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The Axiom of Hemi-Slant 3-Spheres in Almost Hermitian Geometry

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Abstract. The axiom of hemi-slant 3-spheres is introduced. It is proved that if an almost Hermitian manifold M with dimension $2m \geq 6$ satisfies this axiom for some slant angle $\theta \in (0, \pi/2)$, then M has pointwise constant type α if and only if M has pointwise constant anti-holomorphic sectional curvature α, and using this result some conditions for constancy of sectional curvature of a considered almost Hermitian manifold are given.

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1. Introduction

In [2], Cartan defined *the axiom of n-planes*. A Riemannian manifold M of dimension $m \geq 3$ is said to satisfy *the axiom of n-planes*, where n is a fixed integer $2 \leq n \leq m-1$, if for each point $p \in M$ and any n-dimensional subspace σ of the tangent space T_pM there exists an n-dimensional totally geodesic submanifold N such that $p \in N$ and $T_pN = \sigma$. He gave a criterion for constancy of sectional curvature in the following theorem.

Theorem 1.1. *Let M be a Riemannian manifold of dimension $m \geq 3$. If M satisfies the axiom of n-planes for some $n, 2 \leq n \leq m-1$, then M has constant sectional curvature.*

In [21], Yano and Mogi applied Cartan's idea to Kaehlerian manifolds. A Kaehlerian manifold M is said to satisfy the *axiom of holomorphic planes* if for each point $p \in M$ and each holomorphic plane $\sigma \subset T_pM$, there exists a totally geodesic submanifold N such that $p \in N$ and $T_pN = \sigma$. They proved the following theorem.

Theorem 1.2. *A Kaehlerian manifold satisfying the axiom of holomorphic planes is a complex space form.*

In [12], Leung and Nomizu defined *the axiom of n-spheres* by taking totally umbilical submanifold N with parallel mean curvature vector field instead of totally geodesic submanifold N in the axiom of n-planes. They proved the following theorem.

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Theorem 1.3. *If a Riemannian manifold M of dimension $m \geq 3$ satisfies the axiom of n-spheres for some $n, 2 \leq n \leq m-1$, then M has constant sectional curvature.*

Afterwards, many studies have been made in this direction. Kaehlerian manifolds were studied in [4, 6, 8, 11, 19], the papers [17] and [19] discussed nearly Kaehlerian (almost Tachibana) manifolds, and results concerning larger classes of almost Hermitian manifolds can be found in [9, 10, 16, 17].

In this paper, we shall introduce the axiom of hemi-slant 3-spheres and as an application, we shall give an interesting relation between the notion of constant type and anti-holomorphic sectional curvature for a $2m(m \geq 3)$-dimensional almost Hermitian manifold satisfying this axiom for some slant angle $\theta \in (0, \pi/2)$. Using this fact, we shall prove some theorems related to sectional curvature for a considered almost Hermitian manifold. We shall also give some results related to the Weyl conformal curvature tensor and the Bochner curvature tensor of a certain almost Hermitian manifold satisfying the axiom of hemi-slant 3-spheres. Our work is motivated by the above-cited papers.

2. Preliminaries

A C^∞-manifold M is called *almost Hermitian* if its tangent bundle has an almost complex structure J and a Riemannian metric g such that $g(JX, JY) = g(X,Y)$ for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of C^∞ vector fields on M. Let ∇ be the covariant derivative on M, the Riemannian curvature tensor R associated with ∇ defined by $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. We denote $g(R(X,Y)Z,U)$ by $R(X,Y,Z,U)$. The sectional curvature K of M determined by orthonormal vector fields X and Y is given by $K(X,Y) = R(X,Y,X,Y)$. The Weyl conformal curvature tensor W is defined by

$$\begin{aligned} &W(X,Y,Z,U) \\ &= R(X,Y,Z,U) - \frac{1}{2m-2}\{g(X,U)Ric(Y,Z) - g(X,Z)Ric(Y,U) + g(Y,Z)Ric(X,U) \\ &\quad - g(Y,U)Ric(X,Z)\} + \frac{S}{(2m-1)(2m-2)}\{g(X,U)g(Y,Z) - g(X,Z)g(Y,U)\} \end{aligned}$$

for all $X,Y,Z,U \in T_pM$, where Ric and S are the *Ricci tensor* and the *scalar curvature* of M, respectively. A $2m$-dimensional almost Hermitian manifold with $m \geq 2$ is *conformally flat* if and only if $W = 0$ identically [10, 20].

