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Coupled Fixed Point Theorems in Partially Ordered Menger Spaces

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Abstract. In this paper, we prove several coupled common fixed point theorems for nonlinear mappings in a partially ordered probabilistic metric space. An example is presented to illustrate the main result of this paper.

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1. Introduction

The Banach contractive mapping principle is an important result of analysis and it has been applied widely in a number of branches of mathematics. Recently, some new results for contractions in partially ordered metric spaces were presented and applied to the periodic boundary value problem for different equations; see [1, 2, 4, 9, 12, 13] and the references cited therein.

Recently, Bhaskar and Lakshmikantham [7] introduced the concepts of a mixed monotone mapping and a coupled fixed point. Let (X, \leq) be a partially ordered set. A sequence $\{x_n\} \subset X$ is called nondecreasing if

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$$

and nonincreasing if

$$x_1 \succeq x_2 \succeq \cdots \succeq x_n \succeq \cdots$$

where $x \succeq y$ denotes $y \preceq x$ for all $x, y \in X$. A mapping $T : X \times X \to X$ is called to be mixed monotone if T(x, y) is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,

$$\begin{aligned} & x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow T(x_1, y) \preceq T(x_2, y), \\ & y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow T(x, y_2) \preceq T(x, y_1). \end{aligned}$$

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An element $(x, y) \in X \times X$ is said a coupled fixed point of the mapping *T* if T(x, y) = x and T(y, x) = y. In [7], Bhaskar and Lakshmikantham proved the following coupled fixed point theorem:

Theorem 1.1. [7, Theorem 2.2] *Let* (X, \preceq) *be partially ordered set and* (X,d) *be a complete metric space. Let* $T : X \times X \to X$ *be a mapping having the mixed monotone property on* X*. Assume that there exists a* $k \in [0, 1)$ *such that*

$$d(T(x,y),T(u,v)) \le [k(d(x,u)+d(y,v))]/2, \quad \forall u \le x, \ y \le v.$$

Suppose either

- (a) T is continuous; or
- (b) *X* has the following property:
 - (i) if a nondecreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$, for each $n \geq 1$;
 - (ii) if a nonincreasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$, for each $n \geq 1$.

If there exist $x_0, y_0 \in X$ *such that*

$$x_0 \leq T(x_0, y_0)$$
 and $T(y_0, x_0) \leq y_0$,

then there exist $x, y \in X$ such that T(x, y) = x and T(y, x) = y.

Recently, Lakshmikantham and Ćirić [10] introduced a new concept of commutative mappings with the mixed monotone property. Let (X, \preceq) be a partially ordered set and $T: X \times X \to X$ and $g: X \to X$. *T* is called to commute with *g* if

$$T(g(x),g(y)) = g(T(x,y))$$

for all $x, y \in X$. *T* is said to have the mixed *g*-monotone property if *T* is monotone *g*-nondecreasing in its first argument and is monotone *g*-nonincreasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \preceq g(x_2) \quad \text{implies} \quad T(x_1, y) \preceq T(x_2, y)$$

and

 $y_1, y_2 \in X$, $g(y_1) \preceq g(y_2)$ implies $T(x, y_2) \preceq T(x, y_1)$.

An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings *T* and *g* if

g(x) = T(x, y) and g(y) = T(y, x).

Lakshmikantham and Ćirić [10] proved the following theorem that extended and improved Theorem 1.1 of Bhaskar and Lakshmikantham [7]:

Theorem 1.2. [10] Let (X, \preceq) be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Assume that there is a function $\varphi : [0,\infty) \rightarrow [0,\infty)$ with $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$ for each t > 0 and also suppose that $T : X \times X \rightarrow X$ and $g : X \rightarrow X$ are such that T has the mixed g-monotone property and

$$d(T(x,y),T(u,v)) \le \varphi([d(g(x),g(u)) + d(g(y),g(v))]/2)$$

for all $x, y, u, v \in X$ for which $g(x) \preceq g(u)$ and $g(v) \preceq g(y)$. Suppose that $T(X \times X) \subset g(X)$, *g* is continuous and commutes with *T* and also suppose either

- (a) T is continuous or
- (b) *X* has the following property:
 - (i) if a nondecreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all n,

(ii) if a nonincreasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$ for all n.

If there exist $x_0, y_0 \in X$ *such that*

$$g(x_0) \leq T(x_0, y_0)$$
 and $T(y_0, x_0) \leq g(y_0)$,

then there exist $x, y \in X$ such that

$$g(x) = T(x, y)$$
 and $g(y) = T(y, x)$,

that is, T and g have a coupled coincidence.

In fact, the fixed point theorems for contractions are investigated not only in partially ordered metric spaces but also in partially ordered probabilistic metric spaces. Recently, Ćirić, Miheţ and Saadati [5] considered some fixed point theorems for a class of contractive mappings in partially ordered probabilistic metric spaces. The following theorem is the one of main results of [5]:

Theorem 1.3. [5] Let (X, \preceq) be a partially ordered set and (X, F, Δ) be a complete Menger probabilistic metric space under a t-norm Δ of Hadžić-type. Let $A, h : X \to X$ be two self-mappings of X such that $A(X) \subset h(X)$, A be a h-nondecreasing mapping and, for some $k \in (0, 1)$,

$$F_{A(x),A(y)}(kt) \ge \min\{F_{h(x),h(y)}(t),F_{h(x),A(x)}(t),F_{h(y),A(y)}(t)\},\$$

for all $x, y \in X$ for which $h(x) \leq h(y)$ and all t > 0.

