BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37**(2) (2014), 575–596

Impulsive Periodic Type Boundary Value Problems for Multi-Term Singular Fractional Differential Equations

YUJI LIU
Department of Mathematics, Guangdong University of Finance and Economics, Guangzhou 510320, P. R.China
liuyuji888@sohu.com

Abstract. A class of impulsive periodic type boundary value problems for multi-term singular fractional differential equations is presented. Results on the existence of solutions to these problems are established. The analysis relies on the well known fixed point theorems. An example is given to illustrate the efficiency of the main theorems.

2010 Mathematics Subject Classification: 92D25, 34A37, 34K15

Keywords and phrases: Solution, impulsive singular fractional differential equation, periodic type boundary value problem, fixed point theorem.

1. Introduction

In recent papers [1–6, 8, 11–14, 16–20], the authors studied the existence or uniqueness of positive solutions or solutions of boundary value problems for fractional differential equations. While the existence of solutions of impulsive boundary value problems for Riemann-Liouville fractional differential equations has not been given up to now, the research proceeds slowly and appears some new difficulties.

In [15], the authors studied the existence and uniqueness of solutions of the following periodic boundary value problem of the impulsive fractional differential equation

$$(1.1)\quad \begin{cases} D_{0^+}^{\alpha}u(t)-\lambda u(t)=f(t,u(t)), & t\in(0,1], t\neq t_1\in(0,1), 0<\alpha\leq 1,\\ \lim_{t\to 0}t^{1-\alpha}u(t)=u(1),\\ \lim_{t\to t_1^+}[t-t_1]^{1-\alpha}[u(t)-u(t_1)]=I(u(t_1)), \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order $\alpha, I: R\to R$ is continuous, $f:[0,1]\times R\to R$ is continuous, $\lambda\in R$ is a constant. The existence and uniqueness of solutions of BVP(1.1) are established under some assumptions by using Banach contraction principle. One of the main assumptions in [15] is as follows: there exist positive numbers M and m such that

$$(1.2)\qquad |f(t,x)|\leq M,\quad |I(x)|\leq m,\quad t\in[0,1],\ x\in R.$$

Communicated by Syakila Ahmad.
Received: February 14, 2012; *Revised:* May 30, 2012.

In this paper, we discuss the periodic type boundary value problem of the nonlinear singular fractional differential equation with the multi-terms

$$(1.3)\qquad \begin{cases} D_{0+}^{\beta}[\Phi(\rho(t)D_{0+}^{\alpha}u(t))] = q(t)f(t,u(t),D_{0+}^{\alpha}u(t)), & t\in(0,1),\\ \lim_{t\to 1} t^{1-\alpha}u(t) - \lim_{t\to 0} t^{1-\alpha}u(t) = \int_0^1 G(s,u(s),D_{0+}^{\alpha}u(s))ds, \\ \lim_{t\to 1} t^{1-\beta}\Phi(\rho(t)D_{0+}^{\alpha}u(t)) - \lim_{t\to 0} t^{1-\beta}\Phi(\rho(t)D_{0+}^{\alpha}u(t)) \\ \qquad\qquad = \int_0^1 H(s,u(s),D_{0+}^{\alpha}u(s))ds, \\ \lim_{t\to t_1^+} u(t) = I(t_1,u(t_1),D_{0+}^{\alpha}u(t_1)), \\ \lim_{t\to t_1^+} \Phi(\rho(t)D_{0+}^{\alpha}u(t)) = J(t_1,u(t_1),D_{0+}^{\alpha}u(t_1)). \end{cases}$$

where

- $0<\alpha,\beta\le 1$, D_{0+}^{α} (or D_{0+}^{β}) is the Riemann-Liouville fractional derivative of order α (or β),
- $\Phi: R\to R$ is a sup-multiplicative-like function with supporting function ω, its inverse function is denoted by $\Phi^{-1}: R\to R$ with supporting function ν,
- $0<t_1<1$, $I,J:(0,1)\times R^2\to R$ are continuous functions, **(•)** $\phi,\psi:(0,1)\to R$ with $\phi|_{(0,t_1]},\rho|_{(0,t_1]}\in L^1(0,t_1)$ and $\phi|_{(t_1,1]},\rho|_{(t_1,1]}\in L^1(t_1,1)$,
- $\rho:(0,1)\to[0,+\infty)$ with $\rho|_{(0,t_1]}\in C^0(0,t_1]$ and $\rho|_{(t_1,1]}\in C^0(t_1,1)$ and satisfies that there exist numbers $L>0$ and $k>-\alpha$ such that $\rho(t)\ge(t^{-k}\nu(t^{\beta-1}))/L$ for all $t\in(0,1), t\ne t_1$,
- $q:(0,1)\to R$ with $q|_{(0,t_1]}\in C^0(0,t_1]$ and $q|_{(t_1,1]}\in C^0(t_1,1)$ and there exist numbers $L_1>0$ and $k_1>-\beta$ such that $|q(t)|\le L_1t^{k_1}$ for all $t\in(0,1)$,
- f,G,H defined on $(0,t_1)\bigcup(t_1,1)\times R\times R$ are **impulsive Caratheodory functions** that may be singular at $t=0,t_1$ and 1.

A function x defined on $(0,1)$ is called a solution of BVP(1.3), if $x|_{(0,t_1]}\in C^0(0,t_1]$ and $x|_{(t_1,1]}\in C^0(t_1,1]$, $D_{0+}^{\alpha}x|_{(0,t_1]}\in C^0(0,t_1]$ and $D_{0+}^{\alpha}x|_{(t_1,1]}\in C^0(t_1,1]$, $D_{0+}^{\beta}[\Phi(\rho(t)D_{0+}^{\alpha}x(t))]$ $\in L^1(0,1)$ and x satisfies all equations in (1.3).

We obtain the results on the existence of at least one solution of BVP(1.3). An example is given to illustrate the efficiency of the main theorem. The results in this paper generalize those ones in [18], i.e., assumption (1.2) is replaced by a weaker one. The impulsive functions are different those ones in [18].

Remark 1.1. In the special case: $\alpha=\beta=1$, $g(t,x,y)=h(t,x,y)\equiv 0$ and the impulse disappears, BVP(1.3) becomes the periodic boundary value problem for ordinary differential equation

$$\begin{cases} [\Phi(\rho(t)u'(t))]' = q(t)f(t,u(t),u'(t)), & t\in(0,1),\\ u(0)=u(1),\ u'(0)=u'(1). \end{cases}$$

So we call BVP(1.3) the impulsive periodic type boundary value problem of the nonlinear fractional differential equation.

The remainder of this paper is as follows: in Section 2, we present preliminary results. In Section 3, the main theorems and their proof are given. In Section 4, an example is given to illustrate the main results.

2. Preliminary results

For the convenience of the readers, we present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [7,10]. Let the Gamma and beta functions $\Gamma(\alpha)$ and $\mathbf{B}(p,q)$ be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx, \quad \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \to R$ is given by*

$$I_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds,$$

provided that the right-hand side exists.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0,\infty) \to R$ is given by*

$$D_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n-1 \le \alpha < n$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3. *Let X and Y be Banach spaces. $L : D(L) \subset X \to Y$ is called a Fredholm operator of index zero if Im L is closed in X and* $\dim\mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L < +\infty$.

