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# A Remark on the Axiom of Spheres with Lightlike Submanifolds and Application of Jacobi Equation

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**Abstract.** In this paper we improve the axiom of $r$-planes($r$-spheres) for lightlike submanifolds given by Kumar, Rani and Nagaich. Moreover we introduce another axiom called axiom of $r$-submanifolds. We prove that if a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ satisfies axiom of $r$-submanifolds, then it is a real space form. We also use Jacobi equation and obtain some related results to totally umbilical lightlike submanifold of semi-Riemannian manifold.

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## 1. Introduction

The notion of axiom of planes for Riemannian manifolds was first introduced by Elie Cartan [1] as:

**Axiom 1.** *A Riemannian manifold $M$ of dimension $m \geq 3$ satisfies the axiom of $r$-planes if for each point $p$ in $M$ and for every $r$-dimensional linear subspace $T$ of $T_pM$, there exists an $r$-dimensional totally geodesic submanifold $N$ in $M$ containing $p$ such that $T_pN = T$.*

Then he proved the following:

**Theorem 1.1.** *A Riemannian manifold of dimension $m \geq 3$ satisfies the axiom of $r$-planes for some $r$, $2 \leq r \leq m$ if and only if it is a real space form.*

Further in 1971, Leung and Nomizu [7] generalized this notion by introducing the axiom of $r$-spheres as:

**Axiom 2.** *A Riemannian manifold $M$ of dimension $m \geq 3$ satisfies the axiom of $r$-spheres if for each point $p$ in $M$ and for every $r$-dimensional linear subspace $T$ of $T_pM$, there exists*

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*an r-dimensional totally umbilical submanifold N in M with parallel mean curvature vector field containing p such that* $T_pN = T$.

They proved the following to characterize a real space form:

**Theorem 1.2.** *A Riemannian manifold of dimension* $m \geq 3$ *is a real space form if and only if it satisfies the axiom of r-spheres for some* $r, 2 \leq r \leq m$.

Graves and Nomizu [4] generalized the notions of axioms of planes and spheres for indefinite Riemannian manifolds. Kumar *et al.* [6], in particular studied the axioms of planes and spheres for semi-Riemannian manifolds with lightlike submanifolds (for details of lightlike (null) submanifolds or lightlike (null) curves see [2, 9]) . They gave these axioms as follows:

**Axiom 3.** *(Of r-planes and r-spheres) A semi-Riemannian manifold* $\bar{M}$ *of dimension* $m+n \geq 3$ *satisfies the axiom of r-planes (r-spheres) if for each point* $p \in \bar{M}$ *and for every r-dimensional linear subspace T of* $T_p\bar{M}$, *there exist an r-dimensional totally geodesic lightlike submanifold (totally umbilical lightlike submanifold with parallel transversal curvature vector field) M such that* $p \in M$ *and* $T_pM = T$, $2 \leq r \leq m+n$.

Thus they showed the following:

**Theorem 1.3.** *Let* $(\bar{M}, \bar{g})$ *be a semi-Riemannian manifold of dimension* $m+n \geq 3$ *and* $(M, g, S(TM), S(TM^{\perp}))$ *be a lightlike submanifold of* $\bar{M}$. *If* $\bar{M}$ *satisfies the axiom of r-spheres(r-planes),* $2 \leq r \leq m+n$, *then* $\bar{M}$ *is a real space form.*

But the proof of the above theorem has an error since Kumar *et al.* [6] used the induced connection $\nabla$ as metric connection on lightlike submanifolds though it is not in general metric [2, pg. 158]. However the idea of [6] is still interesting.

