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Anti-Periodic Boundary Value Problems of Second-Order Functional Differential Equations

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Abstract. We deal with anti-periodic problems for second-order functional differential equations. The main tools in our study will be the Schauder's fixed point theorem and the property of the continuous function space with anti-periodic conditions. Some new results on the existence and uniqueness of anti-periodic solutions are obtained, which generalize and extend previously known theorems.

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1. Introduction

Consider the nonlinear second-order differential equations with delays of the form

$$u'' + f(t, x'(t), x(t), x(t-\tau(t))) = 0, \quad t \in R \tag{1.1}$$

where $f : R^4 \to R$ is continuous function,τ is T-periodic with respect to t and $T > 0$ is a constant.

Anti-periodic boundary value problems have been discussed in the past 20 years. Okochi [16,17] initiated the study for anti-periodic solutions of evolution equation in Hilbert spaces. Following Okochi's work, Chen *et al.* [6,7], studied by fixed point theorem the anti-periodic solution for first order semilinear evolution equations in a real separable Hilbert space.

Recently Liu in [13] studied the following anti-periodic problem of nonlinear evolution equations with nonmonotone perturbations

$$\begin{cases} u'(t) + Au(t) + Gu(t) = f, \quad a.e.\ t \in (0,T), \\ u(0) = -u(T), \end{cases}$$

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in a real reflexive Banach space V. A is monotone and G is not. Existence of solutions for anti-periodic problem has been obtained by using the theory of pseudomonotone perturbations of maximal monotone mappings. Sufficient conditions for the existence of anti-periodic solutions of the first order differential equations, we also refer to [2, 3, 8, 12].

Nevertheless second order differential equations with anti-periodic boundary value conditions are discussed in few papers [1, 14, 18, 19]. Aftabizaden, Aizicovici and Pavel [1] studied the anti-periodic solutions of second order evolution equations in Hilbert and Banach spaces by using monotone and accretive operator theory. In [14] the authors have discussed anti-periodic boundary value problems for second order differential equations, and sufficient conditions for existence of coupled solutions and a unique solution are obtained by using the monotone iterative technique. In [18, 19], the existence of at least one solution is obtained by using the Schauder fixed point theorem and the Leray-Schauder topological degree respectively.

Although second order differential equations with anti-periodic boundary conditions have been discussed in [1, 4, 10, 14, 18–20], as we can see, the function f is independent of x' and $x(t-\tau(t))$. Since anti-periodic boundary conditions appear in physics in a variety of situations (see for example, in [5, 11] and the references therein), the development of the general theory of the problem is timely.

However, to the best of our knowledge, few authors have considered the existence and uniqueness of anti-periodic boundary value problems of second order functional differential equations (1.1). Thus it is worth continuing the investigation of the existence and uniqueness of anti-periodic solutions of equation (1.1). A primary purpose of this paper is to study the existence and uniqueness of anti-periodic solutions of equation (1.1). We will establish some sufficient conditions for the existence and uniqueness of anti-periodic solutions of equation (1.1). Our results are different from the references listed above. In particular, an example is also to be given to illustrate the effectiveness of our results.

2. Preliminaries

To prove our existence theorem, we need the following set of hypotheses:

(H_1) $f\in C(R^4,R)$, $\tau\in C(R,R)$, and for all $t,x,y,z\in R$,

$$f\left(t+\frac{T}{2},-x,-y,-z\right)=-f(t,x,y,z),\quad \tau\left(t+\frac{T}{2}\right)=\tau(t).$$

(H_2) There exist three nonnegative constants a,b,c such that

$$a\frac{T}{2\pi}+b\frac{T^2}{4\pi^2}+c\frac{T^2}{4\sqrt{3}\pi}<1,$$

and $\forall t,x_1,x_2,y_1,y_2,z_1,z_2\in R$,

$$|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)|\le a|x_1-x_2|+b|y_1-y_2|+c|z_1-z_2|.$$