It is easy to see that if L is a Fredholm operator of index zero, then there exist the projectors $P : X \to X$, and $Q : Y \to Y$ such that

$$\mathrm{Im}\, P = \mathrm{Ker}\, L, \quad \mathrm{Ker}\, Q = \mathrm{Im}\, L, \quad X = \mathrm{Ker}\, L \oplus \mathrm{Ker}\, P, \quad Y = \mathrm{Im}\, L \oplus \mathrm{Im}\, Q.$$

If $L : D(L) \subset X \to Y$ is called a Fredholm operator of index zero, the inverse of

$$L|_{D(L)\cap \mathrm{Ker}\, P} : D(L) \cap \mathrm{Ker}\, P \to \mathrm{Im}\, L$$

is denoted by K_p.

Definition 2.4. *Suppose that $L : D(L) \subset X \to Y$ is called a Fredholm operator of index zero. The continuous map $N : X \to Y$ is called L-compact if both $QN(\overline{\Omega})$ and $K_p(I-Q)N : \overline{\Omega} \to X$ are compact for each nonempty open subset Ω of X satisfying $D(L) \cap \overline{\Omega} \neq \emptyset$.*

To obtain the main results, we need the abstract existence theorem [9].

Lemma 2.1. Leray-Schauder Nonlinear Alternative. *Let X,Y be Banach spaces and $L : D(L) \bigcap X \to Y$ a Fredholm operator of index zero with* $\mathrm{Ker}L = \{0 \in X\}$*, $N : X \to Y$ L-compact. Suppose Ω is a nonempty open subset of X satisfying $D(L) \cap \overline{\Omega} \neq \emptyset$. Then either there exists $x \in \partial\Omega$ and $\theta \in (0,1)$ such that $Lx = \theta Nx$ or there exists $x \in \overline{\Omega}$ such that $Lx = Nx$.*

Definition 2.5. *An odd homeomorphism Φ of the real line R onto itself is called a sup-multiplicative-like function if there exists a homeomorphism ω of $[0,+\infty)$ onto itself which supports Φ in the sense that for all $v_1, v_2 \ge 0$ it holds*

$$\Phi(v_1 v_2) \ge \omega(v_1)\Phi(v_2). \tag{2.1}$$

ω is called the supporting function of Φ.

Remark 2.1. Note that any sup-multiplicative function is sup-multiplicative-like function. Also any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u, \quad u \in R$$

is sup-multiplicative-like, provided that $c_j \geq 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, u\}$, $u \geq 0$.

Remark 2.2. It is clear that a sup-multiplicative-like function Φ and any corresponding supporting function ω are increasing functions vanishing at zero and moreover their inverses Φ^{-1} and ν respectively are increasing and such that

$$\Phi^{-1}(w_1 w_2) \leq \nu(w_1)\Phi^{-1}(w_2), \tag{2.2}$$

for all $w_1, w_2 \geq 0$ and ν is called the supporting function of Φ^{-1}.

Remark 2.3. If $\Phi_p(x) = |x|^{p-2}x$ for $p > 1$, we call Φ_p a one-dimensional p-Laplacian. By Remark 2.1, Φ_p is a sup-multiplicative-like function with its supporting function $\omega(x) = |x|^{p-2}x$ and its inverse function $\Phi_p^{-1}(x) = |x|^{q-2}x$. The supporting function of Φ_p^{-1} is $\nu(x) = |x|^{q-2}x$. Here q satisfies $1/p + 1/q = 1$. It is easy to see that

$$\lim_{t\to 0} \frac{1}{\nu(t^{1-\beta})\nu(t^{\beta-1})} = \lim_{t\to 0} \frac{\nu(t^{\beta-1})}{\nu(t^{\beta-1})} = 1 < +\infty.$$

In this paper we always suppose that Φ is a sup-multiplicative-like function with its supporting function ω, the inverse function Φ^{-1} has its supporting function ν.

Definition 2.6. *We call $F : (0,t_1) \cup (t_1,1) \times R^2 \to R$ an* **impulsive Caratheodory function** *if it satisfies the following items:*

(i) $t \to F\left(t, t^{\alpha-1}u, (\Phi^{-1}(t^{\beta-1}v))/(\rho(t))\right)$ *is continuous both on* $(0,t_1]$ *and* $(t_1,1)$ *and the limits exist*

$\lim_{t\to 0^+} F\left(t, t^{\alpha-1}u, \frac{\Phi^{-1}(t^{\beta-1}v)}{\rho(t)}\right)$, $\quad \lim_{t\to t_1^+} F\left(t, t^{\alpha-1}u, \frac{\Phi^{-1}(t^{\beta-1}v)}{\rho(t)}\right)$,
$\lim_{t\to 1^-} F\left(t, t^{\alpha-1}u, \frac{\Phi^{-1}(t^{\beta-1}v)}{\rho(t)}\right)$

for any $(u,v) \in R^2$,

(ii) $(u,v) \to F\left(t, t^{\alpha-1}u, (\Phi^{-1}(t^{\beta-1}v))/(\rho(t))\right)$ *is continuous on* R^2 *for all* $t \in (0,t_1) \cup (t_1,1)$.

Define

$$x(t) = u(t), \quad y(t) = \Phi(\rho(t) D_{0^+}^{\alpha} x(t)).$$

Then BVP (1.3) is transformed to

$$\begin{cases} D_{0^+}^{\alpha} x(t) = \frac{\Phi^{-1}(y(t))}{\rho(t)}, & t \in (0,1), t \neq t_1, \\ D_{0^+}^{\beta} y(t) = q(t) f\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right), & t \in (0,1), t \neq t_1, \\ \lim_{t\to 1} t^{1-\alpha} x(t) - \lim_{t\to 0} t^{1-\alpha} x(t) = \int_0^1 \phi(t) G\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right) dt, \\ \lim_{t\to 1} t^{1-\beta} y(t) - \lim_{t\to 0} t^{1-\beta} y(t) = \int_0^1 \psi(t) H\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right) dt, \\ \lim_{t\to t_1^+} x(t) = I\left(t_1, x(t_1), \frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right), \\ \lim_{t\to t_1^+} y(t) = J\left(t_1, x(t_1), \frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right). \end{cases} \tag{2.3}$$

It is easy to see that if (x,y) is a solution of BVP (2.3), then x is a solution of BVP (1.3).

We use the Banach spaces

$$X=\left\{x:(0,1]\to R:\begin{array}{c}x|_{(0,t_1]}\in C^0(0,t_1],x|_{(t_1,1]}\in C^0(t_1,1]\\ \text{there exist the limits}\\ \lim_{t\to 0^+}t^{1-\alpha}x(t),\ \lim_{t\to t_1^+}x(t)\end{array}\right\}$$

with the norm

$$||x||=||x||_\infty=\sup_{t\in(0,1]}t^{1-\alpha}|x(t)|,$$

$$Y=\left\{y:(0,1]\to R:\begin{array}{c}y|_{(0,t_1]}\in C^0(0,t_1],y|_{(t_1,1]}\in C^0(t_1,1]\\ \text{there exist the limits}\\ \lim_{t\to 0^+}t^{1-\beta}y(t),\ \lim_{t\to t_1^+}y(t)\end{array}\right\}$$

with the norm

$$||y||=||y||_\infty=\sup_{t\in(0,1)}t^{1-\beta}|y(t)|,$$

$L^1[0,1]$ with the norm

$$||u||_1=\int_0^1|u(s)|ds.$$

Choose $E=X\times Y$ with the norm

$$||(x,y)||=\max\{||x||_\infty,||y||_\infty\}\quad\text{for}\quad(x,y)\in X\times Y.$$