By an *r-plane* we mean an r-dimensional linear subspace of a tangent space $T_pM, p \in M$. Motivated from [3], we have the following definition.

Definition 2.1. *Let σ be a 2-plane. The angle $\theta \in [0, \pi/2]$ between σ and $J\sigma$ is defined by*

$$\cos\theta = |g(X, JY)|,$$

where $\{X,Y\}$ is an orthonormal basis of σ. If $\theta =$ constant, then σ is called a slant-plane *and θ is called* slant angle *of σ.*

This is a generalization of holomorphic and anti-holomorphic planes. In fact, holomorphic and anti-holomorphic planes are slant planes with slant angle θ equal to 0 and $\pi/2$, respectively, see [4,6,8]. Now, motivated from [1] and [14] we have the following definition.

Definition 2.2. *A 3-plane σ in T_pM is called* hemi-slant *if it contains a slant 2-plane with slant angle $\theta \in [0, \pi/2)$ and a nonzero vector $Z \in T_pM$ such that JZ is perpendicular to σ, in which case $\sigma = D^\theta \oplus \{Z\}$ with $JZ \perp \sigma$, where D^θ is the corresponding slant 2-plane.*

The sectional curvature of M restricted to a holomorphic (resp. an anti-holomorphic) plane σ is called *holomorphic* (resp. *anti-holomorphic*) *sectional curvature*. If the holomorphic (resp. anti-holomorphic) sectional curvature at each point $p \in M$, does not depend on σ, then M is said to be *pointwise constant holomorphic* (resp. *pointwise constant anti-holomorphic) sectional curvature*. A connected Riemannian (resp. Kaehlerian) manifold of (global) constant sectional curvature (resp. of constant holomorphic sectional curvature) is called a *real space form* (resp. a *complex space form*) [9, 20]. The following useful notion was defined by Gray in [7].

Definition 2.3. *Let M be an almost Hermitian manifold. Then M is said to be of* constant type *at $p \in M$ provided that for all $X \in T_pM$, we have $\lambda(X,Y) = \lambda(X,Z)$ whenever the planes* span$\{X,Y\}$ *and* span$\{X,Z\}$ *are anti-holomorphic and $g(Y,Y) = g(Z,Z)$, where the function λ is defined by $\lambda(X,Y) = R(X,Y,X,Y) - R(X,Y,JX,JY)$. If this holds for all $p \in M$, then we say that M has* (pointwise) constant type. *Finally, if for $X,Y \in \chi(M)$ with $g(X,Y) = g(JX,Y) = 0$, the value $\lambda(X,Y)$ is constant whenever $g(X,X) = g(Y,Y) = 1$, then we say that M has* global constant type.

Vanhecke introduced to the notion of *RK-manifold* in [16]. An almost Hermitian manifold M is called an *RK*-manifold if

$$R(X,Y,Z,U) = R(JX,JY,JZ,JU) \tag{2.1}$$

for all $X,Y,Z,U \in \chi(M)$. He proved many theorems. Recall some of them.

Theorem 2.1. [16] *Let M be an RK-manifold. Then M has (pointwise) constant type if and only if there exists $\alpha \in \mathscr{F}(M)$ such that*

$$\lambda(X,Y) = \alpha\{g(X,X)g(Y,Y) - g^2(X,Y) - g^2(X,JY)\},$$

for all $X,Y \in \chi(M)$. Furthermore, M has global constant type if and only if α is a constant function.

