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Approximately Midconvex Set-Valued Functions

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Abstract. We will show that if F is a set-valued mapping which satisfies

$$F(x)+F(y)\subseteq 2F\left((x+y)/2\right)+K$$

for some convex compact set K, then under some restrictions, there are maximal superadditive and midconvex mappings which are K-subclose to F.

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1. Introduction

The notion of stability of functional equations has its origins with Ulam [25], who posed the fundamental problem in 1940 and with Hyers [6], who gave the first significant partial solution in 1941. A generalized version of Hyers theorem for approximately linear mappings was given by Rassias [19]. Since then, the stability problems of various functional equations have been extensively investigated by a number of authors (e.g. [1, 7–12, 18, 22, 26]).

Functional inclusion is a tool for defining many notions of set-valued analysis, e.g. linear, affine, convex, midconvex, concave, superadditive and subadditive maps.

In set-valued analysis, a functional inclusion is called stable if any function which satisfies this inclusion approximately is near to a true solution of the functional inclusion. The Hyers-Ulam stability is discussed for set-valued functional equations and inclusions by some mathematicians [3, 15–17, 24].

Let X and Y be semigroups and $F:X\to 2^Y$. If F satisfies

$$F(x)+F(y)\subseteq F(x+y)\quad (x\in X), \tag{1.1}$$

then F is called superadditive. A function $F:X\to 2^Y$ is called midconvex if

$$F(x)+F(y)\subseteq 2F\left(\frac{x+y}{2}\right)\quad (x,y\in X). \tag{1.2}$$

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Note that these notions are different. For example, if $F,G:[0,\infty)\to 2^{\mathbb{R}}$ are defined by $F(x)=[0,\sqrt{x}\ ]$ and $G(x)=[0,x^2]$ for each $x\in[0,\infty)$, then F is midconvex but it is not superadditive, while the converse holds for G.

Some authors studied different properties of midconvex and additive set-valued functions (e.g. [2, 5, 14, 23]). In this paper, we will show that, under certain circumstances, every approximately midconvex function F from an abelian semigroup to compact convex subsets of a topological vector space can be approximated by a set-valued additive mapping. We also prove that there exists a maximal midconvex set-valued mapping which approximates F.

2. Results

Throughout the paper, unless otherwise state, we will assume that X is an abelian semigroup divisible by two and Y is a topological vector space. If $A,B\subset Y$ and $\lambda\in\mathbb{R}$, we define

$$A+B=\{a+b : a\in A,\ b\in B\},\quad \lambda A=\{\lambda a : a\in A\}.$$

One can easily see that for each $A,B\subset Y$ and $\lambda,\mu\geq 0$,

$$\lambda(A+B)=\lambda A+\lambda B,\quad (\lambda+\mu)A\subseteq\lambda A+\mu A.$$

Moreover, if A is convex, then $(\lambda+\mu)A=\lambda A+\mu A$. We denote by $C(Y)$ and $CC(Y)$ the collection of all non-empty compact subsets and all non-empty compact convex subsets of Y respectively.

Definition 2.1. *If K is a subset of Y and $F:X\to 2^Y$, we say that F is K-midconvex if*

$$F(x)+F(y)\subseteq 2F\left(\frac{x+y}{2}\right)+K\quad (x,y\in X). \tag{2.1}$$

The above definition is known in the case where K is a convex cone. Many properties of such set-valued functions can be found, for instance in [13].

We need some axillary results. The first one is due to Rådström [20].

Lemma 2.1. *Let A,B and C be nonempty subsets of a topological vector space Y. Suppose that B is closed and convex and C is bounded. If $A+C\subseteq B+C$, then $A\subseteq B$. If moreover, A is closed and convex and $A+C=B+C$, then $A=B$.*

The following result may be found in [4, Lemma 29.2].

Lemma 2.2. *Assume that $\{A_n\}$ and $\{B_n\}$ are deceasing sequences of closed subsets of topological vector space and A_1 is compact. Then*

$$\bigcap_{n=1}^{\infty}\Big(A_n+B_n\Big)=\bigcap_{n=1}^{\infty}A_n+\bigcap_{n=1}^{\infty}B_n.$$

Definition 2.2. *Let $F,G:X\to C(Y)$ be two set valued functions, for subset K of Y we say that F is K-subclose to G if $F(x)\subseteq G(x)+K\quad (x\in X)$.*

Theorem 2.1. *Let $F:X\to CC(Y)$ be a K-midconvex set-valued function, $K\in CC(Y)$ and $0\in F(0)$. Then there exists a superadditive set-valued function $A:X\to CC(Y)$ which is maximal K-subclose to F and $A(2x)=2A(x)$ for each $x\in X$.*

Proof. We divide the proof into three steps.