Choose $Z=L^1(0,1)\times L^1(0,1)\times R^4$ with the norm

$$\left|\left|\begin{pmatrix}u\\v\\c\\d\\c\\d\end{pmatrix}\right|\right|=\max\{||u||_1,\ ||v||_1,\ |a|,\ |b|,|c|,|d|\}\quad\text{for}\quad\begin{pmatrix}u\\v\\c\\d\\c\\d\end{pmatrix}\in Z.$$

Define L to be the linear operator from $D(L)\bigcap E$ to Z with

$$D(L)=\left\{(x,y)\in E:D_{0^+}^{\alpha}x,D_{0^+}^{\beta}y\in L^1(0,1)\right\}$$

and

$$L(x,y)(t)=\begin{pmatrix}D_{0^+}^{\alpha}x(t)\\ D_{0^+}^{\beta}y(t)\\ \lim_{t\to 1}t^{1-\alpha}x(t)-\lim_{t\to 0}t^{1-\alpha}x(t)\\ \lim_{t\to 1}t^{1-\beta}y(t)-\lim_{t\to 0}t^{1-\beta}y(t)\\ \lim_{t\to t_1^+}x(t)\\ \lim_{t\to t_1^+}y(t)\end{pmatrix}^T$$

for $(x,y)\in D(L)\bigcap E$. Define $N:E\to Z$ by

$$N(x,y)(t)=\begin{pmatrix} \frac{\Phi^{-1}(y(t))}{\rho(t)} \\ q(t)f\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right) \\ \int_0^1 \phi(t)G\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)dt \\ \int_0^1 \psi(t)H\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)dt \\ I\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right) \\ J\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right) \end{pmatrix}^T \quad \text{for} \quad (x,y)\in E.$$

Then BVP(2.3) can be written as

$$L(x,y)=N(x,y), \quad (x,y)\in E.$$

Lemma 2.2. *Suppose that f,G,H are* **impulsive Caratheodory functions***, I,J are continuous functions and Φ is a sup-multiplicative-like function with ν satisfying*

$$\lim_{t\to 0}\frac{1}{\nu(t^{1-\beta})\nu(t^{\beta-1})}=\lim_{t\to 0}\frac{\nu(t^{\beta-1})}{\nu(t^{\beta-1})}<+\infty. \tag{2.4}$$

Then L is a Fredhold operator with index zero and $N:E\to Z$ is L-compact.

Proof. To prove that L is a Fredhold operator with index zero, we should do the following three steps.

Step (i) Prove that $\operatorname{Ker} L=\{(0,0)\in E\}$.

We know that $(x,y)\in \operatorname{Ker} L$ if and only if

$$\begin{cases} D_{0^+}^{\alpha}x(t)=0, \\ D_{0^+}^{\beta}y(t)=0, \\ \lim_{t\to 1}t^{1-\alpha}x(t)-\lim_{t\to 0}t^{1-\alpha}x(t)=0, \\ \lim_{t\to 1}t^{1-\beta}y(t)-\lim_{t\to 0}t^{1-\beta}y(t)=0 \\ \lim_{t\to t_1^+}x(t)=0 \\ \lim_{t\to t_1^+}y(t)=0. \end{cases} \tag{2.5}$$

Hence $(x,y)\in \operatorname{Ker} L$ if and only if $x(t)=0$ and $y(t)=0$. Thus $\operatorname{Ker} L=\{(0,0)\in E\}$.

Step (ii) Prove that $\operatorname{Im} L=\{\,(u,v,a,b,c,d)\in Z\}=Z$.

For $(u,v,a,b,c,d)\in Z$, we know that $(u,v,a,b,c,d)\in \operatorname{Im} L$ if and only if there exist $(x,y)\in E$ such that

$$\begin{cases} D_{0^+}^{\alpha}x(t)=u(t), \\ D_{0^+}^{\beta}y(t)=v(t), \\ \lim_{t\to 1}t^{1-\alpha}x(t)-\lim_{t\to 0}t^{1-\alpha}x(t)=a, \\ \lim_{t\to 1}t^{1-\beta}y(t)-\lim_{t\to 0}t^{1-\beta}y(t)=b, \\ \lim_{t\to t_1^+}x(t)=c, \\ \lim_{t\to t_1^+}y(t)=d. \end{cases} \tag{2.6}$$

So we get

(2.7)
$$x(t)=\begin{cases}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds+\Bigg(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds \\ \qquad +\frac{c\Gamma(\alpha)-\int_0^{t_1}(t_1-s)^{\alpha-1}u(s)ds}{\Gamma(\alpha)t_1^{\alpha-1}}-a\Bigg)t^{\alpha-1}, & t\in(0,t_1], \\ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)ds+\frac{c\Gamma(\alpha)-\int_0^{t_1}(t_1-s)^{\alpha-1}u(s)ds}{\Gamma(\alpha)t_1^{\alpha-1}}t^{\alpha-1}, & t\in(t_1,1]\end{cases}$$
$$y(t)=\begin{cases}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds+\Bigg(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds \\ \qquad +\frac{d\Gamma(\beta)-\int_0^{t_1}(t_1-s)^{\beta-1}v(s)ds}{\Gamma(\beta)t_1^{\beta-1}}-b\Bigg)t^{\beta-1}, & t\in(0,t_1], \\ \frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}v(s)ds+\frac{d\Gamma(\beta)-\int_0^{t_1}(t_1-s)^{\beta-1}v(s)ds}{\Gamma(\beta)t_1^{\beta-1}}t^{\beta-1}, & t\in(t_1,1].\end{cases}$$

It is easy to show that $(x,y)\in D(L)\bigcap E$. Then $\mathrm{Im}\,L=Z$. Furthermore, (x,y) satisfies (2.7) if and only if (x,y) satisfies (2.6).

Step (iii) Prove that $\mathrm{Im}\,L$ is closed in X and $\dim\mathrm{Ker}\,L=\mathrm{co}\dim\mathrm{Im}\,L<+\infty$.

From Step (ii) $\mathrm{Im}\,L=Z$ is closed in Z. It follows from $\mathrm{Ker}\,L=\{(0,0)\in E\}$ that $\dim\mathrm{Ker}\,L=0$. Define the projector $P:\ E\to E$ by

(2.8)
$$P(x,y)(t)=(0,\ 0)\quad\text{for}\quad(x,y)\in E.$$

It is easy to prove that

(2.9)
$$\mathrm{Im}\,P=\mathrm{Ker}\,L,\quad X=\mathrm{Ker}\,L\oplus\mathrm{Ker}\,P.$$

Define the projector $Q:\ Z\to Z$ by

(2.10)
$$Q(u,v,a,b,c,d)(t)=(0,0,0,0,0,0)$$

for $(u,v,a,b,c,d)\in Z$.

It is easy to show that

(2.11)
$$\mathrm{Im}\,L=\mathrm{Ker}\,Q,\quad Y=\mathrm{Im}\,Q\oplus\mathrm{Im}\,L.$$

From above discussion, we see that $\dim\mathrm{Ker}\,L=\mathrm{co}\dim\mathrm{Im}\,L=0<+\infty$. So L is a Fredholm operator of index zero.