$(\bar{H}_2)$ There exist three nonnegative constants $\bar{a},\bar{b},\bar{c}$ such that

$$\frac{\bar{b}}{2\pi}+\frac{\bar{c}}{2\sqrt{3}}<\frac{\bar{a}}{T},$$

and

$$(f(t,x_1,y,z)-f(t,x_2,y,z))(x_1-x_2)\ge\bar{a}|x_1-x_2|^2,$$
$$|f(t,x,y_1,z_1)-f(t,x,y_2,z_2)|\le\bar{b}|y_1-y_2|+\bar{c}|z_1-z_2|,$$

$\forall t, x, y, z, x_1, x_2, y_1, y_2, z_1, z_2 \in R.$

(H_3) There exist two nonnegative continuous functions $p(t), q(t)$ and a nonnegative constant L such that

$$|f(t,u,0,0)| \le p(t)|u| + q(t),\ \forall t \in [0,T], |u| > L.$$

Let $u(t): R \to R$ be continuous in t. $u(t)$ is said to be anti-periodic on R if,

$$u(t+T) = u(t), \quad u\left(t + \frac{T}{2}\right) = -u(t), \quad \forall t \in R.$$

We shall adopt the following notations:

$$C_T^k := \{x \in C^k(R,R), x \text{ is } T-\text{periodic}\}, \quad k = \{0,1,\cdots\}$$

$$|x|_p = (\int_0^T |x(t)|^p dt)^{\frac{1}{p}},\ \forall p \ge 1, \quad |x|_0 = \max_{t \in [0,T]} |x(t)|,$$

$$C_T^{k,\frac{1}{2}} := \{x \in C_T^k, x(t + \frac{T}{2}) = -x(t), \quad \forall t \in R\},$$

which is a linear normed space endowed with the norm $\|\cdot\|_{C_T^{k,1/2}}$ defined by

$$\|x\|_{C_T^{k,1/2}} = \max\{|x|_0, |x'|_0, \cdots, |x^{(k)}|_0\}, \forall x \in C_T^{k,1/2}.$$

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.1. [15] *If $x \in C_T^1$ and $\int_0^T x(t)dt = 0$, then*

$$\int_0^T |x(t)|^2 dt \le \left(\frac{T^2}{4\pi^2}\right) \int_0^T |x'(t)|^2 dt \tag{2.1}$$

(Wirtinger inequality) and

$$|x(t)|_0^2 \le \left(\frac{T}{12}\right) \int_0^T |x'(t)|^2 dt \tag{2.2}$$

(Sobolev inequality).

Lemma 2.2. [18] *Let $\lambda > 0, \delta(t) \in C_T^{0,1/2}$ and x is an anti-periodic solution of*

$$-x''(t) + \lambda^2 x(t) = \delta(t) \tag{2.3}$$

if and only if x satisfies

$$x(t) = \int_0^{\frac{T}{2}} \int_0^{\frac{T}{2}} G(t,s) G^*(s,u)(-\delta(u)) du ds,\ \forall t \in [0, T/2] \tag{2.4}$$

where

$$G(t,s) = \begin{cases} \frac{e^{\lambda(\frac{T}{2}-t+s)}}{e^{\lambda\frac{T}{2}}+1}, & 0 \le s < t \le \frac{T}{2}, \\ \frac{-e^{\lambda(s-t)}}{e^{\lambda\frac{T}{2}}+1}, & 0 \le t \le s \le \frac{T}{2}, \end{cases} \qquad G^*(t,s) = \begin{cases} \frac{e^{\lambda(t-s)}}{e^{\lambda\frac{T}{2}}+1}, & 0 \le s < t \le \frac{T}{2}, \\ \frac{-e^{\lambda(\frac{T}{2}+t-s)}}{e^{\lambda\frac{T}{2}}+1}, & 0 \le t \le s \le \frac{T}{2}. \end{cases}$$

Lemma 2.3. [9] *Let A be a continuous and compact mapping of a Banach space B into itself, and suppose there exists a constant M such that*

$$\|x\|_B < M$$

for all $x \in B$ and $\sigma \in [0,1]$ satisfying $x = \sigma A x$. Then A has a fixed point.

3. Main results

In this section we study the existence and uniqueness of anti-periodic solutions to problem (1.1).