Also suppose that h(X) is closed and if $\{h(x_n)\} \subset X$ is a nondecreasing sequence with $h(x_n) \rightarrow h(z)$ in h(X), then $h(z) \preceq h(h(z))$ and $h(x_n) \preceq h(z)$ for all n hold. If there exists an $x_0 \in X$ with $h(x_0) \preceq A(x_0)$, then A and h have a coincidence. Further, if A and h commute at their coincidence points, then A and h have a common fixed point.

For the recent results on fixed point theorems in partially ordered probabilistic metric spaces, we refer the reader to [6, 14].

In this paper, motivated and inspired by the results of Lakshmikantham and Ćirić [10] and Ćirić, Mihet and Saadati [5], we prove several coupled common fixed point theorems for nonlinear contractive mappings in a partially ordered probabilistic metric space. An example is presented to illustrate the main result of this paper.

2. Preliminaries

In this section, we recall some definitions and results in the theory of probabilistic metric spaces. For more details, the readers are referred to [3, 8, 15].

Definition 2.1. A mapping $F : (0,\infty) \to [0,1]$ is called a distribution function if it is nondecreasing and left-continuous with $\inf_{x \in \mathbb{R}} F(x) = 0$. If in addition F(0) = 0, then F is called a distance distribution function.

Definition 2.2. A distance distribution function F satisfying $\lim_{t\to\infty} F(t) = 1$ is called a Menger distance distribution function.

The set of all Menger distance distribution functions is denoted by D^+ . This space D^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only

if $F(t) \leq G(t)$ for all $t \in [0,\infty)$. The maximal element for D^+ in this order is the distance distribution function ε_0 , given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Definition 2.3. A triangular norm (shortly, t-norm) is a binary operation Δ on [0,1] satisfying the following conditions:

- (1) Δ is associative and commutative;
- (2) Δ is continuous;
- (3) $\Delta(a, 1) = a$ for all $a \in [0, 1]$;
- (4) $\Delta(a,b) \leq \Delta(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Two typical examples of the continuous *t*-norm are $\Delta_P(a,b) = ab$, $\Delta_M(a,b) = \min\{a,b\}$ for all $a, b \in [0, 1]$. By (4), it is easy to see that Δ satisfies the following

(2.1)
$$\min\{\Delta(a,b),\Delta(c,d)\} \ge \Delta(\min\{a,c\},\min\{b,d\}), \quad \forall a,b,c,d \in [0,1].$$

Now, the *t*-norm is recursively defined by $\Delta^1 = \Delta$ and

$$\Delta^n(x_1,\ldots,x_{n+1}) = \Delta(\Delta^{n-1}(x_1,\ldots,x_n),x_{n+1})$$

for all $n \ge 2$ and $x_i \in [0,1]$, i = 1, 2, ..., n + 1. A *t*-norm Δ is said to be of Hadžić-type if the family $\{\Delta^n\}$ is equicontinuous at x = 1, that is, for any $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that

$$a > 1 - \delta \implies \Delta^n(a) > 1 - \varepsilon, \quad \forall n \ge 1.$$

 Δ_M is a trivial example of a *t*-norm of Hadžić-type [8].

Definition 2.4. A Menger probabilistic space (briefly, Menger PM-space) is a triple (X, F, Δ) , where X is a nonempty set, Δ is a continuous t-norm and F is a mapping from $X \times X \to D^+$ $(F_{x,y}$ denotes the value of F at the pair (x, y)) satisfying the following conditions:

(PM-1) $F_{x,y}(t) = 1$ for all $x, y \in X$ and t > 0 if and only if x = y; (PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and t > 0;

(PM-3) $F_{x,z}(t+s) \ge \Delta(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \ge 0$.

Definition 2.5. Let (X, F, Δ) be a Menger PM-space.

- (1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (write $x_n \to x$) if, for any t > 0 and $0 < \varepsilon < 1$, there exists a positive integer N such that $F_{x_n,x}(t) > 1 \varepsilon$ whenever $n \ge N$.
- (2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any t > 0 and $0 < \varepsilon < 1$, there exists a positive integer N such that $F_{x_n,x_m}(t) > 1 \varepsilon$ whenever $m, n \ge N$.
- (3) A Menger PM-space (X, F, Δ) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 2.1. [15] If (X, F, Δ) is a Menger PM-space and $\{x_n\}$ and $\{y_n\}$ are sequences such that $x_n \to x$ and $y_n \to y$, then $\lim_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t)$ for every continuity point of $F_{x,y}$.

