdots \to G^{n-1} \to X \to 0,$$
$$0 \to N \to \tilde{G}^0 \to \tilde{G}^1 \to \cdots \to \tilde{G}^{n-1} \to \tilde{X} \to 0,$$

where $G^0, G^1, \cdots, G^{n-1}$ and $\tilde{G}^0, \tilde{G}^1, \cdots, \tilde{G}^{n-1}$ are FP-Gorenstein cotorsion R-modules. Then X is FP-Gorenstein cotorsion if and only if $\tilde{X}$ is FP-Gorenstein cotorsion.

(2) *Dually, consider the following two exact sequences*

$$0 \to K \to F_{m-1} \to F_{m-2} \to \cdots \to F_0 \to N \to 0,$$
$$0 \to \tilde{K} \to \tilde{F}_{m-1} \to \tilde{F}_{m-2} \to \cdots \to \tilde{F}_0 \to N \to 0,$$

where $F_0, \cdots, F_{m-1}$ and $\tilde{F}_0, \cdots, \tilde{F}_{m-1}$ are all in ${}^{\perp}\mathcal{FGC}$. Then $K \in {}^{\perp}\mathcal{FGC}$ if and only if $\tilde{K} \in {}^{\perp}\mathcal{FGC}$.

Proof. (1). Clearly, we can construct the following diagram:

$$\begin{array}{ccccccccccccc}
0 & \to & N & \to & G^0 & \to & G^1 & \to \cdots \to & G^{n-1} & \to & X & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & E^0 & \to & E^1 & \to \cdots \to & E^{n-1} & \to & L & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & N & \to & \tilde{G}^0 & \to & \tilde{G}^1 & \to \cdots \to & \tilde{G}^{n-1} & \to & \tilde{X} & \to & 0
\end{array}$$

where E^i is injective for $0 \leq i \leq n-1$. By mapping cone, we get the following two exact sequences:

$$0 \to N \to N \oplus G^0 \to E^0 \oplus G^1 \to \cdots \to E^{n-2} \oplus G^{n-1} \to E^{n-1} \oplus X \to L \to 0,$$
$$0 \to N \to N \oplus \tilde{G}^0 \to E^0 \oplus \tilde{G}^1 \to \cdots \to E^{n-2} \oplus \tilde{G}^{n-1} \to E^{n-1} \oplus \tilde{X} \to L \to 0.$$

Then we get two exact sequences by [7, Remark 1.4.14]:

$$0 \to G^0 \to E^0 \oplus G^1 \to \cdots \to E^{n-2} \oplus G^{n-1} \to E^{n-1} \oplus X \to L \to 0,$$
$$0 \to \tilde{G}^0 \to E^0 \oplus \tilde{G}^1 \to \cdots \to E^{n-2} \oplus \tilde{G}^{n-1} \to E^{n-1} \oplus \tilde{X} \to L \to 0.$$

By Corollary 2.1, X is FP-Gorenstein cotorsion if and only if L is FP-Gorenstein cotorsion if and only if $\tilde{X}$ is FP-Gorenstein cotorsion.

(2). The proof is dual to that of (1). ∎

Over coherent rings, it is easily to see $\mathrm{Gfd}^*(M) = \mathrm{Gfd}(M)$ for every finitely presented R-module M.

Theorem 4.1. *Let R be a coherent ring.*

(1) FP-$\mathrm{Gcd}(M) = 0$ *or* ∞ *for an R-module M.*
(2) FP-G-$\mathrm{cot.D}(R) = 0$ *or* ∞.
(3) $(\mathcal{FGC}, \mathcal{FGC}^{\perp})$ *is a perfect, hereditary cotorsion pair.*

Proof. (1). Suppose that FP-$\mathrm{Gcd}(M) = n < \infty$ for some nonnegative integer n. Let F be a finitely presented Gorenstein flat R-module. Then there exists an exact sequence

$$0 \to F \to P^0 \to P^1 \to P^2 \to \cdots \to P^{n-1} \to F' \to 0$$

such that each P^i is finitely generated projective for $0 \le i \le n-1$ and F' is a finitely presented Gorenstein flat. So we get $\mathrm{Ext}^1_R(F, M) \cong \mathrm{Ext}^{n+1}_R(F', M) = 0$. Hence M is FP-Gorenstein cotorsion.

(2) is clear by (1).