Theorem 3.1. *Assume that the condition (H_1) and one of the two conditions $(H_2),(\bar{H}_2)$ hold. Then equation (1.1) has at most one anti-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two anti-periodic solutions of equation (1.1). Then, we have

$$(x_1(t)-x_2(t))''+f(t,x_1'(t),x_1(t),x_1(t-\tau(t)))-f(t,x_2'(t),x_2(t),x_2(t-\tau(t)))=0. \tag{3.1}$$

Since $X(t)=x_1(t)-x_2(t)$ is an anti-periodic function on R, then

$$\int_0^T X(t)dt=\int_0^{\frac{T}{2}} X(t)dt+\int_{\frac{T}{2}}^T X(t)dt=\int_0^{\frac{T}{2}} X(t)dt+\int_0^{\frac{T}{2}} X(t+\frac{T}{2})dt=0.$$

By using the Sobolev inequality, we can get

$$|X|_0\leq\sqrt{\frac{T}{12}}|X'|_2. \tag{3.2}$$

Now suppose that (H_2) (or $(\bar{H}_2)$) holds. We shall consider two cases as follows.

Case (i) If (H_2) holds, multiplying both sides of (3.1) by $-X(t)$ and then integrating it from 0 to T, we have from (2.1), (2.2) and Schwarz inequality

$$\begin{aligned}
|X'|_2^2&=-\int_0^T X''(t)X(t)dt\\
&=\int_0^T[f(t,x_1'(t),x_1(t),x_1(t-\tau(t)))-f(t,x_2'(t),x_2(t),x_2(t-\tau(t)))]X(t)dt\\
&\leq\int_0^T\{a|X'(t)||X(t)|+b|X(t)|^2+c|X(t-\tau(t))||X(t)|\}dt\\
&\leq a|X'|_2|X|_2+b|X|_2^2+c|X|_0\int_0^T|X(t)|dt\leq\left[a\frac{T}{2\pi}+b\frac{T^2}{4\pi^2}+c\frac{T^2}{4\sqrt{3}\pi}\right]|X'|_2^2.
\end{aligned} \tag{3.3}$$

It follows from (H_2) that $X(t)\equiv 0$, $\forall t\in R$. Thus $x_1(t)\equiv x_2(t),\forall t\in R$.

Case (ii) If $(\bar{H}_2)$ holds, multiplying both sides of (3.1) by $X'(t)$ and then integrating it from 0 to T, we obtain from (2.1), (2.2) and Schwarz inequality

$$\begin{aligned}
\bar{a}|X'|_2^2&\leq\int_0^T[f(t,x_1'(t),x_1(t),x_1(t-\tau(t))-f(t,x_2'(t),x_1(t),x_1(t-\tau(t)))]X'(t)dt\\
&=\int_0^T[f(t,x_2'(t),x_2(t),x_2(t-\tau(t)))-f(t,x_2'(t),x_1(t),x_1(t-\tau(t)))]X'(t)dt\\
&\leq\int_0^T\{\bar{b}|X'(t)||X(t)|+\bar{c}|X(t-\tau(t))||X'(t)|\}dt\\
&\leq\bar{b}|X'|_2|X|_2+\bar{c}|X|_0\int_0^T|X'(t)|dt\leq\left[\bar{b}\frac{T}{2\pi}+\bar{c}\frac{T}{2\sqrt{3}}\right]|X'|_2^2.
\end{aligned}$$

It follows from $(\bar{H}_2)$ that $X(t)\equiv 0,\forall t\in R$. Thus $x_1(t)\equiv x_2(t),\forall t\in R$. Therefore, equation (1.1) has at most one anti-periodic solution. The proof of Theorem 3.1 is now complete. ∎

Now we show the existence of solutions for the anti-periodic problem (1.1).