3. Main results

Lemma 3.1. For $n \in \mathbb{N}$, let $g_n : (0, \infty) \to (0, \infty)$ and $F_n, G_n, F, G : \mathbb{R} \to [0, 1]$. Assume that $\sup_{t>0} \{\min\{F(t), G(t)\}\} = 1$ and for any t > 0,

$$\lim_{n\to\infty}g_n(t)=0 \quad and \quad \min\{F_n(g_n(t)),G_n(g_n(t))\}\geq\min\{F(t),G(t)\}$$

Proof. Fix t > 0 and $\varepsilon > 0$. By hypothesis, there is $t_0 > 0$ such that $\min\{F(t_0), G(t_0)\} > 1 - \varepsilon$. Since $g_n(t_0) \to 0$, there is $n_0 \in \mathbb{N}$ such that $g_n(t_0) < t$ for all $n \ge n_0$. By monotonicity,

$$\min\{F_n(t), G_n(t)\} \ge \min\{F_n(g_n(t_0)), G_n(g_n(t_0))\}$$
$$\ge \min\{F(t_0), G(t_0)\} > 1 - \varepsilon, \quad \forall n \ge n_0.$$

Hence we infer that $\lim_{n\to\infty} F_n(t) = 1$ and $\lim_{n\to\infty} G_n(t) = 1$, since $F_n(t), G_n(t) \le 1$.

Theorem 3.1. Let (X, \preceq) be partially ordered set and (X, F, Δ) be a complete Menger PMspace under a t-norm Δ of Hadžić-type. Let a function $\varphi : [0, \infty) \to [0, \infty)$ satisfy that, for any t > 0,

$$0 < \varphi(t) < t$$
 and $\lim_{n \to \infty} \varphi^n(t) = 0.$

Let $T: X \times X \to X$ and $h: X \to X$ be such that T has the mixed h-monotone property and

(3.1)
$$F_{T(x,y),T(u,v)}(\varphi(t)) \ge \min\{F_{h(x),h(u)}(t),F_{h(y),h(v)}(t)\}$$

for all t > 0 and all $x, y, u, v \in X$ with $h(x) \succeq h(u)$ and $h(y) \preceq h(v)$. Suppose $T(X \times X) \subset h(X)$, *h* is continuous and commutes with *T* and also suppose either

- (a) T is continuous or
- (b) *X* has the following property:
 - (i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, for each $n \ge 1$;
 - (ii) if a nonincreasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$, for each $n \geq 1$.

If there exist $x_0, y_0 \in X$ *such that*

$$h(x_0) \leq T(x_0, y_0)$$
 and $T(y_0, x_0) \leq h(y_0)$,

then there exist $x, y \in X$ such that T(x, y) = h(x) and T(y, x) = h(y), that is, T and h have a coupled coincidence.

Proof. By hypothesis we have $h(x_0) \leq T(x_0, y_0)$ and $T(y_0, x_0) \leq h(y_0)$. Since $T(X \times X) \subset h(X)$, there exist $x_1, y_1 \in X$ such that $T(x_0, y_0) = h(x_1)$ and $T(y_0, x_0) = h(y_1)$. Further, there exist $x_2, y_2 \in X$ such that $T(x_1, y_1) = h(x_2)$ and $T(y_1, x_1) = h(y_2)$. Continuing this process, we can choose two sequences $\{x_n\}$ and $\{y_n\}$ such that

(3.2)
$$h(x_{n+1}) = T(x_n, y_n)$$
 and $h(y_{n+1}) = T(y_n, x_n), \quad \forall n \ge 0,$

where \mathbb{N} denotes the set of all positive integers.

Next we show by induction that

(3.3)
$$h(x_n) \leq h(x_{n+1})$$
 and $h(y_{n+1}) \leq h(y_n), \quad \forall n \geq 0.$

First, by hypothesis we have $h(x_0) \leq T(x_0, y_0) = h(x_1)$ and $h(y_1) = T(y_0, x_0) \leq h(y_0)$. So, (3.3) holds for n = 0. Assume that (3.3) holds for some positive integer n, i.e., $h(x_n) \leq h(x_{n+1})$ and $h(y_{n+1}) \leq h(y_n)$. Since T is the mixed h-monotone, one has

$$T(x_n, y_n) \preceq T(x_{n+1}, y_n) \preceq T(x_{n+1}, y_{n+1})$$

and

$$T(y_{n+1}, x_{n+1}) \preceq T(y_{n+1}, x_n) \preceq T(y_n, x_n),$$

i.e.,

$$h(x_{n+1}) \leq h(x_{n+2})$$
 and $h(y_{n+2}) \leq h(y_{n+1})$

Hence, (3.3) holds for all $n \ge 0$.

For each $n \in \mathbb{N}$, we prove by induction that

(3.4) $\min\{F_{h(x_{n+1}),h(x_n)}(\varphi^n(t)),F_{h(y_{n+1}),h(y_n)}(\varphi^n(t))\} \ge \min\{F_{h(x_1),h(x_0)}(t),F_{h(y_1),h(y_0)}(t)\}.$ For n = 1, by (3.1)–(3.3) we have

$$F_{h(x_2),h(x_1)}(\varphi(t)) = F_{T(x_1,y_1),T(x_0,y_0)}(\varphi(t)) \ge \min\{F_{h(x_1),h(x_0)}(t),F_{h(y_1),h(y_0)}(t)\}$$

and

$$F_{h(y_2),h(y_1)}(\varphi(t)) = F_{T(y_1,x_1),T(y_0,x_0)}(\varphi(t)) \ge \min\{F_{h(x_1),h(x_0)}(t),F_{h(y_1),h(y_0)}(t)\}$$