(3). We first prove that $(\mathcal{FGC}, \mathcal{FGC}^{\perp})$ is a cotorsion pair. Note that $({}^{\perp}(\mathcal{FGC}^{\perp}), \mathcal{FGC}^{\perp})$ is a cotorsion pair, then we must prove $\mathcal{FGC} = {}^{\perp}(\mathcal{FGC}^{\perp})$. $\mathcal{FGC} \subseteq {}^{\perp}(\mathcal{FGC}^{\perp})$ is clear, so we need to prove $\mathcal{FGC} \supseteq {}^{\perp}(\mathcal{FGC}^{\perp})$. For any R-module $M \in {}^{\perp}(\mathcal{FGC}^{\perp})$, there exists an exact sequence $0 \to K \to C \to M \to 0$, where $C \to M$ is the FP-Gorenstein cotorsion cover of M by Theorem 3.1. Then $K \in \mathcal{FGC}^{\perp}$ by [20, Lemma 2.1.1] and so $\mathrm{Ext}^1_R(M, K) = 0$. Hence $0 \to K \to C \to M \to 0$ splits and then $M \in \mathcal{FGC}$. So $\mathcal{FGC} \supseteq {}^{\perp}(\mathcal{FGC}^{\perp})$.

Note that $\mathcal{FGC}$ is resolving by Remark 2.1 and Theorem 4.1, then $(\mathcal{FGC}, \mathcal{FGC}^{\perp})$ is a complete, hereditary cotorsion pair by Theorem 3.1 and [7, Proposition 7.1.7].

Since $\mathcal{FGC}$ is closed under direct limits by Proposition 2.4, $(\mathcal{FGC}, \mathcal{FGC}^{\perp})$ is a perfect cotorsion pair by [7, Theorem 7.2.6]. ∎

Proposition 4.2. *Let R be a coherent ring and M an R-module. Then the following are equivalent for a nonnegative integer n:*

(1) $\mathrm{Gfd}^*(M) \le n$.
(2) $\mathrm{Ext}^{n+1}_R(M, C) = 0$ *for all FP-Gorenstein cotorsion R-modules C.*
(3) $\mathrm{Ext}^{i}_R(M, C) = 0$ *for all FP-Gorenstein cotorsion R-modules C and all $i \ge n+1$.*
(4) *If the sequence $0 \to G^n \to G^{n-1} \to \cdots \to G^0 \to M \to 0$ is exact such that $G^0, G^1, \cdots, G^{n-1}$ are all in ${}^{\perp}\mathcal{FGC}$, then G^n is also in ${}^{\perp}\mathcal{FGC}$.*
(5) *If $f : M \to C$ is a special $\mathcal{FGC}$-preenvelope, then* $\mathrm{Gfd}^*(C) \le n$.

Consequently, the ${}^{\perp}\mathcal{FGC}$ dimension of M is determined by the formula:

$$\mathrm{Gfd}^*(M) = \sup\{i \in \mathbb{N}_0 | \exists C \in \mathcal{FGC} : \mathrm{Ext}^i_R(M, C) \neq 0\}.$$

Proof. By Definition 4.1, Proposition 4.1 and Theorem 3.1. ∎

Corollary 4.1. *Let R be a coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of R-modules. If two of* $\mathrm{Gfd}^*(A)$, $\mathrm{Gfd}^*(B)$ *and* $\mathrm{Gfd}^*(C)$ *are finite, so does the third. Moreover,*

(1) $\mathrm{Gfd}^*(B) \le \max\{\mathrm{Gfd}^*(A), \mathrm{Gfd}^*(C)\}$,
(2) $\mathrm{Gfd}^*(C) \le \max\{\mathrm{Gfd}^*(A) + 1, \mathrm{Gfd}^*(B)\}$,
(3) $\mathrm{Gfd}^*(A) \le \max\{\mathrm{Gfd}^*(B), \mathrm{Gfd}^*(C) - 1\}$.

In particular, if B is in $^{\perp}\mathcal{FGC}$ *and* $\mathrm{Gfd}^*(C) > 0$*, then* $\mathrm{Gfd}^*(C) = \mathrm{Gfd}^*(A) + 1$.

Corollary 4.2. *Let R be a coherent ring with* $\mathrm{D}(R) < \infty$*. Then* $G\text{-wD}^*(R) = \mathrm{D}(R)$*. In particular, R is left hereditary if and only if* $G\text{-wD}^*(R) \leq 1$.