Theorem 3.2. *Let (H_1) hold. Assume that either the condition (H_2) or the conditions $(\bar{H}_2),(H_3)$ are satisfied. Then equation (1.1) has at least one anti-periodic solution.*

Proof. Define a mapping $A : C_T^{1,1/2} \to C_T^{1,1/2}$ by

$$Ax(t) = \int_0^{\frac{T}{2}} \int_0^{\frac{T}{2}} G(t,s)G^*(s,u)(-f(u,x'(u),x(u),x(u-\tau(u))) - \lambda^2 x(u))duds, \tag{3.4}$$

$\forall t \in [0,T/2]$, while $Ax(t) = -Ax(t-T/2)$ $\forall t \in [T/2,T]$. Lemma 2.2 implies that solving problem (1.1) is equivalent to finding an $x \in C_T^{1,1/2}$ such that $x = Ax$.

In the following we shall use the well-known fixed point theorem, Lemma 2.3 to complete our proof.

We firstly show that A is completely continuous.
(i) $A : C_T^{1,1/2} \to C_T^{1,1/2}$ is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C_T^{1,1/2}$ as $n \to \infty$. Since $f \in C(R^4,R)$, we easily obtain $\forall u \in [0,T/2]$

$$\begin{aligned} \lim_{n\to\infty} f(u,x_n'(u),x_n(u),x_n(u-\tau(u))) + \lambda^2 x_n(u) \\ = f(u,x'(u),x(u),x(u-\tau(u))) - \lambda^2 x(u)]. \end{aligned} \tag{3.5}$$

From the definitions of the functions G,G^* in Lemma 2.2, it is easy to get that

$$|G(t,s)| \le \frac{e^{\lambda\frac{T}{2}}}{e^{\lambda\frac{T}{2}}+1}, \qquad |G^*(t,s)| \le \frac{e^{\lambda\frac{T}{2}}}{e^{\lambda\frac{T}{2}}+1}.$$

In virtue of Governed Convergence Theorem, we obtain

$$\lim_{n\to\infty} |A(x_n) - A(x)|_0 = 0. \tag{3.6}$$

By (3.4), we have $\forall t \in [0,T/2]$

$$\begin{aligned} Ax(t) = &\int_0^t \frac{e^{\lambda(\frac{T}{2}-t+s)}}{e^{\lambda\frac{T}{2}}+1} \int_0^{\frac{T}{2}} G^*(s,u)(-f(u,x'(u),x(u),x(u-\tau(u))) - \lambda^2 x(u))duds \\ &- \int_t^{\frac{T}{2}} \frac{e^{\lambda(s-t)}}{e^{\lambda\frac{T}{2}}+1} \int_0^{\frac{T}{2}} G^*(s,u)(-f(u,x'(u),x(u),x(u-\tau(u))) - \lambda^2 x(u))duds. \end{aligned}$$

Differentiating both sides of the above identity,we have

$$(Ax(t))' = -\lambda Ax(t) + \int_0^{\frac{T}{2}} G^*(t,u)(-f(u,x'(u),x(u),x(u-\tau(u))) - \lambda^2 x(u))du. \tag{3.7}$$

$$\begin{aligned} (Ax(t))'' = &-\lambda(Ax(t))' + \lambda \int_0^{\frac{T}{2}} G^*(t,u)(-f(u,x'(u),x(u),x(u-\tau(u))) \\ &- \lambda^2 x(u))du - f(t,x'(t),x(t),x(t-\tau(t)) - \lambda^2 x(t)) - \lambda(Ax(t))' \\ &+ \lambda(('Ax(t)) + \lambda Ax(t)) - f(t,x'(t),x(t),x(t-\tau(t))) - \lambda^2 x(t)) \\ = &\lambda^2 Ax(t) - f(t,x'(t),x(t),x(t-\tau(t))) - \lambda^2 x(t)). \end{aligned} \tag{3.8}$$