Hence, (3.4) holds for n = 1. Now, assume that (3.4) holds for some $n \in \mathbb{N}$. Then, by (3.1)–(3.4) and assumption we have

$$F_{h(x_{n+2}),h(x_{n+1})}(\varphi^{n+1}(t)) = F_{T(x_{n+1},y_{n+1}),T(x_n,y_n)}(\varphi^{n+1}(t))$$

$$\geq \min\{F_{h(x_{n+1}),h(x_n)}(\varphi^n(t)),F_{h(y_{n+1}),h(y_n)}(\varphi^n(t))\}$$

$$\geq \min\{F_{h(x_1),h(x_0)}(t),F_{h(y_1),h(y_0)}(t)\}$$

and

$$\begin{aligned} F_{h(y_{n+2}),h(y_{n+1})}(\varphi^{n+1}(t)) &= F_{T(y_{n+1},x_{n+1}),T(y_n,x_n)}(\varphi^{n+1}(t)) \\ &\geq \min\{F_{h(x_{n+1}),h(x_n)}(\varphi^n(t)),F_{h(y_{n+1}),h(y_n)}(\varphi^n(t))\} \\ &\geq \min\{F_{h(x_1),h(x_0)}(t),F_{h(y_1),h(y_0)}(t)\}, \end{aligned}$$

which implies that (3.4) holds for n + 1. Therefore, (3.4) holds for all $n \ge 1$. By Lemma 3.1 and (3.4) we have, for any t > 0

(3.5) $\lim_{n \to \infty} F_{h(x_{n+1}, h(x_n))}(t) = 1$

and

(3.6)
$$\lim_{n \to \infty} F_{h(y_{n+1}, h(y_n))}(t) = 1.$$

Now let $n \in \mathbb{N}$ and t > 0. We show by induction that, for any $n \ge 0$,

(3.7)
$$\min\{F_{h(x_n),h(x_{n+i})}(t),F_{h(y_n),h(y_{n+i})}(t)\} \\ \ge \Delta^i(\min\{F_{h(x_n),h(x_{n+1})}(t-\varphi(t)),F_{h(y_n),h(y_{n+1})}(t-\varphi(t))\})$$

This is obvious for i = 0, since $F_{h(x_n),h(x_n)}(t) = 1$ and $F_{h(y_n),h(y_n)}(t) = 1$. Assume that (3.7) holds for some *i*. Hence, by (3.1), (2.1), commutativity of *T* and *h* and the monotonicity of Δ , we have

$$\begin{split} \min\{F_{h(x_n),h(x_{n+i+1})}(t),F_{h(y_n),h(y_{n+i+1})}(t)\}\\ &=\min\{F_{h(x_n),h(x_{n+i+1})}(t-\varphi(t)+\varphi(t)),F_{h(y_n),h(y_{n+i+1})}(t-\varphi(t)+\varphi(t))\}\\ &\geq \min\{\Delta(F_{h(x_n),h(x_{n+1})}(t-\varphi(t)),F_{h(x_{n+1}),h(x_{n+i+1})}(\varphi(t))),\Delta(F_{h(y_n),h(y_{n+1})}(t-\varphi(t)),\\ &F_{h(y_{n+1}),h(y_{n+i+1})}(\varphi(t)))\}\\ &\geq \Delta(\min\{F_{h(x_n),h(x_{n+1})}(t-\varphi(t)),F_{h(y_n),h(y_{n+1})}(t-\varphi(t))\},\min\{F_{h(x_{n+1}),h(x_{n+i+1})}(\varphi(t)),\\ \end{split}$$

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$$\begin{split} & F_{h(y_{n+1}),h(y_{n+i+1})}(\varphi(t))\} \Big) \\ &= \Delta \Big(\min\{F_{h(x_n),h(x_{n+1})}(t-\varphi(t)),F_{h(y_n),h(y_{n+1})}(t-\varphi(t))\},\min\{F_{T(x_n,y_n),T(x_{n+i},y_{n+i})}(\varphi(t)),\\ & F_{T(y_n,x_n),T(y_{n+i},x_{n+i})}(\varphi(t))\} \Big) \\ &\geq \Delta \Big(\min\{F_{h(x_n),h(x_{n+1})}(t-\varphi(t)),F_{y_n,y_{n+1}}(t-\varphi(t))\},\min\{F_{h(x_n),h(x_{n+i})}(t),F_{h(y_n),h(y_{n+i})}(t)\} \Big) \\ &\geq \Delta \Big(\min\{F_{h(x_n),h(x_{n+1})}(t-\varphi(t)),F_{h(y_n),h(y_{n+1})}(t-\varphi(t))\},\\ & \Delta^i(\min\{F_{h(x_n),h(x_{n+1})}(t-\varphi(t)),F_{h(y_n),h(y_{n+1})}(t-\varphi(t))\}) \Big) \\ &= \Delta^{i+1} \Big(\min\{F_{h(x_n),h(x_{n+1})}(t-\varphi(t)),F_{h(y_n),h(y_{n+1})}(t-\varphi(t)) \Big), \end{split}$$

which completes the induction.

We show that $\{h(x_n)\}$ and $\{h(y_n)\}$ are the Cauchy sequences, i.e., $\lim_{m,n\to\infty} F_{h(x_n),h(x_m)}(t) = 1$ and $\lim_{m,n\to\infty} F_{h(y_n),h(y_m)}(t) = 1$ for any t > 0. Let t > 0 and $\varepsilon > 0$. By hypothesis, $\{\Delta^n : n \in \mathbb{N}\}$ is equicontinuous at 1 and $\Delta^n(1) = 1$, so there exists $\delta > 0$ such that

(3.8) if
$$s \in (1 - \delta, 1]$$
, then $\Delta^n(s) > 1 - \varepsilon$ for all $n \in \mathbb{N}$.