The purpose of the present paper is to improve the proposed axiom and rewrite the proof of the main theorem in [6]. We rewrite the axiom-3 (*r*-planes and *r*-spheres) of [6] as follows:

**Axiom 3$^*$:** *A semi-Riemannian manifold* $\bar{M}$ *of dimension* $m+n \geq 3$ *satisfies the axiom of r-planes (r-spheres) if for each point* $p \in \bar{M}$ *and for every r-dimensional linear subspace T of* $T_p\bar{M}$, *there exist an r-dimensional totally geodesic lightlike submanifold (totally umbilical lightlike submanifold with parallel second fundamental form) M such that* $p \in M$ *and* $T_pM = T$, $2 \leq r \leq m+n$.

Further, in this article we study a little more generalized form of axiom in some sense called axiom of *r*-submanifolds given by the following:

**Axiom 4.** *A semi-Riemannian manifold* $\bar{M}$ *of dimension* $m+n \geq 3$ *satisfies the axiom of r-submanifolds if for each point* $p \in \bar{M}$ *and for every r-dimensional linear subspace T of* $T_p\bar{M}$, *there exist an r-dimensional screen totally umbilical lightlike submanifold M with parallel second fundamental form such that* $T_pM = T$, $\forall p \in M$ *and* $h^s(X, \xi) = 0$, $\forall X \in TM$, $\forall \xi \in Rad(TM)$.

Then we prove the following:

**Theorem 1.4.** *Let* $(\bar{M}, \bar{g})$ *be a semi-Riemannian manifold of dimension* $m+n \geq 3$ *and* $(M, g, S(TM), S(TM^{\perp}))$ *be a lightlike submanifold of* $\bar{M}$. *If* $\bar{M}$ *satisfies the axiom of r-submanifolds,* $2 \leq r \leq m+n$, *then* $\bar{M}$ *is a real space form.*

## 2. Lightlike submanifolds

Let $(\bar{M},\bar{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold and $\bar{g}$ be a semi-Riemannian metric on $\bar{M}$. Let $M$ be a submanifold of $\bar{M}$ of codimension $n$ and said to be a lightlike submanifold if it admits a degenerate metric $g$ induced from $\bar{g}$ whose radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is of rank $s$, where $TM^{\perp}$ of tangent space $TM$ is defined by

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u,v) = 0, \forall v \in T_xM, x \in M\}.$$

As for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exist a local frame $\{N_i\}$ in the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that $\bar{g}(\xi_i,N_j) = \delta_{ij}$ and $\bar{g}(N_i,N_j) = 0$, it follows that there exist a complementary vector bundle to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ on which $\bar{g}$ vanishes and is called lightlike transversal vector bundle of $M$ which is locally spanned by $\{N_i\}$ and is denoted by $ltr(TM)$.

Consider the vector bundle which is complementary but not orthogonal to $TM$ in $T\bar{M}|_M$, that is,

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.1}$$

which is called transversal vector bundle of $M$.

Also we have

$$T\bar{M}|_M = TM \oplus tr(TM).$$

Suppose $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. As $TM$ and $tr(TM)$ are complementary vector subbundles of $T\bar{M}|_M$, the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_XY = \nabla_XY + h(X,Y),\ \forall X,Y \in \Gamma(TM),$$

$$\bar{\nabla}_XV = -A_VX + \nabla^t_XV, \forall X \in \Gamma(TM), V \in \Gamma(tr(TM)),$$

where $\{\nabla_XY, A_VX\}$ and $\{h(X,Y), \nabla^t_XV\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. According to decomposition (2.1) we have the following relations:

$$\bar{\nabla}_XY = \nabla_XY + h^l(X,Y) + h^s(X,Y), \tag{2.2}$$

$$\bar{\nabla}_XN = -A_NX + \nabla^l_XN + D^s(X,N), \tag{2.3}$$

$$\bar{\nabla}_XW = -A_WX + D^l(X,W) + \nabla^s_XW, \tag{2.4}$$

$\forall\, X,Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. By using equations (2.2), (2.3), (2.4) and taking into account that $\bar{\nabla}$ is a metric connection, we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_WX,Y), \tag{2.5}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X\xi) = 0,$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(N, A_WX), \tag{2.6}$$

$\forall\, X,Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(S(TM^{\perp}))$ and $\xi \in \Gamma(Rad(TM))$.