In virtue of (3.5)–(3.7) and Governed Convergence Theorem again, we have

$$\lim_{n\to\infty} |(A(x_n))' - (A(x))'|_0 = 0 \tag{3.9}$$

which shows that $A : C_T^{1,1/2} \to C_T^{1,1/2}$ is continuous.
(ii) Let D be a bounded set in $C_T^{1,1/2}$, that is, there exists a $d > 0$ for any $x \in D$ such that $|x|_0 \le d$ and $|x'|_0 \le d$. Thus,

$$\begin{aligned}|Ax|_0 &= \max_{t\in\left[0,\frac{T}{2}\right]} |\int_0^{\frac{T}{2}}\int_0^{\frac{T}{2}} G(t,s)G^*(s,u)(-f(u,x'(u),x(u),x(u-\tau(u))) - \lambda^2 x(u))duds| \\ &\le \frac{T^2 e^{\lambda T}}{4(e^{\lambda\frac{T}{2}}+1)^2}\max\left\{|f(s,u_1,u_2,u_3)|+\lambda^2 u_2| : s\in\left[0,\frac{T}{2}\right], |u_i|\le d, i=1,2,3.\right\} \\ &=: M_1.\end{aligned}$$

$$\begin{aligned}&|(Ax)'|_0 \\ &= \max_{t\in\left[0,\frac{T}{2}\right]} |-\lambda Ax(t) + \int_0^{\frac{T}{2}} G^*(t,u)(-f(u,x'(u),x(u),x(u-\tau(u))) - \lambda^2 x(u))du| \\ &\le \lambda|Ax|_0 + \frac{Te^{\lambda\frac{T}{2}}}{2(e^{\lambda\frac{T}{2}}+1)}\max\left\{|f(s,u_1,u_2,u_3)|+\lambda^2 u_2| : s\in\left[0,\frac{T}{2}\right], |u_i|\le d, i=1,2,3.\right\} \\ &\le M_1 + \frac{Te^{\lambda\frac{T}{2}}}{2(e^{\lambda\frac{T}{2}}+1)}\max\left\{|f(s,u_1,u_2,u_3)|+\lambda^2 u_2| : s\in\left[0,\frac{T}{2}\right], |u_i|\le d, i=1,2,3.\right\} \\ &=: M_2.\end{aligned}$$

By (3.8) and the above two inequalities, we obtain

$$\begin{aligned}|(Ax(t))''|_0 = \lambda^2|Ax|_0 &+ \max\left\{|f(s,u_1,u_2,u_3)|+\lambda^2 u_2| : s\in\left[0,\frac{T}{2}\right], |u_i|\le d, i=1,2,3.\right\} \\ &+ \lambda^2|x|_0 \le \text{Const}.\end{aligned}$$

which implies that $A : C_T^{1,1/2} \to C_T^{2,1/2}$ $(\subset C_T^{1,1/2})$ is bounded. Therefore, by compact embedding theorem we get that $A : C_T^{1,1/2} \to C_T^{1,1/2}$ is compact. Hence we have shown that A is completely continuous.

In view of the fixed point theorem of Lemma 2.3, Assume that x is a solution of the equation

$$x = \sigma Ax, \quad \sigma \in (0,1]. \tag{3.10}$$

Then, like (3.8) we easily get

$$x''(t) + \sigma f(t,x'(t),x(t),x(t-\tau(t))) = 0, \quad t\in[0,T]. \tag{3.11}$$

We shall consider two cases as follows:

Case (i) If (H_2) holds, multiplying both sides of (3.11) by $-x(t)$ and then integrating it from 0 to T, we have from (2.1), (2.2) and Schwarz inequality

$$\begin{aligned} |x'|_2^2 &= -\int_0^T x''(t)x(t)dt = \sigma\int_0^T (f(t,x'(t),x(t),x(t-\tau(t)))x(t)dt \\ &\le \int_0^T (a|x'(t)||x(t)| + b|x(t)|^2 + c|x(t-\tau(t))||x(t)|dt + \int_0^T |f(t,0,0,0)x(t)|dt \\ &\le a|x'|_2|x|_2 + b|x|_2^2 + c|x|_0\int_0^T |x(t)|dt + (\int_0^T |f(t,0,0,0)|^2dt)^{\frac{1}{2}}|x|_2 \\ &\le \left[a\frac{T}{2\pi} + b\frac{T^2}{4\pi^2} + c\frac{T^2}{4\sqrt{3}\pi}\right]|x'|_2^2 + \frac{T}{2\pi}\left(\int_0^T |f(t,0,0,0)|^2dt\right)^{\frac{1}{2}}|x'|_2. \end{aligned} \tag{3.12}$$

It follows from (H_2) that $|x'|_2$ is bounded if $x(t)$ is a solution (3.10). Therefore, we may assume that

$$|x'|_2 \le C_1 \tag{3.13}$$

if x is a solution (3.10), where C_1 is a positive constant. Using inequality (2.2), we obtain

$$|x|_0 \le \frac{\sqrt{T}}{2\sqrt{3}}C_1 \tag{3.14}$$

where x is any solutions to (3.10).