Since, by (3.5) and (3.6), $\lim_{n\to\infty} F_{h(x_n),h(x_{n+1})}(t-\varphi(t)) = 1$ and $\lim_{n\to\infty} F_{h(y_n),h(y_{n+1})}(t-\varphi(t)) = 1$, there is $n_0 \in \mathbb{N}$ such that, for any $n \ge n_0$, $\min\{F_{h(x_n),h(x_{n+1})}(t-\varphi(t)),F_{h(y_n),h(y_{n+1})}(t-\varphi(t))\} \in (1-\delta,1]$. Hence, by (3.7) and (3.8), we get $\min\{F_{h(x_n),h(x_{n+i})}(t),F_{h(y_n),h(y_{n+i})}(t)\} > 1-\varepsilon$ for any $i \in \mathbb{N} \cup \{0\}$. This shows that $F_{h(x_n),h(x_{n+i})}(t) > 1-\varepsilon$ and $F_{h(y_n),h(y_{n+i})}(t) > 1-\varepsilon$ for every $i \in \mathbb{N} \cup \{0\}$. This proves that $\{h(x_n)\}$ and $\{h(y_n)\}$ are the Cauchy sequences.

Since *X* is complete, there exist $x, y \in X$ such that

(3.9)
$$\lim_{n\to\infty}F_{h(x_n),x}(t)=1 \quad \text{and} \quad \lim_{n\to\infty}F_{h(y_n),y}(t)=1, \quad \forall t>0.$$

From (3.9) and the continuity of h,

(3.10)
$$\lim_{n \to \infty} F_{h(h(x_n)),h(x)}(t) = 1 \quad \text{and} \quad \lim_{n \to \infty} F_{h(h(y_n)),h(y)}(t) = 1, \quad \forall t > 0.$$

From (3.2) and commutativity of *T* and *h*, for any t > 0,

(3.11)
$$F_{h(h(x_{n+1})),T(x,y)}(t) = F_{h(T(x_n,y_n)),T(x,y)}(t) = F_{T(h(x_n),h(y_n)),T(x,y)}(t)$$

and

(3.12)
$$F_{h(h(y_{n+1})),T(y,x)}(t) = F_{h(T(y_n,x_n)),T(y,x)}(t) = F_{T(h(y_n),h(x_n)),T(y,x)}(t).$$

Now we show that h(x) = T(x, y) and h(y) = T(y, x). Suppose that the assumption (a) holds. Taking the limit as $n \to \infty$ in (3.11) and (3.12), by (3.10) and the continuity of *T* we get

$$F_{h(x),T(x,y)}(t) = \lim_{n \to \infty} F_{h(h(x_{n+1})),T(x,y)}(t) = \lim_{n \to \infty} F_{T(h(x_n),h(y_n)),T(x,y)}(t) = F_{T(x,y),T(x,y)}(t) = 1$$

and

$$F_{h(y),T(y,x)}(t) = \lim_{n \to \infty} F_{h(h(y_{n+1})),T(y,x)}(t) = \lim_{n \to \infty} F_{T(h(y_n),h(x_n)),T(y,x)}(t) = F_{T(y,x),T(y,x)}(t) = 1$$

Hence, h(x) = T(x, y) and h(y) = T(y, x).

Now suppose that (b) holds. By hypothesis, we have $h(x_n) \leq x$ and $h(y_n) \geq y$ for all $n \geq 1$. Hence, by (3.2), commutativity of *T* and *h* and (3.1) we have

$$F_{T(x,y),h(x)}(t) \geq \Delta(F_{T(x,y),h(h(x_{n+1}))}(\varphi(t)),F_{h(h(x_{n+1})),h(x)}(t-\varphi(t)))$$

$$= \Delta(F_{T(x,y),h(T(x_{n},y_{n}))}(\varphi(t)),F_{h(h(x_{n+1})),h(x)}(t-\varphi(t)))$$

$$= \Delta(F_{T(x,y),T(h(x_{n}),h(y_{n}))}(\varphi(t)),F_{h(h(x_{n+1})),h(x)}(t-\varphi(t)))$$

$$\geq \Delta(\min\{F_{h(x),h(h(x_{n}))}(t),F_{h(y),h(h(y_{n}))}(t)\},F_{h(h(x_{n+1})),h(x)}(t-\varphi(t)))).$$

Letting $n \to \infty$ in (3.13) and noting that *h* is continuous, so it follows from (3.10) and (3.13) that

$$F_{T(x,y),h(x)}(t) \ge \Delta(1,1).$$

This implies that T(x, y) = h(x). Similarly, we can prove that T(y, x) = h(y). Thus we prove that *T* and *h* have a coupled coincidence point.

Now we shall prove the existence and uniqueness theorem of a coupled common fixed point. Note that if (X, \preceq) is a partially ordered set, then we endow the product $X \times X$ with the following partial order:

for all
$$(x,y), (u,v) \in X \times X, (u,v) \preceq (x,y) \iff u \succeq x, v \preceq y.$$

Theorem 3.2. In addition to the hypothesis of Theorem 3.1, suppose that for every (x,y), $(y^*,x^*) \in X \times X$, there exists a $(u,v) \in X \times X$ such that (T(u,v),T(v,u)) is comparable to (T(x,y),T(y,x)) and $(T(x^*,y^*),T(y^*,x^*))$. Then T and h have a unique coupled common fixed point. That is, there exist a unique $(x,y) \in X \times X$ such that

$$x = h(x) = T(x, y)$$
 and $y = h(y) = T(y, x)$.