Now, we prove that N is L-compact. This is divided into three steps.

Step (i) We prove that N is continuous. Let $(x_n,y_n)\in E$ with $(x_n,y_n)\to(x_0,y_0)$ as $n\to\infty$. We will show that $N(x_n,y_n)\to N(x_0,y_0)$ as $n\to\infty$.

In fact, we have

$$||(x_n,y_n)||=\sup_{n=0,1,2,\dots}\left\{\sup_{t\in(0,1]}t^{1-\alpha}|x_n(t)|,\ \sup_{t\in(0,1]}t^{1-\beta}|y_n(t)|\right\}=r<+\infty$$

and

(2.12)
$$\sup_{t\in(0,1]}t^{1-\alpha}|x_n(t)-x_0(t)|\to0,\quad\sup_{t\in(0,1]}t^{1-\beta}|y_n(t)-y_0(t)|\to0,\ n\to\infty.$$

By

$$N(x_n,y_n)(t)=\begin{pmatrix} \frac{\Phi^{-1}(y_n(t))}{\rho(t)} \\ q(t)f\left(t,x_n(t),\frac{\Phi^{-1}(y_n(t))}{\rho(t)}\right) \\ \int_0^1\phi(t)G\left(t,x_n(t),\frac{\Phi^{-1}(y_n(t))}{\rho(t)}\right)dt \\ \int_0^1\psi(t)H\left(t,x_n(t),\frac{\Phi^{-1}(y_n(t))}{\rho(t)}\right)dt \\ I\left(t_1,x_n(t_1),\frac{\Phi^{-1}(y_n(t_1))}{\rho(t_1)}\right) \\ J\left(t_1,x_n(t_1),\frac{\Phi^{-1}(y_n(t_1))}{\rho(t_1)}\right) \end{pmatrix}^T \quad \text{for } (x,y)\in E.$$

Since Φ is a sup-multiplicative-like function, we get from (2.2) that

$$\frac{1}{\nu(t^{1-\beta})\nu(t^{\beta-1})}\Phi^{-1}(x)\le\frac{\Phi^{-1}(t^{\beta-1}x)}{\nu(t^{\beta-1})}\le\frac{\nu(t^{\beta-1})}{\nu(t^{\beta-1})}\Phi^{-1}(x),x\ge 0$$

and

$$\frac{1}{\nu(t^{1-\beta})\nu(t^{\beta-1})}\Phi^{-1}(x)\ge\frac{\Phi^{-1}(t^{\beta-1}x)}{\nu(t^{\beta-1})}\ge\frac{\nu(t^{\beta-1})}{\nu(t^{\beta-1})}\Phi^{-1}(x),x\le 0$$

Then (2.4) implies that $\frac{\Phi^{-1}(t^{\beta-1}x)}{\nu(t^{\beta-1})}$ is continuous on $(0,t_1]\times R^2$ and there exists the limit $\lim_{t\to0^+}\frac{\Phi^{-1}(t^{\beta-1}x)}{\nu(t^{\beta-1})}$. Hence $\frac{\Phi^{-1}(t^{\beta-1}x)}{\nu(t^{\beta-1})}$ is continuous on $[0,t_1]\times R^2$. It follows that $\frac{\Phi^{-1}(t^{\beta-1}x)}{\nu(t^{\beta-1})}$ is uniformly continuous on $[0,t_1]\times[-r,r]\times[-r,r]$.

Similarly, we can see that $\frac{\Phi^{-1}(t^{\beta-1}x)}{\nu(t^{\beta-1})}$ is uniformly continuous on $[t_1,1]\times[-r,r]\times[-r,r]$.

For any $\varepsilon>0$ there exists $\delta>0$ such that

$$\left|\frac{\Phi^{-1}(t^{\beta-1}u_1)}{\nu(t^{\beta-1})}-\frac{\Phi^{-1}(t^{\beta-1}u_2)}{\nu(t^{\beta-1})}\right|<\frac{\varepsilon}{\int_0^1\frac{\nu(t^{\beta-1})}{\rho(t)}dt},t\in(0,1],|u_1-u_2|<\delta. \tag{2.13}$$

From (2.12), there exists N such that

$$t^{1-\alpha}|x_n(t)-x_0(t)|<\delta,\ t^{1-\beta}|y_n(t)-y_0(t)|<\delta,\ t\in(0,1],\ n>N. \tag{2.14}$$

Hence

$$\begin{aligned}&\int_0^1\left|\frac{\Phi^{-1}(y_n(t))}{\rho(t)}-\frac{\Phi^{-1}(y_0(t))}{\rho(t)}\right|dt\\ &=\int_0^1\frac{\nu(t^{\beta-1})}{\rho(t)}\left|\frac{\Phi^{-1}\left(t^{\beta-1}t^{1-\beta}y_n(t)\right)}{\nu(t^{\beta-1})}-\frac{\Phi^{-1}\left(t^{\beta-1}t^{1-\beta}y_0(t)\right)}{\nu(t^{\beta-1})}\right|dt\\ &<\int_0^1\frac{\nu(t^{\beta-1})}{\rho(t)}\frac{\varepsilon}{\int_0^1\frac{\nu(t^{\beta-1})}{\rho(t)}dt}dt=\varepsilon,n>N.\end{aligned}$$

Since f is a Caratheodory function, we know that $f\left(t,t^{\alpha-1}u,\frac{\Phi^{-1}(t^{\beta-1}v)}{\rho(t)}\right)$ is continuous on $[0,t_1]\times R^2$. So $f\left(t,t^{\alpha-1}u,\frac{\Phi^{-1}(t^{\beta-1}v)}{\rho(t)}\right)$ is uniformly continuous on $[0,t_1]\times[-r,r]\times[-r,r]$. Similarly we can see that $f\left(t,t^{\alpha-1}u,\frac{\Phi^{-1}(t^{\beta-1}v)}{\rho(t)}\right)$ is uniformly continuous on $[t_1,1]\times[-r,r]\times[-r,r]$.

So for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| f\left(t, t^{\alpha-1}u_1, \frac{\Phi^{-1}(t^{\beta-1}v_1)}{\rho(t)}\right) - f\left(t, t^{\alpha-1}u_2, \frac{\Phi^{-1}(t^{\beta-1}v_2)}{\rho(t)}\right)\right| < \frac{\varepsilon}{\int_0^1 q(t)dt}$$

for all $t \in (0,1]$ and$|u_1 - u_2| < \delta$ and $|v_1 - v_2| < \delta$. Hence we get

$$\begin{aligned}
&\int_0^1 \left| q(t) f\left(t, x_n(t), \frac{\Phi^{-1}(y_n(t))}{\rho(t)}\right) - q(t) f\left(t, x_0(t), \frac{\Phi^{-1}(y_0(t))}{\rho(t)}\right)\right| dt \\
&= \int_0^1 q(t) \left| f\left(t, t^{\alpha-1}t^{1-\alpha}x_n(t), \frac{\Phi^{-1}\left(t^{\beta-1}t^{1-\beta}y_n(t)\right)}{\rho(t)}\right)\right. \\
&\left. - f\left(t, t^{\alpha-1}t^{1-\alpha}x_0(t), \frac{\Phi^{-1}\left(t^{\beta-1}t^{1-\beta}y_0(t)\right)}{\rho(t)}\right)\right| dt \\
&< \int_0^1 q(t) \frac{\varepsilon}{\int_0^1 q(t)dt} dt = \varepsilon, n > N.
\end{aligned}$$