Now we are going to show $|x'|_0$ is also bounded if x is a solution (3.10). Multiplying both sides of (3.11) by $x''(t)$ and then integrating it from 0 to T,

$$\begin{aligned} |x''|_2^2 &= \int_0^T |x''(t)|^2dt = \sigma\int_0^T (f(t,x'(t),x(t),x(t-\tau(t)))x''(t)dt \\ &\le \int_0^T |f(t,x'(t),x(t),x(t-\tau(t)))x''(t)|dt. \end{aligned} \tag{3.15}$$

By use of (H_2), we get

$$\begin{aligned} |x''|_2^2 &\le \int_0^T |f(t,x'(t),x(t),x(t-\tau(t)))x''(t)|dt \\ &\le \int_0^T [a|x'(t)| + b|x(t)| + c|x(t-\tau(t))|]|x''(t)|dt + \int_0^T |f(t,0,0,0)||x''(t)|dt \\ &\le \left[a|x'|_2 + b|x|_2 + c\sqrt{T}|x|_0 + \left(\int_0^T |f(t,0,0,0)|^2dt\right)^{\frac{1}{2}}\right]|x''|_2 \end{aligned} \tag{3.16}$$

which implies that $|x''|_2$ is bounded from (2.1), (3.13) and (3.14). It follows from (2.2) that there exists a positive constant C_2 such that

$$|x'|_0 \le \frac{\sqrt{T}}{2\sqrt{3}}|x''|_2 \le C_2. \tag{3.17}$$

Therefore, under the assumption (H_2), we have shown that for any $x \in C_T^{1,1/2}$ which is a solution of (3.10), then

$$\|x\|_{C_T^{1,\frac{1}{2}}} < \max\left\{\frac{\sqrt{T}}{2\sqrt{3}}C_1, C_2\right\} + 1. \tag{3.18}$$

Case (ii) If $(\bar{H}_2)$ and (H_3) hold, multiplying both sides of (3.11) by $x'(t)$ and then integrating it from 0 to T, we obtain from (2.1), (2.2) and Schwarz inequality

$$\begin{aligned}
\bar{a}|x'|_2^2 &\le \int_0^T [f(t,x'(t),x(t),x(t-\tau(t))) - f(t,0,x(t),x(t-\tau(t)))]x'(t)dt \\
&= \int_0^T |f(t,0,x(t),x(t-\tau(t)))x'(t)|dt \\
&\le \int_0^T \{\bar{b}|x(t)||x'(t)| + \bar{c}|x(t-\tau(t))||x'(t)|dt + \int_0^T |f(t,0,0,0)||x'(t)|dt \\
&\le \bar{b}|x'|_2|x|_2 + \bar{c}|x|_0 \int_0^T |x'(t)|dt + \int_0^T |f(t,0,0,0)||x'(t)|dt \\
&\le \left[\bar{b}\frac{T}{2\pi} + \bar{c}\frac{T}{2\sqrt{3}}\right]|x'|_2^2 + \left(\int_0^T |f(t,0,0,0)|^2 dt\right)^{\frac{1}{2}} |x'|_2
\end{aligned}$$

It follows from $(\bar{H}_2)$ that hat $|x'|_2$ is bounded if $x(t)$ is a solution (3.10). Therefore, we may assume that

$$|x'|_2 \le \bar{C}_1 \tag{3.19}$$

if x is a solution (3.10), where $\bar{C}_1$ is a positive constant. Using inequality (2.2), we obtain

$$|x|_0 \le \frac{\sqrt{T}}{2\sqrt{3}}\bar{C}_1 \tag{3.20}$$

where x is any solutions to (3.10).