Theorem 2.2. [16] *Let M be an RK-manifold. Suppose that M has constant holomorphic sectional curvature μ at a point $p \in M$, let $X,Y \in T_pM$ be any orthonormal vectors. Then we have*

$$K(X,Y) = \frac{\mu}{4}\{1 + 3g^2(X,JY)\} + \frac{5}{8}\lambda(X,Y) + \frac{1}{8}\lambda(X,JY),$$

where $K(X,Y)$ is sectional curvature determined by X and Y.

Theorem 2.3. [16] *Let M be an RK-manifold with pointwise constant anti-holomorphic (resp. holomorphic) sectional curvature ν (resp. μ). Then M has pointwise constant holomorphic (resp.anti-holomorphic) sectional curvature μ (resp. ν) if and only if M has pointwise constant type α, in which case*

$$4\nu = \mu + 3\alpha.$$

The dimension of M is supposed to be ≥ 6.

We shall call an almost Hermitian manifold M as *Kaehlerian* if $\nabla_X J = 0$ for all $X \in \chi(M)$, *nearly Kaehlerian (almost Tachibana or K-space)* if $(\nabla_X J)X = 0$ for all $X \in \chi(M)$, and *para-Kaehlerian* if $R(X,Y,Z,U) = R(X,Y,JZ,JU)$ for all $X,Y,Z,U \in \chi(M)$. These manifolds satisfy (2.1), so they are *RK*-manifolds. It is easy to see that a para-Kaehlerian manifold has global constant type [16, 17].

For a $2m$-dimensional Kaehlerian manifold, the Bochner curvature tensor B is defined by

$$\begin{aligned}
&B(X,Y,Z,U)\\
&= R(X,Y,Z,U) - \frac{1}{2(m+2)}\{g(X,U)Ric(Y,Z) - g(X,Z)Ric(Y,U) + g(Y,Z)Ric(X,U)\\
&\quad - g(Y,U)Ric(X,Z) + g(X,JU)Ric(Y,JZ) - g(X,JZ)Ric(Y,JU) + g(Y,JZ)Ric(X,JU)\\
&\quad - g(Y,JU)Ric(X,JZ) - 2g(X,JY)Ric(Z,JU) - 2g(Z,JU)Ric(X,JY)\}\\
&\quad + \frac{S}{4(m+1)(m+2)}\{g(X,U)g(Y,Z) - g(X,Z)g(Y,U)\}\\
&\quad + g(X,JU)g(Y,JZ) - g(X,JZ)g(Y,JU) - 2g(X,JY)g(Z,JU)
\end{aligned}$$

for all $X,Y,Z,U \in T_pM$ and $p \in M$, where Ric and S are the *Ricci tensor* and the *scalar curvature* of M, respectively [11]. The following lemma gives a criterion for vanishing of the Bochner curvature tensor of a Kaehlerian manifold.

Lemma 2.1. [11] *A Kaehlerian manifold M of dimension $2m \geq 6$ has a vanishing Bochner curvature tensor, if and only if for each point $p \in M$ and for all unit vectors $X,Y,Z \in T_pM$, which span an anti-holomorphic 3-plane*

$$R(X,JX,Y,Z) = 2R(X,Y,JX,Z)$$

holds.

Now, we give some definitions related to submanifolds.

Let M be a C^∞-Riemannian manifold with metric tensor g and N be a submanifold of M. We denote by ∇ and $\hat{\nabla}$ the covariant derivatives on M and N respectively. For any vector fields X and Y tangent to N, the second fundamental form T is defined by

$$T(X,Y) = \nabla_X Y - \hat{\nabla}_X Y$$

where $\hat{\nabla}_X Y$ is tangent to N and $T(X,Y)$ is normal to N. The normal bundle-valued form T is a symmetric tensor field of type $(0,2)$. We say that N is *totally umbilical* submanifold in M if for all X,Y tangent to N, we have