Proposition 4.3. *Let R be a coherent ring with* $G\text{-wD}^*(R) = n$ *for some nonnegative integer n and M an R-module. Then*

(1) $\mathrm{id}(M) \leq n$ *if* $\mathrm{fd}(M) < \infty$,
(2) $\mathrm{id}(M) \leq n$ *if* $\mathrm{pd}(M) < \infty$,
(3) $\mathrm{id}(M) < \infty$ *if and only if* $\mathrm{id}(M) \leq n$ *if and only if* $FP\text{-id}(M) \leq n$ *if and only if* $FP\text{-id}(M) < \infty$.

Proof. (1). Since $G\text{-wD}^*(R) = n < \infty$, there exists an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to N \to 0$$

for any R-module N such that $F_i \in {}^{\perp}\mathcal{FGC}$ for $0 \leq i \leq n$. Note that $M \in \mathcal{FGC}$ if $\mathrm{fd}(M) < \infty$ by Proposition 2.3, then we have $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for any R-module N. Hence $\mathrm{id}(M) \leq n$.

(2) is a consequence of (1).

(3). $\mathrm{id}(M) < \infty \Rightarrow \mathrm{id}(M) \leq n$ and $FP\text{-id}(M) < \infty \Rightarrow \mathrm{id}(M) \leq n$ are similar to (1).

$\mathrm{id}(M) \leq n \Rightarrow FP\text{-id}(M) \leq n \Rightarrow FP\text{-id}(M) < \infty$ are trivial. ∎

Theorem 4.2. *Let R be a Noetherian ring. Then the following are equivalent:*

(1) *R is quasi-Frobenius (i.e., 0-Gorenstein).*
(2) *Every FP-Gorenstein cotorsion R-module is injective.*
(3) *Every Gorenstein cotorsion R-module is injective.*
(4) $\mathrm{Gfd}^*(M) = 0$ *for any R-module M.*

Proof. (1) ⇒ (2). Since R is quasi-Frobenius, R/I is finitely presented Gorenstein flat for any left ideal I of R. Then for any FP-Gorenstein cotorsion R-module N, we have $\mathrm{Ext}_R^1(R/I, N) = 0$. So N is injective by Bear criterion.

(2) ⇒ (3) and (2) ⇔ (4) are trivial.

(3) ⇒ (1). Since $(\mathcal{GF}, \mathcal{GC})$ is a cotorsion pair, every R-module is Gorenstein flat by (3). Then R is quasi-Frobenius by [7, Theorem 12.3.1]. ∎

Remark 4.1. In general, $G\text{-wD}(R) \leq G\text{-wD}^*(R) \leq \mathrm{D}(R)$. Theorem 4.2 shows that the the second inequality may be strict. In fact, the first inequality may be also strict. For example, consider Small's triangular ring

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

Since $\mathrm{wD}(R) = 1$ and $\mathrm{D}(R) = 2$ by [13, Example (5.62b)], we have $G\text{-wD}(R) = \mathrm{wD}(R) = 1 < G\text{-wD}^*(R) = \mathrm{D}(R) = 2$.

Following [5], a ring R is called an *n-FC ring* if R is left and right coherent with $FP\text{-id}({}_RR) \leq n$ and $FP\text{-id}(R_R) \leq n$ for an integer $n \geq 0$. An R-module M is said to be *torsionless* (or *semi-reflexive*) [13] if the natural map $i : M \to M^{**}$ is a monomorphism and an R-module M is called *reflexive* if $i : M \to M^{**}$ is an isomorphism, where $M^* = \mathrm{Hom}_R(M, R)$.

Theorem 4.3. *Let R be a coherent ring. Then the following are equivalent:*

(1) *R is an FC ring (i.e., 0-FC ring).*
(2) *Every FP-Gorenstein cotorsion R-module is FP-injective.*

Proof. (1) ⇒ (2). Since R is FC, every R-module is Gorenstein flat by [14, Proposition 5.5]. For any FP-Gorenstein cotorsion R-module N, we have $\mathrm{Ext}_R^1(F,N) = 0$ for any finitely presented R-module F. Hence N is FP-injective.