Similarly we get for $n > N$ that

$$\left| \int_0^1 \phi(t) G\left(t, x_n(t), \frac{\Phi^{-1}(y_n(t))}{\rho(t)}\right) dt - \int_0^1 \phi(t) G\left(t, x_0(t), \frac{\Phi^{-1}(y_0(t))}{\rho(t)}\right) dt\right| < \varepsilon,$$

and

$$\left| \int_0^1 \psi(t) H\left(t, x_n(t), \frac{\Phi^{-1}(y_n(t))}{\rho(t)}\right) dt - \int_0^1 \psi(t) H\left(t, x_0(t), \frac{\Phi^{-1}(y_0(t))}{\rho(t)}\right) dt\right| < \varepsilon,$$

and

$$\left| I\left(t_1, x_n(t_1), \frac{\Phi^{-1}(y_n(t_1))}{\rho(t_1)}\right) - I\left(t_1, x_0(t_1), \frac{\Phi^{-1}(y_0(t_1))}{\rho(t_1)}\right)\right| < \varepsilon,$$

$$\left| J\left(t_1, x_n(t_1), \frac{\Phi^{-1}(y_n(t_1))}{\rho(t_1)}\right) - J\left(t_1, x_0(t_1), \frac{\Phi^{-1}(y_0(t_1))}{\rho(t_1)}\right)\right| < \varepsilon.$$

Then

$$||N(x_n, y_n) - N(x_0, y_0)|| \to 0, \, n \to \infty.$$

It follows that N is continuous.

Let $P : X \to X$ and $Q : Y \to Y$ be defined by (2.8) and (2.10). For $(u, v, a, b, c, d) \in \operatorname{Im} L = Z$, let

$$K_P(u, v, a, b, c, d)(t) = (x_1(t), y_1(t)) \tag{2.15}$$

where
(2.16)

$$x_1(t)=\begin{cases}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds\\ +\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds+\frac{c\Gamma(\alpha)-\int_0^{t_1}(t_1-s)^{\alpha-1}u(s)ds}{\Gamma(\alpha)t_1^{\alpha-1}}-a\right)t^{\alpha-1}, t\in(0,t_1],\\ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)ds+\frac{c\Gamma(\alpha)-\int_0^{t_1}(t_1-s)^{\alpha-1}u(s)ds}{\Gamma(\alpha)t_1^{\alpha-1}}t^{\alpha-1}, t\in(t_1,1],\end{cases}$$

$$y_1(t)=\begin{cases}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\\ +\left(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds+\frac{d\Gamma(\beta)-\int_0^{t_1}(t_1-s)^{\beta-1}v(s)ds}{\Gamma(\beta)t_1^{\beta-1}}-b\right)t^{\beta-1}, t\in(0,t_1],\\ \frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}v(s)ds+\frac{d\Gamma(\beta)-\int_0^{t_1}(t_1-s)^{\beta-1}v(s)ds}{\Gamma(\beta)t_1^{\beta-1}}t^{\beta-1}, t\in(t_1,1].\end{cases}$$

One sees $K_P(u,v,a,b,c,d)\in D(L)\bigcap E$ and K_P is the inverse of $L: D(L)\bigcap \operatorname{Ker} P\to \operatorname{Im} L$. The isomorphism $\wedge:\ \operatorname{Ker} L\to Y/\operatorname{Im} L$ is given by

$$\wedge(0,0)=(0,0,0,0,0,0).$$

Furthermore, one has

$$QN(x,y)(t)=Q\left(0,0,0,0,0,0\right), \tag{2.17}$$

and

$$K_p(I-Q)N(x,y)(t)=\left(x_2(t),y_2(t)\right),$$

where

$$x_2(t)=\begin{cases}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(s))}{\rho(s)}ds+\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(s))}{\rho(s)}ds\right.\\ +\frac{I\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\Gamma(\alpha)-\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{\Phi^{-1}(y(s))}{\rho(s)}ds}{\Gamma(\alpha)t_1^{\alpha-1}}\\ \left.-\int_0^1\phi(t)G\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)dt\right)t^{\alpha-1},\quad t\in(0,t_1],\\ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\frac{\Phi^{-1}(y(s))}{\rho(s)}ds\\ +\frac{I\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\Gamma(\alpha)-\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{\Phi^{-1}(y(s))}{\rho(s)}ds}{\Gamma(\alpha)t_1^{\alpha-1}}t^{\alpha-1}, t\in(t_1,1],\end{cases} \tag{2.18}$$

(2.19)

$$y_2(t)=\begin{cases}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds\\ +\left(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds\right.\\ +\frac{J\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\Gamma(\beta)-\int_0^{t_1}(t_1-s)^{\beta-1}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds}{\Gamma(\beta)t_1^{\beta-1}}\\ \left.-\int_0^1\psi(t)H\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)dt\right)t^{\beta-1}, \quad t\in(0,t_1],\\ \frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds+\frac{J\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)}{t_1^{\beta-1}}t^{\beta-1}\\ -\frac{\int_0^{t_1}(t_1-s)^{\beta-1}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds}{\Gamma(\beta)t_1^{\beta-1}}t^{\beta-1}, t\in(t_1,1].\end{cases}$$

Let Ω be a bounded open subset of E with $\overline{\Omega}\bigcap D(L)\neq\emptyset$. We have

$$||(x,y)||=\sup_{n=0,1,2,\ldots}\left\{\sup_{t\in(0,1]}t^{1-\alpha}|x(t)|,\ \sup_{t\in(0,1]}t^{1-\beta}|y(t)|\right\}=r<+\infty,(x,y)\in\Omega. \tag{2.20}$$

Since f,G,H are **impulsive Caratheodory functions**, I,J are continuous functions, together with (2.20), there exists $M>0$ such that

$$\begin{gathered}\left|f\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)\right|=\left|f\left(t,t^{\alpha-1}t^{1-\alpha}x(t),\frac{\Phi^{-1}\left(t^{\beta-1}t^{1-\beta}y(t)\right)}{\rho(t)}\right)\right|\leq M,\\ \left|G\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)\right|\leq M,\\ \left|H\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)\right|\leq M,\\ \left|I\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\right|\leq M,\\ \left|J\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\right|\leq M \text{ hold for all } t\in[0,1].\end{gathered} \tag{2.21}$$

Step (ii) Prove that $QN(\overline{\Omega})$ is bounded.

It is easy to see from (2.17) that $QN(\overline{\Omega})$ is bounded.

Step (iii) Prove that $K_P(I-Q)N:\overline{\Omega}\to E$ is compact, i.e., prove that $K_P(I-Q)N(\overline{\Omega})$ is relatively compact.

We must prove that $K_P(I-Q)N(\overline{\Omega})$ is uniformly bounded and equi-continuous both on each subinterval $[e,f]\subseteq(0,t_1]$ and $(t_1,1]$ respectively, and equi-convergent both at $t=0$ and $t=t_1$ respectively.