$$T(X,Y) = g(X,Y)\eta, \tag{2.2}$$

where η is the mean curvature vector field of N in M. The Codazzi equation is given by

$$(R(X,Y)Z)^\perp = (\nabla_X T)(Y,Z) - (\nabla_Y T)(X,Z) \tag{2.3}$$

for all X,Y,Z tangent to N. Where $^\perp$ denotes the normal component and the covariant derivative of T, denoted by $\nabla_X T$, is defined by

$$(\nabla_X T)(Y,Z) = D_X(T(Y,Z)) - T(\hat{\nabla}_X Y,Z) - T(Y,\hat{\nabla}_X Z) \tag{2.4}$$

for all X,Y,Z tangent to N, where D denotes the operator of covariant derivative in the normal bundle of N [4,6,8,19].

3. Main results

We now introduce the following axiom.

Definition 3.1. ***(Axiom of hemi-slant 3-spheres).*** *An almost Hermitian manifold M is said to satisfy the axiom of hemi-slant 3-spheres if for each point $p \in M$ and each hemi-slant 3-plane σ in T_pM, there exists a 3-dimensional totally umbilical submanifold N such that $p \in N$ and $T_pN = \sigma$.*

Before studying the axiom of hemi-slant 3-spheres, let us note the following.

Remark 3.1. Let M be any $2m$-dimensional almost Hermitian manifold with $m \geq 3$ and let $\{X_1, ..., X_m, JX_1, ..., JX_m\}$ be an orthonormal J-basis of T_pM. Then we always have a hemi-slant 3-plane with the slant angle θ. For example, $\sigma = D^\theta \oplus \{X_3\}$ is a hemi-slant 3-plane with the slant angle θ, where $D^\theta = \operatorname{span}\{X_1, \cos\theta JX_1 + \sin\theta X_2\}$.

Lemma 3.1. *Let M be an almost Hermitian manifold with dimension $2m \geq 6$. If M satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0, \pi/2)$, then we have*

$$\lambda(X,Y) = K(X,Y) \tag{3.1}$$

for all orthonormal vectors $X, Y \in T_pM$ with $g(X, JY) = 0$, where $\lambda(X,Y) = R(X,Y,X,Y) - R(X,Y,JX,JY)$ and K denotes anti-holomorphic sectional curvature.

Proof. Let p be an arbitrary point of M and let X, Y and Z be any orthonormal vectors in T_pM with $g(X,JY) = g(X,JZ) = g(Y,JZ) = 0$. Consider the hemi-slant 3-plane $\sigma = D^\theta \oplus \{Y\}$ with slant angle $\theta \in (0, \pi/2)$, where $D^\theta = \operatorname{span}\{X, \cos\theta JX + \sin\theta Z\}$. By the axiom of hemi-slant 3-spheres, there exists a 3-dimensional totally umbilical submanifold N such that $p \in N$ and $T_pN = \sigma$. Then, with the help of (2.2) and (2.4) from (2.3), we have

$$(R(X,Y)(\cos\theta JX + \sin\theta Z))^\perp = 0. \tag{3.2}$$

Since JY is normal to N, we get

$$R(X,Y,\cos\theta JX + \sin\theta Z, JY) = 0. \tag{3.3}$$

Now, consider the hemi-slant 3-plane $\sigma_2 = D_2^\theta \oplus \{Y\}$ with slant angle $\theta \in (0, \pi/2)$, where $D_2^\theta = \operatorname{span}\{X, \cos\theta JX - \sin\theta Z\}$. By a similar method, we can obtain

$$R(X,Y,\cos\theta JX - \sin\theta Z, JY) = 0. \tag{3.4}$$

From (3.3) and (3.4) we get

$$R(X,Y,JX,JY) = 0. \tag{3.5}$$

From Definition 2.3, and the equation (3.5), we obtain (3.1). ∎

Theorem 3.1. *Let M be an almost Hermitian manifold with dimension $2m \geq 6$. If M satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0, \pi/2)$, then M has pointwise constant type if and only if M has pointwise constant anti-holomorphic sectional curvature.*