Denoting $\bar{R}$ and $R$ as the curvature tensor of $\bar{\nabla}$ and $\nabla$ respectively, then by using (2.2)–(2.4) we get

$$
\begin{aligned}
(2.7)\quad \bar{R}(X,Y)Z =& R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) \\
&- (\nabla_Y h^l)(X,Z) + (\nabla_X h^s)(Y,Z) - (\nabla_X h^s)(X,Z) + D^l(X, h^s(Y,Z)) \\
&- D^l(Y, h^s(X,Z)) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)).
\end{aligned}
$$

**Definition 2.1.** [3] *A lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is totally umbilical if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on $M$ called the transversal curvature vector field of $M$ such that for all $X, Y \in \Gamma(TM)$, $h(X,Y) = Hg(X,Y)$, i.e.*

$$h^l(X,Y) = H^l g(X,Y) \text{ and } h^s(X,Y) = H^s g(X,Y).$$

On the other hand, in case of totally umbilical submanifolds [3] we have

$$h^l(X,\xi) = 0,\ h^s(X,\xi) = 0. \tag{2.8}$$

Then replacing $X$ by $\xi$ in equation (2.6) yields

$$D^s(\xi, N) = 0. \tag{2.9}$$

Since $S(TM) \subset TM$ and by axiom of $r$-submanifolds it is integrable, we introduce the following definition.

**Definition 2.2.** *Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold satisfying the axiom of $r$-submanifolds. A lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is $S(TM)$-**totally umbilical** if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on $M$ called the transversal curvature vector field of $M$ such that for all $X, Y \in S(TM)$, $h(X,Y) = Hg(X,Y)$, i.e.*

$$h^l(X,Y) = H^l g(X,Y) \text{ and } h^s(X,Y) = H^s g(X,Y).$$

**Note:** It is clear that in $(\bar{M}, \bar{g})$ satisfying the axiom of $r$-submanifolds, every totally umbilical lightlike submanifold is $S(TM)$-totally umbilical lightlike submanifold. We will see that the converse is also true, i.e. $S(TM)$-total umbilicity also implies the total umbilicity of lightlike submanifolds.

## 3. Corrected proof of main theorem in [5]

**Proof of Theorem 1.3**

At an arbitrary point $p \in M$, let $X, \xi$ and $V$ be orthonormal at $p$. Let $T$ be an $r$-dimensional subspace of $T_p\bar{M}|_M$ containing $X$ and $\xi$ transversal to $V$. By the axiom $3^*$ there exists an $r$-dimensional totally umbilical lightlike submanifold $M$ with parallel second fundamental form. This means that we have

$$\nabla h = 0,$$

or in particular

$$(\nabla_X h^l)(X,X) = 0, (\nabla_X h^s)(X,X) = 0, (\nabla_X h^l)(\xi,X) = 0, (\nabla_X h^s)(\xi,X) = 0. \tag{3.1}$$

Now from (2.7), for $X, \xi \in \Gamma(TM)$ the transversal form of $\bar{R}(X,\xi)X$ is as below:

$$(\bar{R}(X,\xi)X)^N = (\nabla_X h^l)(\xi,X) - (\nabla_\xi h^l)(X,X) + (\nabla_X h^s)(\xi,X) - (\nabla_\xi h^s)(X,X) + D^l(X,h^s(\xi,X)) - D^l(\xi,h^s(X,X)) + D^s(X,h^l(\xi,X)) - D^s(\xi,h^l(X,X)).$$