Step 1. *There is a superadditive function $A : X \to CC(Y)$ such that $A(x) \subseteq F(x) + K$ for each $x \in X$.*
Put $y = 0$ in (2.1) to obtain

$$F(x) + F(0) \subseteq 2F\left(\frac{x}{2}\right) + K \quad (x \in X).$$

Since $0 \in F(0)$, we have

$$F(x) \subseteq 2F\left(\frac{x}{2}\right) + K \quad (x \in X). \tag{2.2}$$

Replacing x by $2^n x$ in (2.2), we see that

$$F(2^n x) \subseteq 2F\left(2^{n-1}x\right) + K \quad (x \in X, n \in \mathbb{N}). \tag{2.3}$$

By multiplying both sides of (2.3) by 2^{-n}, we get

$$2^{-n}F(2^n x) \subseteq 2^{-(n-1)}F\left(2^{(n-1)}x\right) + \frac{K}{2^n} \quad (x \in X, n \in \mathbb{N}). \tag{2.4}$$

It follows from (2.4) that

$$2^{-n}F(2^n x) + \frac{K}{2^n} \subseteq 2^{-(n-1)}F\left(2^{(n-1)}x\right) + \frac{K}{2^{n-1}} \quad (x \in X, n \in \mathbb{N}). \tag{2.5}$$

Let $A_n(x) = 2^{-n}F(2^n x) + K/2^n \quad (x \in X, n \in \mathbb{N})$. It follows from (2.5) that $\{A_n(x)\}$ is a non-increasing sequence of compact sets in Y for each $x \in X$. Hence

$$A(x) = \bigcap_{n=0}^{\infty} A_n(x) \quad (x \in X)$$

defines a non-empty compact convex valued function on X. In view of (2.5), $A_n(x) \subset A_0(x) = F(x) + K$ for each $n \in \mathbb{N}$ and $x \in X$. Therefore $A(x) \subset F(x) + K$ for each $x \in X$. Moreover,

$$\begin{aligned}
A(x) + A(y) &= \bigcap_{n=0}^{\infty} A_n(x) + \bigcap_{n=0}^{\infty} A_n(y) \\
&\subseteq \bigcap_{n=0}^{\infty} (A_n(x) + A_n(y)) \subseteq \bigcap_{n=1}^{\infty} \left(2^{-n}F(2^n x) + \frac{K}{2^n} + 2^{-n}F(2^n y) + \frac{K}{2^n}\right) \\
&\subseteq \bigcap_{n=1}^{\infty} \left(2^{-n}\left(2F\left(\frac{2^n x + 2^n y}{2}\right) + K\right) + \frac{K}{2^{n-1}}\right) \\
&\subseteq \bigcap_{n=1}^{\infty} \left(2^{-(n-1)}F\left(2^{n-1}x + 2^{n-1}y\right) + \frac{K}{2^{n-1}} + \frac{K}{2^n}\right) \\
&= \bigcap_{n=1}^{\infty} \left(2^{-(n-1)}F\left(2^{n-1}x + 2^{n-1}y\right) + \frac{K}{2^{n-1}}\right) + \bigcap_{n=1}^{\infty} \frac{K}{2^n} \quad \text{by Lemma 2.2} \\
&= \bigcap_{n=1}^{\infty} A_{n-1}(x+y) = A(x+y)
\end{aligned}$$

for each $x, y \in X$. Hence A is superadditive.

Step 2. $A(2x) = 2A(x)$.
For each $x \in X$, we have

$$A(2x) = \bigcap_{n=0}^{\infty} A_n(2x) = \bigcap_{n=0}^{\infty}\left[2^{-n}F(2^{n+1}x) + \frac{K}{2^n}\right] = \bigcap_{n=0}^{\infty}\left[2^{-n}F(2^{n+1}x) + \frac{2K}{2^{n+1}}\right]$$
$$= 2\bigcap_{n=0}^{\infty}\left[2^{-(n+1)}F(2^{n+1}x) + \frac{K}{2^{n+1}}\right] = 2\bigcap_{n=0}^{\infty} A_{n+1}(x) = 2\bigcap_{n=0}^{\infty} A_n(x) = 2A(x).$$

Step 3. *A is maximal superadditive K-subclose to F.*
Let $B : X \to CC(Y)$ be a superadditive K-subclose to F. Then for each $n \in \mathbb{N}$ and $x \in X$

$$2^n B(x) \subseteq B(2^n x) \subseteq F(2^n x) + K.$$

It follows that

$$B(x) \subseteq A_n(x) \quad (x \in X, n \in \mathbb{N}).$$

Therefore $B(x) \subseteq A(x)$ for each $x \in X$. ∎

Definition 2.3. *By a selection f of a mapping $F : X \to 2^Y$ we mean a single-valued mapping $f : X \to Y$ such that $f(x) \in F(x)$ for each $x \in X$.*

Corollary 2.1. *Let $(X,+)$ be an additive group divisible by two and $F : X \to C(Y)$ be a midconvex function such that $0 \in F(0)$. Then F admits an additive selection.*