Proof. Let M be an almost Hermitian manifold with dimension $2m \geq 6$ satisfying the axiom of hemi-slant 3-spheres for some $\theta \in (0, \pi/2)$. If M has pointwise constant type; that is, for all $p \in M$, M has constant type at p, then for all $X, Y, Z \in T_pM$, we have

$$\lambda(X,Y) = \lambda(X,Z), \tag{3.6}$$

whenever the planes $\operatorname{span}\{X,Y\}$ and $\operatorname{span}\{X,Z\}$ are anti-holomorphic and $g(Y,Y) = g(Z,Z)$. Here, we can assume that $g(Y,Y) = g(Z,Z) = 1$. Thus, from Lemma 3.1, we get

$$K(X,Y) = K(X,Z) \tag{3.7}$$

for all orthonormal vectors $X,Y,Z \in T_pM$ with $g(X,JY) = g(X,JZ) = 0$.
On the other hand, since the dimension of M is greater than 6 we can choose a unit vector U in $(\mathrm{span}\{X,JX\})^{\perp} \cap (\mathrm{span}\{Z,JZ\})^{\perp}$. Then, from (3.7), we have

$$K(X,U) = K(X,Z). \tag{3.8}$$

This implies that the sectional curvature is the same for all anti-holomorphic sections which contain any given vector X. Hence we write

$$K(X,Y) = K(Y,Z) = K(Z,U). \tag{3.9}$$

Therefore, we find

$$K(X,Y) = K(Z,U) \tag{3.10}$$

for all $X,Y,Z,U \in T_pM$ whenever the planes $\mathrm{span}\{X,Y\}$ and $\mathrm{span}\{Z,U\}$ are anti-holomorphic. It follows that the sectional curvature is the same for all anti-holomorphic sections at $p \in M$; that is, M has pointwise constant anti-holomorphic sectional curvature.

Conversely, let M be of pointwise constant anti-holomorphic sectional curvature and let p be any point of M. Then for all orthonormal vectors $X,Y,Z \in T_pM$ with $g(X,JY) = g(X,JZ) = 0$, ($\mathrm{span}\{X,Y\}$ and $\mathrm{span}\{X,Z\}$ are anti-holomorphic planes and $g(X,X) = g(Y,Y) = g(Z,Z) = 1$), we have

$$K(X,Y) = K(X,Z). \tag{3.11}$$

From Lemma 3.1, we get

$$\lambda(X,Y) = \lambda(X,Z) \tag{3.12}$$

for all orthonormal vectors $X,Y,Z \in T_pM$ whenever the planes $\mathrm{span}\{X,Y\}$ and $\mathrm{span}\{X,Z\}$ are anti-holomorphic. It is not difficult to see that (3.12) also holds in the case $g(Y,Y) = g(Z,Z) \neq 1$. It follows that M has constant type at p. ∎

With the help of Lemma 3.1, from Theorem 3.1, we have the following result.

Corollary 3.1. *Let M be a $2m$-dimensional almost Hermitian manifold with $m \geq 3$. If M satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0,\pi/2)$, then M has pointwise constant type α if and only if M has pointwise constant anti-holomorphic sectional curvature α.*

We now state the main result of the present work.

Theorem 3.2. *Let M be a $2m$-dimensional (connected) RK-manifold with pointwise constant type α and $m \geq 3$. If M satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0,\pi/2)$, then M is a real space form with constant sectional curvature α and M has global constant type.*

Proof. Let p be any point of M and M has constant type α at p. Then it follows from Corollary 3.1 that M has constant anti-holomorphic sectional curvature α at p. On the other hand, from Theorem 2.3, we see that M has constant holomorphic sectional curvature α at p. With the help of Theorem 2.1, from Theorem 2.2, we obtain

$$K(X,Y) = \alpha \tag{3.13}$$

for all orthonormal vectors $X,Y \in T_p(M)$, where $K(X,Y) = R(X,Y,X,Y)$ is sectional curvature. It is not difficult to see that (3.13) is also true for all $X,Y \in T_p(M)$. By the well-known