By (2.18) and (2.2), we have

$$
\begin{aligned}
& t^{1-\alpha} x_2(t) \\
& \leq t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Phi^{-1}(|y(s)|)}{\rho(s)} ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Phi^{-1}(|y(s)|)}{\rho(s)} ds \\
& \quad + \frac{\left| I\left(t_1, x(t_1), \frac{\Phi^{-1}(|y(t_1)|)}{\rho(t_1)}\right)\right| \Gamma(\alpha) + \int_0^{t_1} (t_1-s)^{\alpha-1} \frac{\Phi^{-1}(|y(s)|)}{\rho(s)} ds}{\Gamma(\alpha) t_1^{\alpha-1}} \\
& \quad + \int_0^1 \left| \phi(t) G\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right)\right| dt \\
& \leq L t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds + L \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds \\
& \quad + \frac{M\Gamma(\alpha) + L\int_0^{t_1} (t_1-s)^{\alpha-1} s^k ds}{\Gamma(\alpha) t_1^{\alpha-1}} + M \int_0^1 |\phi(t)| dt \\
& = L t^{k+1} \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^k dw + L \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds \\
& \quad + \frac{M\Gamma(\alpha) + L\int_0^{t_1} (t_1-s)^{\alpha-1} s^k ds}{\Gamma(\alpha) t_1^{\alpha-1}} + M \int_0^1 |\phi(t)| dt \\
& \leq L \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^k dw + L \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds \\
& \quad + \frac{M\Gamma(\alpha) + L\int_0^{t_1} (t_1-s)^{\alpha-1} s^k ds}{\Gamma(\alpha) t_1^{\alpha-1}} + M \int_0^1 |\phi(t)| dt < +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
& t^{1-\beta} y_2(t) \\
& \leq t^{1-\beta} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left| q(s) f\left(s, x(s), \frac{\Phi^{-1}(y(s))}{\rho(s)}\right)\right| ds \\
& \quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \left| q(s) f\left(s, x(s), \frac{\Phi^{-1}(y(s))}{\rho(s)}\right)\right| ds \\
& \quad + \frac{\left| J\left(t_1, x(t_1), \frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\right| \Gamma(\beta) + \int_0^{t_1} (t_1-s)^{\beta-1} \left| q(s) f\left(s, x(s), \frac{\Phi^{-1}(y(s))}{\rho(s)}\right)\right| ds}{\Gamma(\beta) t_1^{\beta-1}} \\
& \quad + \int_0^1 \left| \psi(t) H\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right)\right| dt \\
& \leq L_1 M t^{k_1+1} \int_0^1 \frac{(1-w)^{\beta-1}}{\Gamma(\beta)} w^{k_1} ds + L_1 M \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} s^{k_1} ds \\
& \quad + \frac{M\Gamma(\beta) + L_1 M \int_0^{t_1} (t_1-s)^{\beta-1} s^{k_1} ds}{\Gamma(\beta) t_1^{\beta-1}} + M \int_0^1 |\psi(t)| dt
\end{aligned}
$$

$$\leq L_1 M \int_0^1 \frac{(1-w)^{\beta-1}}{\Gamma(\beta)} w^{k_1} ds + L_1 M \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} s^{k_1} ds$$
$$+ \frac{M\Gamma(\beta) + L_1 M \int_0^{t_1} (t_1-s)^{\beta-1} s^{k_1} ds}{\Gamma(\beta) t_1^{\beta-1}} + M \int_0^1 |\psi(t)| dt < +\infty.$$

It is easy to see that $K_P(I-Q)N(\overline{\Omega})$ is uniformly bounded.

For each $[e,f] \subseteq (0,t_1]$, and $s_1, s_2 \in [e,f]$ with $s_2 \geq s_1$, use (2.18), we have

$$|s_1^{1-\alpha} x_2(s_1) - s_2^{1-\alpha} x_2(s_2)|$$
$$= \frac{1}{\Gamma(\alpha)} \left| s_1^{1-\alpha} \int_0^{s_1} (s_1-s)^{\alpha-1} \frac{\Phi^{-1}(y(s))}{\rho(s)} ds - s_2^{1-\alpha} \int_0^{s_2} (s_2-s)^{\alpha-1} \frac{\Phi^{-1}(y(s))}{\rho(s)} ds \right|$$
$$\leq \frac{1}{\Gamma(\alpha)} |s_1^{1-\alpha} - s_2^{1-\alpha}| \int_0^{s_1} (s_1-s)^{\alpha-1} \frac{\Phi^{-1}\left(s^{\beta-1} s^{1-\beta} |y(s)|\right)}{\rho(s)} ds$$
$$+ \frac{1}{\Gamma(\alpha)} s_2^{1-\alpha} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} \frac{\Phi^{-1}\left(s^{\beta-1} s^{1-\beta} |y(s)|\right)}{\rho(s)} ds$$
$$+ \frac{1}{\Gamma(\alpha)} s_2^{1-\alpha} \int_0^{s_2} |(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}| \frac{\Phi^{-1}\left(s^{\beta-1} s^{1-\beta} |y(s)|\right)}{\rho(s)} ds$$
$$\leq \frac{1}{\Gamma(\alpha)} |s_1^{1-\alpha} - s_2^{1-\alpha}| \int_0^{s_1} (s_1-s)^{\alpha-1} \frac{\nu(s^{\beta-1}) \Phi^{-1}(||y||)}{\rho(s)} ds$$
$$+ \frac{1}{\Gamma(\alpha)} s_2^{1-\alpha} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} \frac{\nu(s^{\beta-1}) \Phi^{-1}(||y||)}{\rho(s)} ds$$
$$+ \frac{1}{\Gamma(\alpha)} s_2^{1-\alpha} \int_0^{s_2} |(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}| \frac{\nu(s^{\beta-1}) \Phi^{-1}(||y||)}{\rho(s)} ds$$
$$\leq \frac{\Phi^{-1}(r)}{\Gamma(\alpha)} |s_1^{1-\alpha} - s_2^{1-\alpha}| \int_0^{s_1} (s_1-s)^{\alpha-1} \frac{\nu(s^{\beta-1})}{\rho(s)} ds$$
$$+ \frac{\Phi^{-1}(r)}{\Gamma(\alpha)} s_2^{1-\alpha} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} \frac{\nu(s^{\beta-1})}{\rho(s)} ds$$
$$+ \frac{\Phi^{-1}(r)}{\Gamma(\alpha)} s_2^{1-\alpha} \int_0^{s_2} |(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}| \frac{\nu(s^{\beta-1})}{\rho(s)} ds$$
$$\leq \frac{\Phi^{-1}(r)}{L\Gamma(\alpha)} |s_1^{1-\alpha} - s_2^{1-\alpha}| \int_0^{s_1} (s_1-s)^{\alpha-1} s^k ds$$
$$+ \frac{\Phi^{-1}(r)}{L\Gamma(\alpha)} s_2^{1-\alpha} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} s^k ds + \frac{\Phi^{-1}(r)}{L\Gamma(\alpha)} s_2^{1-\alpha} \int_0^{s_2} [(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}] s^k ds$$
$$\leq \frac{\Phi^{-1}(r)}{L\Gamma(\alpha)} \left[|s_1^{1-\alpha} - s_2^{1-\alpha}| s_1^{k+\alpha} \mathbf{B}(\alpha, k+1) + s_2^{k+1} \int_{\frac{k_1}{k_2}}^1 (1-w)^{\alpha-1} w^k dw \right.$$
$$\left. + \left(\frac{s_2}{s_1}\right)^{1-\alpha} s_1^{k+1} \int_0^{\frac{s_2}{s_1}} (1-w)^{\alpha-1} w^k dw - s_2^{k+1} \mathbf{B}(\alpha, k+1) \right] \to 0$$

uniformly as $s_1 \to s_2$. Similarly we get

$$|s_1^{1-\beta} y_2(s_1) - s_2^{1-\beta} y_2(s_2)| \to 0 \text{ uniformly as } s_1 \to s_2.$$

So $K_P(I-Q)N(\overline{\Omega})$ is equi-continuous on each subinterval $[e,f] \subseteq (0,t_1]$. Similarly we can show that $K_P(I-Q)N(\overline{\Omega})$ is equi-continuous on each subinterval $[e,f] \subseteq (t_1,1]$.