(2) ⇒ (1). Let M be a finitely presented R-module. Since every FP-Gorenstein cotorsion R-module is FP-injective by (2), every finitely presented R-module M is Gorenstein flat and hence Gorenstein projective. Then M can be embedded in a free R-module and is torsionless by [13, Remarks 4.65]. By [19, Lemma 4.6], we have an exact sequence

$$0 \to M \to M^{**} \to \mathrm{Ext}_R^1(L,R) \to 0$$

for some finitely presented R-module L. Note that L is finitely presented Gorenstein projective and hence $\mathrm{Ext}_R^1(L,R) = 0$ since R is FP-Gorenstein cotorsion by Remark 2.1. Then M is reflexive and R is an FC ring by [19, Theorem 4.9]. ∎

Example 4.1. By Theorems 4.1, 4.2 and 4.3, we get

(1) If R is quasi-Frobenius (i.e., 0-Gorenstein), then the cotorsion pair $(\mathcal{FGC},\mathcal{FGC}^{\perp})$ is exactly $(\mathcal{P}roj,{}_R\mathfrak{M})$, where $\mathcal{P}roj$ is the class of projective R-modules. In fact, by Theorem 4.2, FP-Gorenstein cotorsion R-modules coincide with injective R-modules. Note that R is quasi-Frobenius, so projective modules coincide with injective modules. Then the result holds.

(2) If R is an FC ring, then the cotorsion pair $(\mathcal{FGC},\mathcal{FGC}^{\perp})$ is exactly $(\mathcal{F}lat,\mathcal{C}ot)$, where $\mathcal{F}lat$ ($\mathcal{C}ot$) is the class of flat (cotorsion) R-modules.

Proposition 4.4. *Let R be a coherent ring. Then the following are equivalent:*

(1) *R is n-FC.*

(2) *FP-id$(M) \leq n$ for any FP-Gorenstein cotorsion (left and right) R-module M.*

Proof. (1) ⇒ (2). Let N be a finitely presented R-module. Since R is n-FC, we get $\mathrm{Gfd}(N) \leq n$ by [5, Theorem 7]. Then $\mathrm{Ext}_R^{n+1}(N,M) = 0$ for any FP-Gorenstein cotorsion R-module M. So FP-id$(M) \leq n$ by [19, Theorem 3.1].

(2) ⇒ (1). Suppose $n \geq 1$. Let N be a finitely presented R-module and M an FP-Gorenstein cotorsion R-module. We get a finitely generated projective resolution of N:

$$0 \to K \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to N \to 0.$$

Since FP-id$(M) \leq n$, $0 = \mathrm{Ext}_R^{n+1}(N,M) \cong \mathrm{Ext}_R^1(K,M)$. Then K is finitely presented Gorenstein flat and hence R is n-FC by [5, Theorem 7] again.

Suppose $n = 0$. By Theorem 4.3, we easily get that R is an FC ring. ∎

Corollary 4.3. *Let R be an n-FC ring. Then the following are equivalent:*

(1) *${}^{\perp}\mathcal{FGC}$ is closed under direct limits.*

(2) *$\mathcal{FGC} = \mathcal{GC}$.*

Proof. (1) ⇒ (2). Since R is an n-FC ring, every Gorenstein flat R-module M is isomorphic to $\varinjlim P_i$ for some inductive system $((P_i),(f_{ji}))$ by [5, Theorem 5], where each P_i is a finitely presented Gorenstein flat R-module. By (1), every Gorenstein flat R-module is in ${}^{\perp}\mathcal{FGC}$, so (2) follows.

(2) ⇒ (1). Since $({}^{\perp}\mathcal{FGC},\mathcal{FGC})$ and $(\mathcal{GF},\mathcal{GC})$ are both cotorsion pairs, we get ${}^{\perp}\mathcal{FGC} = \mathcal{GF}$ by (2). Hence ${}^{\perp}\mathcal{FGC}$ is closed under direct limits by [9, Corollary 2.1.9]. ∎

Theorem 4.4. *Let R be a coherent ring.*

(1) *If every FP-Gorenstein cotorsion R-module is Gorenstein cotorsion, then R is left perfect.*
(2) *If R is an n-FC ring and N is a pure-injective R-module, then N is FP-Gorenstein cotorsion if and only if N is Gorenstein cotorsion.*
(3) *If R is left perfect, then* $\mathrm{Gfd}^*(F) = 0$ *or* ∞ *for any Gorenstein flat R-module F. Furthermore, if* $G\text{-wD}^*(R) < \infty$*, then an R-module M is Gorenstein cotorsion if and only if it is FP-Gorenstein cotorsion.*

Proof. (1). For any flat R-module F, we have a short exact sequence

$$0 \to K \to P \to F \to 0.$$

Note that K is flat and so it is FP-Gorenstein cotorsion by Proposition 2.3. Then we have $\mathrm{Ext}^1_R(F,K) = 0$ and so the sequence splits. Thus F is projective and then R is left perfect.