Since $M$ is totally umbilical, therefore by using (2.5), (2.8), (2.9) and (3.1) in the above equation we obtain $(\bar{R}(X,\xi)X)^N = 0$. Hence

$$\bar{g}(\bar{R}(X,\xi)X),V) = 0.$$

Then the theorem follows from [4]. ∎

**Remark 3.1.** The above proof was not possible by assuming the parallelism of transversal curvature vector in the Axiom-3 since it would give us the following relations

$$(\nabla_X h^l)(Y,Z) = (\nabla_X g)(Y,Z)H^l + g(Y,Z)\nabla^l_X H^l,$$

$$(\nabla_X h^s)(Y,Z) = (\nabla_X g)(Y,Z)H^s + g(Y,Z)\nabla^s_X H^s,$$

from which it is easily seen that parallelism of transversal curvature vector does not imply the parallelism of the second fundamental form because the induced connection $\nabla$ is not metric and hence $(\nabla_X g)(Y,Z) \neq 0$.

## 4. Proof of the main theorem

To prove the main theorem of this article we need the following lemma:

**Lemma 4.1.** *Let $(\bar{M},\bar{g})$ be a semi-Riemannian manifold of dimension $m+n \geq 3$ which satisfies the axiom of r-submanifolds and let M be any r-dimensional $S(TM)$-totally umbilical lightlike submanifold. Then for every r-dimensional linear subspace $T$ of $T_p(\bar{M})$ such that $T = T_pM$, M is totally umbilical.*

*Proof.* We know from the Frobenius theorem that if a submanifold $M$ exists for a linear subspace $T$ of $T_p(\bar{M})$ such that $T = T_p(M)$, then the distribution $T$ is integrable. According to the hypothesis of the lemma, $\bar{M}$ satisfies the axiom of $r$-submanifold which implies that every linear subspace of $T_p(\bar{M})$ is integrable. In particular $S(TM)$ as a linear subspace of $T_p(\bar{M})$ is integrable and hence from the hypothesis there exist a 0-lightlike submanifold $N$ corresponding to $S(TM)$ which is totally umbilical.

If we suppose $N^*$ is any leaf of $S(TM)$, then we have

$$\bar{\nabla}_{PX}PY = \nabla_{PX}PY + h(PX,PY), \ \ \forall PX,PY \in \Gamma(TN^*) = S(TM)$$
$$= \nabla_{PX}PY + h^l(PX,PY) + h^s(PX,PY).$$

But $S(TM)$ is totally umbilical and therefore we have

$$h^l(PX,PY) + h^s(PX,PY) = H\bar{g}(PX,PY). \tag{4.1}$$

Here $H$ is the transversal curvature vector. Now let $X$ and $Y$ be any vector fields belong to $TM$. Then we can decompose these vectors as

$$X = PX + QX \quad and \quad Y = PY + QY,$$

where $P$ and $Q$ are the projection mappings on $S(TM)$ and $Rad(TM)$ respectively. Thus we have

$$\bar{g}(X,Y) = \bar{g}(PX,PY) \tag{4.2}$$

as $S(TM) \perp Rad(TM)$ and $Rad(TM)$ is a lightlike distribution.

Making use of the decomposition of $X$ and $Y$, we find

$$\begin{aligned} h^l(X,Y)+h^s(X,Y) =& h^l(PX,PY)+h^l(PX,QY)+h^l(QX,PY)+h^l(QX,QY) \\ &+h^s(PX,PY)+h^s(PX,QY)+h^s(QX,PY)+h^s(QX,QY). \end{aligned}$$

Now on a lighhtlike submanifold we have [2, pg.157]

(4.3) $$h^l(X,\xi_i)=0, \ \forall X \in \Gamma(TM) \ \ and \ \ \xi_i \in Rad(TM),$$

which in particular gives

(4.4) $$h^l(\xi_i,\xi_j)=0, \ \forall \xi_i, \ \xi_j \in Rad(TM).$$

Also from the axiom we have

(4.5) $$h^s(X,\xi)=0 \text{ or } h^s(PX,\xi)=0 \ \text{ and } \ h^s(QX,\xi)=0.$$

Moreover equations (4.3), (4.4) and (4.5) imply

(4.6) $$h^l(X,Y)+h^s(X,Y)=h^l(PX,PY)+h^s(PX,PY).$$

Finally combining equations (4.1), (4.2) and (4.6) we obtain

$$h^l(X,Y)+h^s(X,Y)=H\bar{g}(X,Y),$$

which shows that $M$ is totally umbilical. ∎

**Example 4.1.** Let $M$ be a surface of $R^4_1$ given by

$$x^3-x^0=0\ ;\ x^2=1+e^{2x_1},$$

where $(x^1,x^2,x^3,x^4)$ is a local coordinate system of $R^4_1$. We take the parametrization as $x^1=u$ and $x^0=v$.