Proof. By Theorem 2.1, there is a superadditive function $A : X \to C(Y)$ such that $A(x) \subseteq F(x)$ for each $x \in X$ and $A(2x) = 2A(x)$ for each $x \in X$. Therefore $A(0)+A(0) \subseteq A(0)+\{0\}$. On account of Lemma 2.1, $A(0) = \{0\}$. It follows that for each $x \in X$, $A(x)+A(-x) \subseteq A(x-x) = \{0\}$. Hence A is single-valued. Let $A(x) = \{f(x)\}$ for each $x \in X$. Then f is a selection of F. Moreover for each $x, y \in X$,

$$f(x)+f(y) \in A(x)+A(y) \subseteq A(x+y) = \{f(x+y)\}.$$

This proves additivity of f. ∎

We need the following well-known result (see e.g. [21, Theorem 1.13(b)]).

Lemma 2.3. *Let X be a topological vector space and $A, B \subseteq X$, then $\overline{A}+\overline{B} \subseteq \overline{A+B}$.*

Theorem 2.2. *Let $F : X \to C(Y)$ be an K-midconvex set-valued function, $K \in CC(Y)$ and $0 \in F(0)$. Then there exists a maximal midconvex set-valued function $M : X \to C(Y)$ which is K-subclose to F.*

Proof. Let

$$\mathscr{P} = \{G : X \to C(Y) : G \text{ is midconvex and } G(x) \subseteq F(x)+K \text{ for each } x \in X\}.$$

The proof of Theorem 2.1 ensures that $\mathscr{P} \neq \emptyset$. Define a binary relation "$\preceq$" on $\mathscr{P}$ as follows.

$$G_1 \preceq G_2 \text{ if and only if } G_1(x) \subseteq G_2(x) \quad \text{for each } x \in X.$$

Then $(\mathscr{P}, \preceq)$ is a partially ordered set. Let $\mathscr{P}_0$ be a chain in $\mathscr{P}$, define

$$H(x) = \overline{\bigcup_{G \in \mathscr{P}_0} G(x)} \quad (x \in X).$$

Since for each $x \in X$ and $G \in \mathscr{P}_0$, $G(x) \subseteq F(x)+K$ and $F(x)+K$ is compact, H is compact-valued. We will show that for each $x,y \in X$,

$$\bigcup_{G\in\mathscr{P}_0} G(x) + \bigcup_{G\in\mathscr{P}_0} G(y) \subseteq 2H\left(\frac{x+y}{2}\right). \tag{2.6}$$

To prove (2.6), take some $x,y \in X$, $z_1 \in \bigcup_{G\in\mathscr{P}_0} G(x)$ and $z_2 \in \bigcup_{G\in\mathscr{P}_0} G(y)$. Then for some $G_1, G_2 \in \mathscr{P}_0$, $z_1 \in G_1(x)$ and $z_2 \in G_2(y)$. Let $G_1 \preceq G_2$, then

$$z_1 + z_2 \in G_1(x) + G_2(y) \subseteq G_2(x) + G_2(y) \subseteq 2G_2\left(\frac{x+y}{2}\right) \subseteq 2H\left(\frac{x+y}{2}\right).$$

This proves (2.6). It follows from (2.6) and Lemma 2.3 that

$$H(x)+H(y) = \overline{\bigcup_{G\in\mathscr{P}_0} G(x)} + \overline{\bigcup_{G\in\mathscr{P}_0} G(y)} \subseteq \overline{\bigcup_{G\in\mathscr{P}_0} G(x) + \bigcup_{G\in\mathscr{P}_0} G(y)} \subseteq 2H\left(\frac{x+y}{2}\right).$$

Therefore H is midconvex. By Zorn's Lemma, $\mathscr{P}$ has a maximal element M. This completes our proof. ∎

Example 2.1. Let $X = [0,\infty)$, $Y = \mathbb{R}$ and $F : X \to CC(Y)$ be defined by

$$F(x) = \begin{cases} [0,\ \sqrt{x}\,] & 0 \le x < 1 \\ [0,\ 2\sqrt{x}\,] & x \ge 1. \end{cases}$$

Since $g(t) = \sqrt{t}$ is concave, $F|_{[0,1)}$ and $F|_{[1,\infty)}$ satisfy (2.1). Since $F(0)+F(1) = [0,2]$ is not subset of $2F((0+1)/2) = [0,\ \sqrt{2}/4]$, F is not midconvex. However,

$$F(x)+F(y) \subseteq [0,1] + [0,2\sqrt{y}] \subseteq 2F\left(\frac{x+y}{2}\right) + [0,1],$$

whenever $0 \le x < 1$ and $y \ge 0$. Hence for $K = [0,1]$, F satisfies (2.1). According to Theorem 2.2, there is a maximal midconvex set-valued map $M : [0,\infty) \to C(Y)$ such that $M(x) \subseteq F(x) + [0,1]$.

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