(2). The sufficiency is trivial.

Necessity. Suppose $n \geq 1$. Let M be a Gorenstein flat R-module. Note that R is n-FC, $M \cong \varinjlim C_i$ for some inductive system $((C_i),(f_{ji}))$, where each C_i is a finitely presented Gorenstein projective R-module by [5, Theorem 5]. Note that N is pure-injective, then [10, Lemma 3.3.4] implies

$$\mathrm{Ext}^1_R(M,N) \cong \mathrm{Ext}^1_R(\varinjlim C_i, N) \cong \varprojlim \mathrm{Ext}^1_R(C_i,N) = 0.$$

So N is Gorenstein cotorsion.

Suppose $n = 0$. Note that an R-module N is FP-Gorenstein cotorsion if and only if it is FP-injective by Theorem 4.3, the rest proof is similar to the case $n \geq 1$.

(3). Let F_0 be a Gorenstein flat R-module. Suppose $\mathrm{Gfd}^*(F_0) = n < \infty$ and let $f : G \to F_0$ be a special ${}^{\perp}\mathcal{FGC}$-precover. Then $K = \ker(f)$ is FP-Gorenstein cotorsion and Gorenstein flat. There exists an exact sequence

$$0 \to F_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to K \to 0$$

with each P_i projective and $F_n \in {}^{\perp}\mathcal{FGC}$. It is easy to see that F_n is FP-Gorenstein cotorsion. Note that there is an exact sequence

$$0 \to L \to P \to F_n \to 0$$

with P projective and $L \in \mathcal{FGC}$. The sequence splits and then F_n is projective. It is not hard to prove that every Gorenstein flat R-module is Gorenstein projective when R is coherent and left perfect. Hence we get that K is projective and so the short exact sequence $0 \to K \to G \to F_0 \to 0$ splits. Hence F_0 is a direct summand of G and so $F_0 \in {}^{\perp}\mathcal{FGC}$. Then $\mathrm{Gfd}^*(F_0) = 0$ or ∞.

Now, the last statement is obvious. ∎

Remark 4.2. The condition $G\text{-wD}^*(R) < \infty$ in Theorem 4.4 (3) can be replaced by $\mathrm{Gfd}^*(F) < \infty$ for all Gorenstein flat R-modules F.

Corollary 4.4. *Let R be a coherent ring. Then the following hold:*

(1) *Every FP-Gorenstein cotorsion R-module is Gorenstein cotorsion if and only if R is left perfect and* $\mathrm{Gfd}^*(F) < \infty$ *for all Gorenstein flat R-modules F.*
(2) $\mathrm{Gfd}(M) \leq \mathrm{Gfd}^*(M) \leq \mathrm{pd}(M)$ *for any R-module M. Furthermore, if R is left perfect, then*
   (a) $\mathrm{Gfd}(M) = \mathrm{Gfd}^*(M)$ *if* $\mathrm{Gfd}^*(M) < \infty$.

(b) $\mathrm{Gfd}(M) = \mathrm{Gfd}^*(M) = \mathrm{pd}(M)$ *if* $\mathrm{pd}(M) < \infty$.

Proof. (1). The sufficiency follows from Theorem 4.4 and Remark 4.2.

Necessity. Since $(\mathcal{GF}, \mathcal{GC})$ and $({}^{\perp}\mathcal{FGC}, \mathcal{FGC})$ are both cotorsion pairs, we easily get $\mathcal{GF} = {}^{\perp}\mathcal{FGC}$ by hypothesis and hence $\mathrm{Gfd}^*(F) = 0 < \infty$ for any Gorenstein flat R-module F.

(2). $\mathrm{Gfd}(M) \le \mathrm{Gfd}^*(M) \le \mathrm{pd}(M)$ are obvious. (a) holds by Theorem 4.4.