The above surface $M$ in $R^4_1$ is lightlike, if and only if, on each coordinate neighborhood of $M$ we have

(4.7) $$\sum_{a=1}^{3}(D^{0a})^2=\sum_{1\le a<b\le 3}(D^{ab})^2,$$

where $D^{AB}=\begin{vmatrix} \frac{\partial x^A}{\partial u} & \frac{\partial x^B}{\partial u} \\ \frac{\partial x^A}{\partial v} & \frac{\partial x^B}{\partial v} \end{vmatrix}$.

Now with the help of the definition of $D^{AB}$ we calculate the values of $D^{0a}$ and $D^{ab}$ for $1\le a<b\le 3$ as given by below:
$D^{00}=0, D^{01}=-1, D^{02}=-2e^{2u}, D^{03}=0, D^{10}=1, D^{11}=0, D^{12}=0, D^{13}=1, D^{20}=2e^{2u}$, $D^{21}=0, D^{22}=0, D^{23}=2e^{2u}$, $D^{30}=0, D^{31}=-1, D^{32}=-2e^{2u}, D^{33}=0$.

It is clear that the equation (4.7) is satisfied for $M$ and hence $M$ is a lightlike surface of $R^4_1$. We calculate now the radical vector $\xi$, the lightlike transversal vector $N$, the vector $U$ spanning screen distribution $S(TM)$ and the vector $W$ spanning $S(TM^\perp)$.

By straightforward calculation, we get [2]

$$\xi=-(1+4e^{2u})(\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^3}),$$

$$N=\frac{1}{2(1+4e^{4u})}(\frac{\partial}{\partial x^0}-\frac{\partial}{\partial x^3}),$$

$$U = \frac{\partial}{\partial x^1} + 2e^{2u}\frac{\partial}{\partial x^2},$$

$$W = 2e^{2u}\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}.$$

Moreover we get [2, Section 5.4] $h^s(X,\xi) = 0$ which is in support of the hypothesis of the above lemma, $h^s(\xi,U) = 0$ and $h^s(U,U) = -\frac{4e^{2u}}{1+4e^{4u}}$. Also from equation (4.3) of [2, Chapter 5], we have $h^l(X,\xi) = 0 \ \forall\ X \in \Gamma(TM)$. Further by standard calculations we see that $h^l(U,U) = 0$ which shows that

$$h(U,U) = h^l(U,U) + h^s(U,U) = H\bar{g}(U,U),$$

where $H = -\frac{4e^{2u}}{(1+4e^{4u})^2}$.

In other words if in general we have two vectors $X = \xi_1 + U_1$ and $Y = \xi_2 + U_2$ in $\Gamma(TM)$, where $\xi_1, \xi_2 \in Rad(TM)$ and $U_1, U_2 \in S(TM)$, then we can write

$$h(X,Y) = h(U_1,U_2) = H\bar{g}(U_1,U_2) = H\bar{g}(X,Y).$$

This enables us to conclude that if an $(m+n)$-dimensional semi-Riemannian manifold $\bar{M}$ satisfies the axiom of $r$-submanifolds, then it always contains totally umbilical submanifold $M$ of dimension $r$, $2 \leq r \leq m+n$.

It means by supposing merely a lightlike submanifold to be $S(TM)$-totally umbilical and $h^s(X,\xi) = 0$. We can say that the lightlike submanifold is totally umbilical.