Since

$$\Bigg|t^{1-\alpha}x_2(t)-\Bigg(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(s))}{\rho(s)}ds + \frac{I\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\Gamma(\alpha)-\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{\Phi^{-1}(y(s))}{\rho(s)}ds}{\Gamma(\alpha)t_1^{\alpha-1}}$$

$$-\int_0^1 \phi(t)G\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)dt\Bigg)\Bigg| \le \frac{t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\frac{\Phi^{-1}\left(s^{\beta-1}s^{1-\beta}|y(s)|\right)}{\rho(s)}ds}{\Gamma(\alpha)}$$

$$\le \frac{t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\frac{\nu(s^{\beta-1})\Phi^{-1}(r)}{\rho(s)}ds}{\Gamma(\alpha)} \le L\Phi^{-1}(r)\frac{t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^k ds}{\Gamma(\alpha)}$$

$$= L\Phi^{-1}(r)t^{k+1}\mathbf{B}(\alpha,k+1) \to 0 \text{ uniformly as } t \to 0.$$

Similarly we can show that

$$\Bigg|t^{1-\beta}y_2(t)-\Bigg(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds$$

$$+\frac{J\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\Gamma(\beta)-\int_0^{t_1}(t_1-s)^{\beta-1}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds}{\Gamma(\beta)t_1^{\beta-1}}$$

$$-\int_0^1 \psi(t)H\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)dt\Bigg)\Bigg| \to 0$$

uniformly as $t \to 0$. Hence $K_P(I-Q)N(\overline{\Omega})$ is equi-convergent at $t=0$. Similarly we can show that $K_P(I-Q)N(\overline{\Omega})$ is equi-convergent at $t=t_1$.

So $K_P(I-Q)N(\overline{\Omega})$ is relatively compact. Then N is L-compact. The proofs are completed. ∎

3. Main results

Now, we prove the main theorem in this paper.

Theorem 3.1. *Suppose that*

(B) *there exist nonnegative numbers $A,B,C,a_i,b_i,c_i (i=1,2)$ and $A_i,B_i,C_i (i=1,2)$ such that*

$$\left|f\left(t,t^{\alpha-1}x,\frac{\Phi^{-1}(t^{\beta-1}y)}{\rho(t)}\right)\right| \le C+B\Phi(|x|)+A|y|,$$

$$\left|G\left(t,t^{\alpha-1}x,\frac{\Phi^{-1}(t^{\beta-1}y)}{\rho(t)}\right)\right| \le c_1+b_1|x|+a_1\Phi^{-1}(|y|),$$

$$\left|H\left(t,t^{\alpha-1}x,\frac{\Phi^{-1}(t^{\beta-1}y)}{\rho(t)}\right)\right| \le c_2+b_2\Phi(|x|)+a_2|y|,$$

$$\left|I\left(t_1,t_1^{\alpha-1}x,\frac{\Phi^{-1}(t_1^{\beta-1}y)}{\rho(t_1)}\right)\right| \le C_1+B_1|x|+A_1\Phi^{-1}(|y|),$$

$$\left|J\left(t_1,t_1^{\alpha-1}x,\frac{\Phi^{-1}(t_1^{\beta-1}y)}{\rho(t_1)}\right)\right| \le C_2+B_2\Phi(|x|)+A_2|y|.$$

Then BVP(1.3) has at least one solution if

$$b_1||\phi||_1 + B_1 t_1^{1-\alpha} < 1,$$

$$\text{(3.1)}\quad \left[L_1 \tfrac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} B + L_1 \mathbf{B}(\beta,k_1+1) B + t_1^{1-\beta} B_2 + \tfrac{L_1 \mathbf{B}(\beta,k_1+1)}{\Gamma(\beta) t_1^{\beta-1}} B + ||\psi||_1 b_2\right] \tfrac{1}{w((2N_1)^{-1})}$$

$$+L_1 \tfrac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} A + L_1 \mathbf{B}(\beta,k_1+1) A + t_1^{1-\beta} A_2 + \tfrac{L_1 \mathbf{B}(\beta,k_1+1)}{\Gamma(\beta) t_1^{\beta-1}} A + ||\psi||_1 a_2 < 1.$$

Proof. To apply Lemma 2.1, we should define an open bounded subset Ω of E centered at zero such that assumptions in Lemma 2.1 hold. To obtain Ω.

Let $\Omega_1 = \{(x,y) \in E \cap D(L) \setminus \operatorname{Ker} L,\ L(x,y) = \lambda N(x,y)$ for some $\lambda \in (0,1)\}$. We prove that Ω_1 is bounded.

For $(x,y) \in \Omega_1$, we get $L(x,y) = \lambda N(x,y)$ and $N(x,y) \in \operatorname{Im} L$. Then

$$\text{(3.2)}\quad \begin{cases} D_{0^+}^{\alpha} x(t) = \lambda \frac{\Phi^{-1}(y(t))}{\rho(t)}, \\ D_{0^+}^{\beta} y(t) = \lambda q(t) f\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right), \\ \lim_{t\to 1} t^{1-\alpha} x(t) - \lim_{t\to 0} t^{1-\alpha} x(t) = \lambda \int_0^1 \phi(t) G\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right) dt, \\ \lim_{t\to 1} t^{1-\beta} y(t) - \lim_{t\to 0} t^{1-\beta} y(t) = \lambda \int_0^1 \psi(t) H\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right) dt, \\ \lim_{t\to t_1^+} x(t) = \lambda I\left(t_1, x(t_1), \frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right), \\ \lim_{t\to t_1^+} y(t) = \lambda J\left(t_1, x(t_1), \frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right). \end{cases}$$

So

$$\text{(3.3)}\quad x(t) = \begin{cases} \lambda \Bigg[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Phi^{-1}(y(s))}{\rho(s)} ds + \Bigg(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\Phi^{-1}(y(s))}{\rho(s)} ds \\ \quad + \frac{I\left(t_1, x(t_1), \frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\Gamma(\alpha) - \int_0^{t_1} (t_1-s)^{\alpha-1} \frac{\Phi^{-1}(y(s))}{\rho(s)} ds}{\Gamma(\alpha) t_1^{\alpha-1}} \\ \quad - \int_0^1 \phi(t) G\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right) dt\Bigg) t^{\alpha-1}\Bigg], \quad t \in (0, t_1], \\ \Bigg[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\Phi^{-1}(y(s))}{\rho(s)} ds \\ \quad + \frac{I\left(t_1, x(t_1), \frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\Gamma(\alpha) - \int_0^{t_1} (t_1-s)^{\alpha-1} \frac{\Phi^{-1}(y(s))}{\rho(s)} ds}{\Gamma(\alpha) t_1^{\alpha-1}} t^{\alpha-1}\Bigg], t \in (t_1, 1], \end{cases}$$

$$
(3.4)\qquad y(t)=\begin{cases}\lambda\Big[\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds \\ +\left(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds\right. \\ +\frac{J\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\Gamma(\beta)-\int_0^{t_1}(t_1-s)^{\beta-1}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds}{\Gamma(\beta)t_1^{\beta-1}} \\ \left.-\int_0^1\psi(t)H\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)dt\right)t^{\beta-1}\Big], \quad t\in(0,t_1], \\ \lambda\Big[\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds \\ +\frac{J\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)}{t_1^{\beta-1}}t^{\beta-1} \\ -\frac{\int_0^{t_1}(t_1-s)^{\beta-1}q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)ds}{\Gamma(\beta)t_1^{\beta-1}}t^{\beta-1}\Big], t\in(t_1,1].\end{cases}
$$