Now we are going to show $|x'|_0$ is also bounded if x is a solution (3.10).
Like (3.15), multiplying both sides of (3.11) by $x''(t)$ and then integrating it from 0 to T,

$$\begin{aligned}
|x''|_2^2 = \int_0^T |x''(t)|^2 dt &= \sigma \int_0^T (f(t,x'(t),x(t),x(t-\tau(t)))x''(t)dt \\
&\le \int_0^T |f(t,x'(t),x(t),x(t-\tau(t)))x''(t)|dt.
\end{aligned} \tag{3.21}$$

In virtue of $(\bar{H}_2)$ and (H_3), we get

$$\begin{aligned}
|x''|_2^2 &\le \int_0^T |f(t,x'(t),x(t),x(t-\tau(t)))x''(t)|dt \\
&\le \int_0^T [\bar{b}|x(t)| + \bar{c}|x(t-\tau(t))|]|x''(t)|dt + \int_0^T |f(t,x'(t),0,0)||x''(t)|dt \\
&\le \left[\bar{b}|x|_2 + \bar{c}\sqrt{T}|x|_0]|x''|_2 + \left(\int_0^T (\max\{|f(t,u,0,0)| : |u| \le L\})^2 dt\right)^{\frac{1}{2}}\right]|x''|_2 \\
&\quad + \int_0^T [p(t)|x'(t)| + q(t)]|x''|dt \\
&\le \{\bar{b}|x|_2 + \bar{c}\sqrt{T}|x|_0 + \left(\int_0^T (\max\{|f(t,u,0,0)| : |u| \le L\})^2 dt\right)^{\frac{1}{2}} \\
&\quad + |p|_0|x'(t)|_2 + |q(t)|_2\}|x''|_2
\end{aligned} \tag{3.22}$$

which implies that $|x''|_2$ is bounded from (2.1), (3.19) and (3.20). It follows from (2.2) that there exists a positive constant $\bar{C}_2$ such that

$$|x'|_0 \le \bar{C}_2 \tag{3.23}$$

Therefore, under the assumption $(\bar{H}_2)$ and (H_3), we also have shown that for any $x \in C_T^{1,1/2}$ which is a solution of (3.10),then

$$\|x\|_{C_T^{1,\frac{1}{2}}} < \max\{\frac{\sqrt{T}}{2\sqrt{3}}\bar{C}_1, \bar{C}_2\} + 1. \tag{3.24}$$

In view of (3.18) and (3.24), we can choose a positive constant M such that

$$\|x\|_{C_T^{1,\frac{1}{2}}} < M,$$

for all x in the Banach space $C_T^{1,1/2}$ and $\sigma \in [0,1]$ satisfying $x = \sigma Ax$. Therefore, by Lemma 2.3, A has a fixed point. The proof is complete. ∎

4. An example

We conclude with a simple example which can be treated by the methods developed above.

Example 4.1. Consider the following nonlinear second-order differential equations with delays of the form (the Rayleigh equation with delays)

$$x''(t) + \frac{1}{2}(\sin^2 t)x'(t) + \frac{1}{7}x(t) + \frac{1+\sin^4 t}{2\sqrt{3}\pi}\sin(x(t-\sin^2 t)) - \cos t = 0. \tag{4.1}$$

Then the above problem has a unique anti-periodic solution with periodic 2π.

Proof. By (24), we have

$$f(t, x'(t), x(t), x(t-\tau(t))) = \frac{1}{2}(\sin^2 t)x'(t) + \frac{1}{7}x(t) + \frac{1+\sin^4 t}{2\sqrt{3}\pi}\sin(x(t-\sin^2 t)) - \cos t,$$

and $a = 1/2, b = 1/7, c = 1/(\sqrt{3}\pi)$. It is obvious that assumption (H_1) and (H_2) hold. Hence, by Theorem 3.1 and 3.2, equation (4.1) has a unique anti-periodic solution with periodic 2π. ∎

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