**Proof of the main theorem***:*

Let $p$ be any arbitrary point of $\bar{M}$ and $X, \xi$ and $V$ be orthonormal vectors at $p$. Let $T$ be an $r$-dimensional subspace of $T_p\bar{M}|_M$ containing $X$ and $\xi$ transversal to $V$. Now $\bar{M}$ satisfies the axiom of $r$-submanifolds and hence arbitrary $r$-dimensional linear subspace $T$ of $T_p\bar{M}$, there exist a $S(TM)$-totally umbilical $r$-dimensional submanifold $M$ with parallel second fundamental form such that $T_pM = T, \ \ \forall p \in M$ and $h^s(X,\xi) = 0, \ \ \forall X \in \Gamma(TM)$, $\xi \in Rad(TM)$.

But from Lemma 4.1, $M$ is totally umbilical lightlike submanifold with parallel second fundamental form $h$.

Now as a consequence of equation (2.7), it follows that the transversal component of $\bar{R}(X,\xi)X$ is given by

(4.8)

$$\begin{aligned}(\bar{R}(X,\xi)X)^N =& (\nabla_X h^l)(\xi,X) - (\nabla_\xi h^l)(X,X) + (\nabla_X h^s)(\xi,X) - (\nabla_\xi h^s)(X,X) \\ &+ D^l(X,h^s(\xi,X)) - D^l(\xi,h^s(X,X)) + D^s(X,h^l(\xi,X)) - D^s(\xi,h^l(X,X)).\end{aligned}$$

Taking into account that the submanifold $M$ is totally umbilical with parallel second fundamental form and equations (2.5), (2.8), (2.9) and (3.1), the above equation implies that

$$(\bar{R}(X,\xi)X)^N = 0$$

or

$$g(\bar{R}(X,\xi)X,V) = 0. \tag{4.9}$$

Therefore our theorem follows immediately from a result of Graves and Nomizu [4].

As a special case this theorem is true for totally geodesic submanifolds.

## 5. Application of Jacobi equation to lightlike submanifolds

To study the geometry of manifolds (or submanifolds) by means of Jacobi equation or Jacobi vector fields has been a motivation for Differential Geometers (for example see [5]). Following this, in this section we discuss the application of Jacobi differential equation to totally umbilical lightlike submanifold $M$ of a semi-Riemannian manifold $\bar{M}$. Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold satisfying the axiom of $r$-submanifolds. It is known that the Jacobi differential equation is given by $Y'' = R(Y, \gamma')\gamma'$, where $\gamma$ is a geodesic on $M$ and $Y$ is a vector field on $\gamma$ called Jacobi vector field. Now from equations (2.8), $h^l(X, \xi)$, $h^s(X, \xi)$, $\nabla_X h^l$ and $\nabla_X h^s$ vanish for all $X \in TM$ and $\xi \in Rad(TM)$, thus we have

$$\bar{R}(X, \xi)\xi = R(X, \xi)\xi \text{ and } \bar{R}(\xi, X)X = R(\xi, X)X.$$

In particular we can write

$$\bar{R}(PX, \xi)\xi = R(PX, \xi)\xi \text{ and } \bar{R}(\xi, PX)PX = R(\xi, PX)PX. \tag{5.1}$$

**Definition 5.1.** [8] *Let $v$ be any subspace of the tangent space $TM$ of $M^n$ and $v^\perp$ denotes its orthogonal complement in $TM$. Then $f_X : \omega \longrightarrow TM$ is defined to be the tidal operator in lightlike submanifold $M^n$ given by*

$$f_X Y = R(Y, X)X,$$

*where $\omega = v$ or $v^\perp$ with $X \in v^\perp$ or $v$ respectively.*

Now let $X \in TM$ such that $X$ is orthogonal to $\gamma'$ satisfies the Jacobi differential equation [8]

$$X'' = R(X, \gamma')\gamma', \tag{5.2}$$

where $\gamma$ is any geodesic of $M$. As $TM = S(TM) \perp Rad(TM)$, therefore we can consider two cases: (i) $X = PX \in S(TM)$ and $\gamma' = \xi \in Rad(TM)$, (ii) $X = \xi \in Rad(TM)$ and $\gamma' = PX \in S(TM)$.