Proof. Existence of the set of coupled coincidence points is due to Theorem 3.1. Let $(x,y), (x^*,y^*) \in X \times X$ be the coupled coincidence points, that is h(x) = T(x,y), h(y) = T(y,x) and $h(x^*) = T(x^*,y^*), h(y^*) = T(y^*,x^*)$. We shall show that

(3.14)
$$h(x) = h(x^*)$$
 and $h(y) = h(y^*)$.

By the assumption, there is $(u, v) \in X \times X$ such that (T(u, v), T(v, u)) is comparable to (T(x, y), T(y, x)) and $(T(x^*, y^*), T(y^*, x^*))$. Let $u_0 = u$ and $v_0 = v$ and choose $u_1, v_1 \in X$ so that $h(u_1) = F(u_0, v_0)$ and $h(v_1) = F(v_1, u_1)$. Then, similarly as in the proof of Theorem 3.1, we can construct sequences $\{h(u_n)\}$ and $\{h(v_n)\}$ such that $h(u_{n+1}) = T(u_n, v_n)$, $h(v_{n+1}) = T(v_n, u_n)$. Further, set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$ and $y_0^* = y^*$ and, on the same way, define the sequences $\{h(x_n)\}, \{h(y_n)\}, \{h(x_n^*)\}$ and $\{h(y_n^*)\}$. Since $(T(x, y), T(y, x)) = (h(x_1), h(y_1)) = (h(x), h(y))$ and

$$(T(u,v),T(v,u)) = (h(u_1),h(v_1))$$

are comparable, then $h(x) \succeq h(u_1)$ and $h(y) \preceq h(v_1)$. Since *T* is mixed *h*-monotone, we have $h(x) = T(x,y) \succeq T(u_1,y) \succeq T(u_1,v_1) = h(u_2)$ and $h(y) = T(y,x) \preceq T(v_1,x) \preceq T(v_1,u_1) = h(v_2)$. Similarly, we can prove that $h(x) \succeq h(u_n)$ and $h(y) \preceq h(v_n)$ for all $n \ge 3$. Thus from (3.1), it follows that

$$F_{T(x,y),T(u_n,v_n)}(\varphi^n(t)) \ge \min\{F_{h(x),h(u_n)}(\varphi^{n-1}(t)),F_{h(y),h(v_n)}(\varphi^{n-1}(t))\}$$

and

$$F_{T(v_n,u_n),T(y,x)}(\varphi^n(t)) \ge \min\{F_{h(y),h(v_n)}(\varphi^{n-1}(t)),F_{h(x),h(u_n)}(\varphi^{n-1}(t))\}$$

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By induction we can prove that for any t > 0,

$$\min\{F_{T(x,y),T(u_n,v_n)}(\boldsymbol{\varphi}^n(t)),F_{T(v_n,u_n),T(y,x)}(\boldsymbol{\varphi}^n(t))\}\geq\min\{F_{h(x),h(u_1)}(t),F_{h(y),h(v_{n+1})}(t)$$

By Lemma 3.1, we get that $\lim_{n\to\infty} F_{T(x,y),T(u_n,v_n)}(t) = 1$ and $\lim_{n\to\infty} F_{T(v_n,u_n),T(y,x)}(t) = 1$. That is, $\lim_{n\to\infty} F_{h(x),h(u_{n+1})}(t) = 1$ and $\lim_{n\to\infty} F_{h(y),h(v_{n+1})}(t) = 1$ for any t > 0. Similarly, we also have $\lim_{n\to\infty} F_{h(x^*),h(u_{n+1})}(t) = 1$ and $\lim_{n\to\infty} F_{h(y^*),h(v_{n+1})}(t) = 1$.

By the triangle inequality and the continuity of Δ , we have, for any t > 0,

$$F_{h(x),h(x^*)}(t) \ge \Delta \big(F_{h(x),h(u_{n+1})}(t/2), F_{h(x^*),h(u_{n+1})}(t) \big) \to \Delta(1,1),$$

$$F_{h(y),h(y^*)}(t) \ge \Delta \big(F_{h(y),h(v_{n+1})}(t/2), F_{h(y^*),h(v_{n+1})}(t) \big) \to \Delta(1,1).$$

This shows that $h(x) = h(x^*)$ and $h(y) = h(y^*)$. Hence we proved (3.14).

Since h(x) = T(x, y) and h(y) = T(y, x), by commutativity of *T* and *h* we have

(3.15)
$$h(h(x)) = h(T(x,y)) = T(h(x),h(y)), h(h(y)) = h(T(y,x)) = T(h(y),h(x)).$$

Denote h(x) = z and h(y) = w. Then from (3.15),

(3.16)
$$h(z) = T(z, w)$$
 and $h(w) = T(w, z)$.

Thus (z, w) is a coupled coincidence point. Then from (3.14) with $x^* = z$ and $y^* = w$ it follows that h(z) = h(x) and h(w) = h(y), that is,

(3.17)
$$h(z) = z \text{ and } h(w) = w.$$

From (3.16) and (3.17), we have

$$z = h(z) = F(z, w)$$
 and $w = h(w) = T(w, z)$.