Schur's theorem [20] it follows that M has constant sectional curvature α and M has global constant type. ∎

Corollary 3.2. *Let M be a $2m$-dimensional (connected) RK-manifold with vanishing constant type and $m \geq 3$. If M satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0,\pi/2)$, then M is flat.*

Now, suppose that M is a Kaehlerian manifold. Then, for all $X,Y,Z \in \chi(M)$, as a result of the Kaehler identity $R(X,Y)JZ = JR(X,Y)Z$, we get $R(JX,JY)Z = R(X,Y)Z$ [20]. In this case, we have

$$R(JX,JY,JX,JY) = R(X,Y,JX,JY). \tag{3.14}$$

Using (2.1) and (3.14), we can see that $\lambda(X,Y) = 0$. Thus, any Kaehlerian manifold has (global) vanishing constant type. Thus, it follows from Corollary 3.2 that:

Corollary 3.3. *Let M be a $2m$-dimensional (connected) Kaehlerian or para-Kaehlerian manifold with $m \geq 3$. If M satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0,\pi/2)$, then M is flat.*

Theorem 3.3. *Let M be a $2m$-dimensional (connected) non-Kaehlerian nearly Kaehlerian manifold with constant type α and $m \geq 3$. If M satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0,\pi/2)$, then M has constant sectional curvature $\alpha > 0$ and $m = 3$.*

Proof. In [7], for a nearly Kaehlerian manifold M, A. Gray proved the following.

$$\lambda(X,Y) = R(X,Y,X,Y) - R(X,Y,JX,JY) = \|(\nabla_X J)Y\|^2, \tag{3.15}$$

where $X,Y \in \chi(M)$. Now, let M be a $2m$-dimensional non-Kaehlerian nearly Kaehlerian manifold with constant type α and $m \geq 3$. Then, it follows from (3.15) that $\alpha = \lambda(X,Y) = \|(\nabla_X J)Y\|^2 > 0$ due to M is non-Kaehlerian. On the other hand, since M satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0,\pi/2)$, by Theorem 3.2, we have M has constant sectional curvature α. Hence, we see that M has constant holomorphic sectional curvature α. Thus, the assertion $m = 3$ follows from the following theorem. ∎

Theorem 3.4. [15] *Except for the 6-dimensional one, there does not exist a non-Kaehlerian nearly Kaehlerian manifold of constant holomorphic sectional curvature.*

Now, we give a result related to the Weyl conformal curvature tensor.

Theorem 3.5. *Let M be an almost Hermitian manifold with dimension $2m \geq 8$. If M satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0,\pi/2)$, then M is conformally flat.*

Proof. Let p be an arbitrary point of M and let X,Y and Z be any orthonormal vectors of T_pM with $g(X,JY) = g(X,JZ) = g(Y,JZ) = 0$. Consider the hemi-slant 3-plane $\sigma_1 = D_1^\theta \oplus \{Z\}$ with slant angle $\theta \in (0,\pi/2)$, where $D_1^\theta = \text{span}\{X, \cos\theta JX + \sin\theta Y\}$ and the hemi-slant 3-plane $\sigma_2 = D_2^\theta \oplus \{Z\}$ with slant angle $\theta \in (0,\pi/2)$, where $D_2^\theta = \text{span}\{X, \cos\theta JX - \sin\theta Y\}$. As in the proof of Lemma 3.1, by the axiom of hemi-slant 3-spheres and by the equation (2.3), we have

$$(R(X,\cos\theta JX + \sin\theta Y)Z)^\perp = 0 \tag{3.16}$$

and

$$(R(X,\cos\theta JX - \sin\theta Y)Z)^\perp = 0. \tag{3.17}$$

On the other hand, since the dimension of M is greater than 8 we can choose a unit vector U in $(\text{span}\{X,JX\})^{\perp}\cap(\text{span}\{Y,JY\})^{\perp}\cap(\text{span}\{Z,JZ\})^{\perp}$. Thus, we write