In the previous section, we have proved that $\bar{M}$ is a space of constant sectional curvature $c$, therefore from the above definition and equations (5.1) we have

$$g(f_{PX}\xi, PX) = g(\bar{R}(\xi, PX)PX, \xi) = g(R(\xi, PX)PX, \xi) = \text{ a constant} \tag{5.3}$$

and

$$g(f_\xi PX, \xi) = g(\bar{R}(PX, \xi)\xi, PX) = g(R(PX, \xi)\xi, PX) = \text{ a constant.} \tag{5.4}$$

If we consider the geodesic variation $\chi$ of $\gamma$ as a one parameter family of freely falling particles, then the variational vector field $X \in TM$ satisfying Jacobi equation (5.2) gives the position of arbitrarily nearby particle relative to $\gamma$. Thus the derivation $X'$ gives the relative velocity and $X''$ as relative acceleration. Then from equations (5.2), (5.3) and (5.4) we derive that

$$g(\xi'', \xi) = g(PX'', PX) = \text{ constant,}$$

which is possible only if the acceleration is orthogonal to the position vector or parallel to the position vector. If the acceleration is orthogonal to the position vector, then obviously the lightlike submanifold would be a plane and if the acceleration is parallel to the position vector, then the lightlike submanifold would be a sphere [8].

Hence we have

**Theorem 5.1.** *Let* $(\bar{M}, \bar{g})$ *be a semi-Riemannian manifold of dimension* $m+n \geq 3$ *and* $(M, g, S(TM), S(TM^{\perp}))$ *be a totally umbilical lightlike submanifold of* $\bar{M}$. *If* $\bar{M}$ *satisfies the axiom of r-submanifolds,* $(2 \leq r \leq m+n)$ *and* $\gamma$ *is a geodesic in M on which the tangent vector field* $X \in TM$ *orthogonal to* $\gamma'$ *and satisfies the Jacobi differential equation, then the lightlike submanifold* $(M, g, S(TM), S(TM^{\perp}))$ *is either a plane or a sphere.*

Further let $\{V_i\}, i = 1, 2, \ldots\ldots, p$ be the orthonormal basis of the transversal vector bundle $tr(TM) = ltr(TM) \perp S(TM^{\perp})$ and $\gamma$ be the null geodesic. This gives that $\gamma' = \xi \in Rad(TM)$. It is also known from the previous discussion that for the manifold $\bar{M}$ satisfying the axiom of $r$-submanifolds we have

$$\bar{R}(X, \xi)\xi = R(X, \xi)\xi.$$

or

$$g(\bar{R}(X, \xi)\xi, V) = g(R(X, \xi)\xi, V) = 0, \quad \forall V \in tr(TM).$$

If $R^N$ denotes the projection of Jacobi operator $\bar{R}$ to the transversal vector bundle $tr(TM)$, then we have the matrix of $R^N$ as $p \times p$ null matrix by virtue of equation (4.9), i.e.

$$R^N = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 0 \end{bmatrix}_{p \times p}. \tag{5.5}$$

Now let $J(t)$ be a solution of the Jacobi equation given by

$$J''(t) + R^N \circ J(t) = 0 \tag{5.6}$$

satisfying the conditions $J(0) = 0$ and $J'(0) = id$, where $id$ denotes the identity matrix of order $p$. Then from equations (5.5) and (5.6) we get $J(t) = t \cdot id$. Therefore

$$\det J(t) = t^p,$$

which shows that $J(t)$ is never zero except at $t = 0$. This implies that for null geodesics there exists a conjugate point only at $t = 0$. This shows that null geodesics never meet except at the starting point.

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