Therefore, (z, w) is a coupled common fixed point of *T* and *h*. To prove the uniqueness, assume that (p,q) is another coupled common fixed point. Then by (3.14) we have p = h(p) = h(z) = z and q = h(q) = q(w) = w. This completes the proof.

Theorem 3.3. Assume that the hypothesis of Theorem 3.1 hold. If x_0 and y_0 are comparable, then x = y, where x and y are the coupled common fixed points of h and T, that is, h(x) = T(x,x).

Proof. Suppose that $x_0 \leq y_0$. We show by induction that

$$(3.18) h(x_n) \preceq h(y_n), \quad \forall n \ge 1,$$

where x_n and y_n satisfy that $h(x_n) = T(x_{n-1}, y_{n-1})$ and $h(y_n) = T(y_{n-1}, x_{n-1})$. For n = 1, by the mixed monotone property of *T* and *h* we have

$$h(x_1) = T(x_0, y_0)) \preceq T(y_0, y_0) \preceq T(y_0, x_0) = h(y_1).$$

This shows that (3.18) holds for n = 1. Assume that (3.18) holds for some $n \ge 1$, i.e., $h(x_n) \preceq h(y_n)$. Then, by the mixed monotone property of *T* and *h*, we have

$$h(x_{n+1}) = T(x_n, y_n) \preceq T(y_n, y_n) \preceq T(y_n, x_n) = h(y_{n+1}).$$

Therefore, (3.18) holds for all $n \ge 1$.

Finally, we prove that x = y. Indeed, for all $n \ge 1$ and any t > 0, we have

$$F_{x,y}(t) \geq \Delta(F_{x,h(x_{n+1})}(t/2), F_{h(x_{n+1}),y}(t/2))$$

$$\geq \Delta(F_{x,h(x_{n+1})}(t/2), \Delta(F_{h(x_{n+1}),h(y_{n+1})}(\varphi(t/2)), F_{h(y_{n+1}),y}(t/2 - \varphi(t/2))))$$

$$= \Delta(F_{x,h(x_{n+1})}(t/2), \Delta(F_{h(y_{n+1}),h(x_{n+1})}(\varphi(t/2)), F_{h(y_{n+1}),y}(t/2 - \varphi(t/2))))$$

$$= \Delta(F_{x,h(x_{n+1})}(t/2), \Delta(F_{T(y_n,x_n),T(x_n,y_n)}(\varphi(t/2)), F_{h(y_{n+1}),y}(t/2 - \varphi(t/2))))$$

$$\geq \Delta(F_{x,h(x_{n+1})}(t/2), \Delta(\min\{F_{h(y_n),h(x_n)}(t/2), F_{h(x_n),h(y_n)}(t/2)\}), F_{h(y_{n+1}),y}(t/2 - \varphi(t/2))))$$

$$= \Delta(F_{x,h(x_{n+1})}(t/2), \Delta(F_{h(y_n),h(x_n)}(t/2), F_{h(y_{n+1}),y}(t/2 - \varphi(t/2))))$$

$$\geq \Delta(F_{x,h(x_{n+1})}(t/2), \Delta(\Delta(\Delta(F_{h(y_n),y}(t/8), F_{y,x}(t/8))), F_{x,h(x_n)}(t/4)), F_{h(y_{n+1}),y}(t/2 - \varphi(t/2)))).$$

Letting $n \to \infty$ in (3.19), since Δ is continuous, $h(x_n) \to x$ and $h(y_n) \to y$, we have

$$F_{x,y}(t) \geq F_{x,y}(t/8), \quad \forall t > 0,$$

which implies that

(3.20)
$$F_{x,y}(t) \ge F_{x,y}(t/8^n), \quad \forall t > 0, \quad n \ge 1.$$

Letting $n \to \infty$ in (3.20), since $F_{x,y}(t/8^n) \to 1$, we have

$$F_{x,y}(t) \geq 1.$$

This shows that x = y. Similarly, if $y_0 \leq x_0$, we also can prove that x = y. Hence, we have proved that h(x) = T(x,x). This completes the proof.

If h = I, where I denotes the identity mapping, then we have the following corollaries:

Corollary 3.1. Let (X, \preceq) be partially ordered set and (X, F, Δ) be a complete Menger PM-space under a t-norm Δ of Hadžić-type. Let a function $\varphi : [0, \infty) \to [0, \infty)$ satisfy that, for any t > 0,

$$0 < \varphi(t) < t$$
 and $\lim_{n \to \infty} \varphi^n(t) = 0.$

Let $T: X \times X \to X$ be such that T has the mixed monotone property and satisfies that

$$F_{T(x,y),T(u,v)}(\varphi(t)) \ge \min\{F_{x,u}(t),F_{y,v}(t)\}$$

for all t > 0 and $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$). Suppose either

- (a) T is continuous or
- (b) *X* has the following property:
 - (i) *if a nondecreasing sequence* $\{x_n\} \rightarrow x$, *then* $x_n \leq x$, $n \geq 1$;
 - (ii) if a nonincreasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$, $n \geq 1$.

If there exist $x_0, y_0 \in X$ *such that*

 $x_0 \leq T(x_0, y_0)$ and $T(y_0, x_0) \leq y_0$,

then there exist $x, y \in X$ such that T(x, y) = x and T(y, x) = y.