$$R(X,\cos\theta JX+\sin\theta Y,Z,U)=0 \tag{3.18}$$

and

$$R(X,\cos\theta JX-\sin\theta Y,Z,U)=0. \tag{3.19}$$

From (3.18) and (3.19), we have

$$R(X,JX,Z,U)=0. \tag{3.20}$$

On the other hand, (3.18) and (3.20) give

$$R(X,Y,Z,U)=0, \tag{3.21}$$

where $X,Y,Z,U\in T_pM$ span an anti-holomorphic 4-plane. According to a well-known theorem of Schouten [13], the Weyl conformal curvature tensor W of M vanishes. This completes the proof. ∎

Next, we will give a result related to the Bochner curvature tensor.

Theorem 3.6. *Let M be a Kaehlerian manifold with dimension $2m\geq 6$. If M satisfies the axiom of hemi-slant 3-spheres for some $\theta\in(0,\pi/2)$, then M has a vanishing Bochner curvature tensor.*

Proof. Let p be any point of M and let X,Y and Z be any orthonormal vectors of T_pM with $g(X,JY)=g(X,JZ)=g(Y,JZ)=0$; that is, they span an anti-holomorhic 3-plane. Then the 3-plane $\sigma_1=D_1^{\theta}\oplus\{Y\}$ is a hemi-slant 3-plane with slant angle $\theta\in(0,\pi/2)$, where $D_1^{\theta}=\text{span}\{X,\cos\theta JX+\sin\theta JZ\}$. By the axiom of hemi-slant 3-spheres, there exists a 3-dimensional totally umbilical submanifold N_1 such that $p\in N_1$ and $T_pN_1=\sigma_1$. Then, with the help of (2.2) and (2.4) from (2.3), we have

$$(R(X,\cos\theta JX+\sin\theta JZ)Y)^{\perp}=0 \tag{3.22}$$

and

$$(R(X,Y)(\cos\theta JX+\sin\theta JZ))^{\perp}=0. \tag{3.23}$$

Since Z is normal to N_1, from (3.22) and (3.23) we get

$$R(X,\cos\theta JX+\sin\theta JZ,Y,Z)=0 \tag{3.24}$$

and

$$R(X,Y,\cos\theta JX+\sin\theta JZ,Z)=0. \tag{3.25}$$

Now, consider the hemi-slant 3-plane $\sigma_2=D_2^{\theta}\oplus\{Y\}$ with slant angle $\theta\in(0,\pi/2)$, where $D_2^{\theta}=\text{span}\{X,\cos\theta JX-\sin\theta JZ\}$. Again by the axiom of hemi-slant 3-spheres, there exists a 3-dimensional totally umbilical submanifold N_2 such that $p\in N_2$ and $T_pN_2=\sigma_2$. Then, with the help of (2.2) and (2.4) from (2.3), we have

$$(R(X,\cos\theta JX-\sin\theta JZ)Y)^{\perp}=0 \tag{3.26}$$

and

$$(R(X,Y)(\cos\theta JX-\sin\theta JZ))^{\perp}=0. \tag{3.27}$$

Since Z is normal to N_2, from (3.26) and (3.27) we get

$$R(X, \cos\theta JX - \sin\theta JZ, Y, Z) = 0 \tag{3.28}$$

and

$$R(X, Y, \cos\theta JX - \sin\theta JZ, Z) = 0. \tag{3.29}$$

From (3.24) and (3.28) we obtain

$$R(X, JX, Y, Z) = 0. \tag{3.30}$$

On the other hand, from (3.25) and (3.29) we obtain

$$R(X, Y, JX, Z) = 0. \tag{3.31}$$

Thus, our assertion follows from (3.30), (3.31) and Lemma 2.1. ∎

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