For (b), we claim that if an R-module is Gorenstein flat, then it is Gorenstein projective. Let F be a Gorenstein flat R-module. Note that R is left perfect, then we get an exact sequence of projective R-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $F = \ker(P^0 \to P^1)$ such that $E \otimes_R -$ is exact for any injective right R-module E. For any projective R-module Q, Q^+ is right injective, then

$$\mathrm{Ext}^i_R(F, Q^{++}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_i(Q^+, F), \mathbb{Q}/\mathbb{Z}) = 0$$

for all $i \ge 1$ by [7, Theorem 3.2.1] and [11, Theorem 3.6]. Since

$$0 \to Q \to Q^{++} \to Q^{++}/Q \to 0$$

is a pure short exact sequence, Q^{++}/Q is flat by [13, Corollary 4.86] and hence projective. This sequence splits and so Q is a direct summand of Q^{++}. We get $\mathrm{Ext}^i_R(F, Q) = 0$ for all $i \ge 1$ and then F is Gorenstein projective by [11, Proposition 2.3]. Thus (b) follows. ∎

Proposition 4.5. *If R is an n-FC ring with $n \ge 0$, then the following are equivalent:*

(1) $\mathrm{wD}(R) < \infty$.
(2) *Every finitely presented Gorenstein flat R-module is projective.*
(3) *Every R-module is FP-Gorenstein cotorsion.*
(4) *Every quotient of an FP-Gorenstein cotorsion R-module is FP-Gorenstein cotorsion.*
(5) *Every submodule of an FP-Gorenstein cotorsion R-module is FP-Gorenstein cotorsion.*
(6) *The left/right symmetric of* (1) ~ (5).

Proof. (1) ⇒ (2). Since $\mathrm{fd}(M) = 0$ or ∞ for any Gorenstein flat R-module M, M is flat by hypothesis. Hence every finitely presented Gorenstein flat R-module is projective.

(2) ⇒ (3) is trivial.

(3) ⇔ (4) ⇔ (5) hold by Theorems 3.1 and 4.1.

(3) ⇒ (1). Since ${}^{\perp}\mathcal{FGC} \subseteq \mathcal{GF}$, we easily get every finitely presented Gorenstein flat R-module is projective by hypothesis. For a Gorenstein flat R-module F, $F = \varinjlim G_i$ for some direct system $((G_i), (f_{ji}))$ by [5, Theorem 5], where each G_i is finitely presented Gorenstein flat. Note that each G_i is projective and hence $F = \varinjlim G_i$ is flat, then $\mathrm{wD}(R) < \infty$ by [5, Theorem 13].

(1) ⇔ (6). The proofs are similar to those of (1) ~ (5). ∎

Proposition 4.6. *Let R be a commutative coherent ring and M an R-module. Then the following are equivalent:*

(1) $M \in \mathcal{FGC}$.
(2) $\mathrm{Hom}_R(P, M) \in \mathcal{FGC}$ *for any projective R-module P.*

(3) *$G \otimes_R M \in \mathcal{FGC}$ for any flat R-module G.*

Proof. (1) ⇒ (2). Let P be a projective R-module and F a finitely presented Gorenstein flat R-module. Then there exists another projective R-module Q such that $P \oplus Q = R^{(X)}$ for some set X. So we have

$$\begin{aligned}\mathrm{Ext}^1_R(F,\ \mathrm{Hom}_R(P\oplus Q, M)) &\cong \mathrm{Ext}^1_R(F,\ \mathrm{Hom}_R(R^{(X)}, M)) \cong \mathrm{Ext}^1_R(F, (\mathrm{Hom}_R(R,M))^X) \\ &\cong (\mathrm{Ext}^1_R(F,M))^X = 0.\end{aligned}$$

Hence $\mathrm{Hom}_R(P,M) \in \mathcal{FGC}$ by Proposition 2.1.

(1) ⇒ (3). Let G be a flat R-module. Then $G = \varinjlim F_i$ for some direct system $((F_i),(f_{ji}))$, where each F_i is a free R-module. For any finitely presented Gorenstein flat R-module F, we have

$$\begin{aligned}\mathrm{Ext}^1_R(F, G\otimes_R M) &\cong \mathrm{Ext}^1_R(F, \varinjlim F_i \otimes_R M) \cong \mathrm{Ext}^1_R(F, \varinjlim (F_i\otimes_R M)) \\ &\cong \varinjlim \mathrm{Ext}^1_R(F, F_i\otimes_R M) \cong \varinjlim \mathrm{Ext}^1_R(F, M^{(X)}) = 0.\end{aligned}$$

The second isomorphism holds since $-\otimes_R -$ commutes with $\varinjlim$, the third follows by [10, Lemma 3.1.6] and the fourth holds since $\mathcal{FGC}$ is closed under direct sums. Hence $G\otimes_R M$ is FP-Gorenstein cotorsion.

(2) ⇒ (1) holds by letting $P = R$ and (3) ⇒ (1) holds by letting $G = R$. ∎

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