Corollary 3.2. In addition to the hypothesis of Corollary 3.1, suppose that for every (x,y), $(y^*,x^*) \in X \times X$, there exists a $(u,v) \in X \times X$ such that (T(u,v),T(v,u)) is comparable to

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(T(x,y),T(y,x)) and $(T(x^*,y^*),T(y^*,x^*))$. Then T has a unique couple fixed point. that is, there exist a unique $(x,y) \in X \times X$ such that

$$x = T(x, y)$$
 and $y = T(y, x)$.

Corollary 3.3. Assume that the hypothesis of Corollary 3.1 hold. If x_0 and y_0 are comparable, then x = y, where x and y are the coupled fixed points of T, that is, x = T(x, x).

Finally, we illustrate Theorem 3.1 of this paper by an example as follows.

Example 3.1. Let $X = [0, \infty)$. Define a partially order \leq by the usual order. That is, $x \leq y$ if and only if $x \leq y$ for all $x, y \in X$. Take $\Delta(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Define $F_{x,y}(t)$ by

$$F_{x,y}(t) = \frac{t}{t+|x-y|}, \quad \forall x, y \in X, \ t \in (0,\infty).$$

Then (X, F, Δ) is a complete Menger PM-space. Define the mappings $T : X \times X \to X$ and $h : X \to X$ by

$$T(x,y) = x/2$$
 and $h(x) = 2x$

for all $x, y \in X$, respectively. It is easy to see that *T* is a mapping having mixed *h*-monotone property. Indeed, if $h(x_1) \le h(x_2)$, i.e., $x_1 \le x_2$, then $T(x_1, y) = x_1/2 \le x_2/2 = T(x_2, y)$ for all $y \in X$ and if $h(y_1) \ge h(y_2)$, i.e., $y_1 \ge y_2$, then $T(x, y_2) = x/2 = T(x, y_1)$ for all $x \in X$. Hence, *T* is a mapping having mixed *h*-monotone property.

Take k = 1/4. For all $x, y, u, v \in X$ with $h(x) \ge h(u)$ and $h(y) \le h(v)$, i.e., $x \ge u$ and $y \le v$, find that

$$F_{T(x,y),T(u,v)}(kt) = \frac{t/4}{t/4 + |x-u|/2} = \frac{t}{t+2|x-u|}$$

$$\geq \min\{F_{h(x),h(u)}(t), F_{h(y),h(v)}(t)\} = \min\left\{\frac{t}{t+2|x-u|}, \frac{t}{t+2|y-v|}\right\}.$$

This shows that *T* and *h* satisfy the condition (3.1). On the other hand, it is easy to see that $T(X \times X) \subset h(X)$ and *h* is continuous and commutes with *T*. Moreover, the hypothesis (a) and (b) are satisfied. Also, $(x_0, y_0) = (0, 0)$ is such that

$$h(x_0) \le T(x_0, y_0)$$
 and $h(y_0) \ge T(y_0, x_0)$.

Therefore, we show that all the hypothesis in Theorem 3.1 are satisfied. By Theorem 3.1, *T* and *h* have a coupled common coincidence point, which is (x, y) = (0, 0).

In fact, this example also may be used to illustrate Theorem 3.2 and 3.3.

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References

- R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.* 87 (2008), no. 1, 109–116.
- [2] D. Burgić, S. Kalabušić and M. R. S. Kulenović, Global attractivity results for mixed-monotone mappings in partially ordered complete metric spaces, *Fixed Point Theory Appl.* 2009, Art. ID 762478, 17 pp.
- [3] S. S. Chang, Y. J. Cho and S. M. Kang, Nonlinear Operator Theory in Probabilistic Metric Spaces, Nova Sci. Publ., Huntington, NY, 2001.

- [4] L. Ćirić, N. Cakić, M. Rajović and J. S. Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, *Fixed Point Theory Appl.* 2008, Art. ID 131294, 11 pp.
- [5] L. Ćirić, D. Miheţ and R. Saadati, Monotone generalized contractions in partially ordered probabilistic metric spaces, *Topology Appl.* 156 (2009), no. 17, 2838–2844.
- [6] L. Ćirić, R. P. Agarwal and B. Samet, Mixed monotone-generalized contractions in partially ordered probabilistic metric spaces, *Fixed Point Theory Appl.* 2011, 2011:56, 13 pp.
- [7] T. Gnana Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.* 65 (2006), no. 7, 1379–1393.
- [8] O. Hadžić and E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Mathematics and its Applications, 536, Kluwer Acad. Publ., Dordrecht, 2001.
- [9] J. Harjani and K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, *Nonlinear Anal.* 72 (2010), no. 3-4, 1188–1197.
- [10] V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal.* 70 (2009), no. 12, 4341–4349.
- [11] N. V. Luong and N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74 (2011), no. 3, 983–992.
- [12] J. J. Nieto and R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order* 22 (2005), no. 3, 223–239 (2006).
- [13] J. J. Nieto and R. Rodríguez-López, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sin. (Engl. Ser.)* 23 (2007), no. 12, 2205–2212.
- [14] R. Saadati, Generalized distance and fixed point theorems in partially ordered probabilistic metric spaces, *Matematicki Vesnik*, to appear.
- [15] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland, New York, 1983.