From (3.3), we have

$$
\begin{aligned}
t^{1-\alpha}|x(t)| &\le t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(|y(s)|)}{\rho(s)}ds+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(|y(s)|)}{\rho(s)}ds \\
&+\frac{\left|I\left(t_1,x(t_1),\frac{\Phi^{-1}(|y(t_1)|)}{\rho(t_1)}\right)\right|\Gamma(\alpha)+\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{\Phi^{-1}(|y(s)|)}{\rho(s)}ds}{\Gamma(\alpha)t_1^{\alpha-1}} \\
&+\int_0^1\left|\phi(t)G\left(t,x(t),\frac{\Phi^{-1}(y(t))}{\rho(t)}\right)\right|dt \\
&\le t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(s^{\beta-1}||y||)}{\rho(s)}ds+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(s^{\beta-1}||y||)}{\rho(s)}ds \\
&+\frac{\left|I\left(t_1,t_1^{\alpha-1}t_1^{1-\alpha}x(t_1),\frac{\Phi^{-1}\left(t_1^{\beta-1}t_1^{1-\beta}y(t_1)\right)}{\rho(t_1)}\right)\right|\Gamma(\alpha)+\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{\Phi^{-1}(s^{\beta-1}||y||)}{\rho(s)}ds}{\Gamma(\alpha)t_1^{\alpha-1}} \\
&+\int_0^1\left|\phi(t)G\left(t,t^{\alpha-1}t^{1-\alpha}x(t),\frac{\Phi^{-1}\left(t^{\beta-1}t^{1-\beta}y(t)\right)}{\rho(t)}\right)\right|dt \\
&\le t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\nu(s^{\beta-1})\Phi^{-1}(||y||)}{\rho(s)}ds+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\nu(s^{\beta-1})\Phi^{-1}(||y||)}{\rho(s)}ds \\
&+\frac{\left[C_1+B_1|t_1^{1-\alpha}x(t_1)|+A_1\Phi^{-1}\left(t_1^{1-\beta}|y(t_1)|\right)\right]\Gamma(\alpha)+\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{\nu(s^{\beta-1})\Phi^{-1}(||y||)}{\rho(s)}ds}{\Gamma(\alpha)t_1^{\alpha-1}} \\
&+\int_0^1|\phi(t)|\left[c_1+b_1|t^{1-\alpha}x(t)|+a_1\Phi^{-1}\left(t^{1-\beta}|y(t)|\right)\right]dt
\end{aligned}
$$

$$\le Lt^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds \Phi^{-1}(||y||) + L\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds \Phi^{-1}(||y||)$$
$$+ \frac{\left[C_1 + B_1||x|| + A_1\Phi^{-1}(||y||)\right]\Gamma(\alpha) + L\int_0^{t_1}(t_1-s)^{\alpha-1}s^k ds \Phi^{-1}(||y||)}{\Gamma(\alpha)t_1^{\alpha-1}}$$
$$+ ||\phi||_1\left[c_1 + b_1||x|| + a_1\Phi^{-1}(||y||)\right]$$
$$\le L\int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^k dw \Phi^{-1}(||y||) + L\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds \Phi^{-1}(||y||)$$
$$+ \frac{\left[C_1 + B_1||x|| + A_1\Phi^{-1}(||y||)\right]\Gamma(\alpha) + L\int_0^{1}(1-w)^{\alpha-1}w^k dw \Phi^{-1}(||y||)}{\Gamma(\alpha)t_1^{\alpha-1}}$$
$$+ ||\phi||_1\left[c_1 + b_1||x|| + a_1\Phi^{-1}(||y||)\right].$$

Then

$$||x|| \le \frac{2L}{\Gamma(\alpha)}\mathbf{B}(\alpha, k+1)\Phi^{-1}(||y||)$$
$$+ \frac{\left[C_1 + B_1||x|| + A_1\Phi^{-1}(||y||)\right]\Gamma(\alpha) + L\mathbf{B}(\alpha, k+1)\Phi^{-1}(||y||)}{\Gamma(\alpha)t_1^{\alpha-1}}$$
$$+ ||\phi||_1\left[c_1 + b_1||x|| + a_1\Phi^{-1}(||y||)\right].$$

It follows that

$$||x|| \le \frac{C_1 t_1^{1-\alpha} + c_1||\phi||_1}{1 - b_1||\phi||_1 - B_1 t_1^{1-\alpha}} + \frac{a_1||\phi||_1 + A_1 t_1^{1-\alpha} + (2 + t_1^{1-\alpha})\frac{L}{\Gamma(\alpha)}\mathbf{B}(\alpha, k+1)}{1 - b_1||\phi||_1 - B_1 t_1^{1-\alpha}}\Phi^{-1}(||y||)$$

(3.5)

$$=: M_1 + N_1\Phi^{-1}(||y||).$$

From (3.4), we get

$$|t^{1-\beta}y(t)| = \left| t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} q(s) f\left(s, x(s), \frac{\Phi^{-1}(y(s))}{\rho(s)}\right) ds \right.$$
$$+ \left(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} q(s) f\left(s, x(s), \frac{\Phi^{-1}(y(s))}{\rho(s)}\right) ds \right.$$
$$+ \frac{J\left(t_1, x(t_1), \frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\Gamma(\beta) - \int_0^{t_1}(t_1-s)^{\beta-1}q(s) f\left(s, x(s), \frac{\Phi^{-1}(y(s))}{\rho(s)}\right) ds}{\Gamma(\beta)t_1^{\beta-1}}$$
$$\left.\left. - \int_0^1 \psi(t) H\left(t, x(t), \frac{\Phi^{-1}(y(t))}{\rho(t)}\right) dt\right)\right|$$
$$\le t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left| q(s) f\left(s, x(s), \frac{\Phi^{-1}(y(s))}{\rho(s)}\right)\right| ds$$
$$+ \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \left| q(s) f\left(s, x(s), \frac{\Phi^{-1}(y(s))}{\rho(s)}\right)\right| ds$$

$$+\frac{\left|J\left(t_1,x(t_1),\frac{\Phi^{-1}(y(t_1))}{\rho(t_1)}\right)\right|\Gamma(\beta)+\int_0^{t_1}(t_1-s)^{\beta-1}\left|q(s)f\left(s,x(s),\frac{\Phi^{-1}(y(s))}{\rho(s)}\right)\right|ds}{\Gamma(\beta)t_1^{\beta-1}}$$

$$+\int_0^1\left|\psi(t)H\left(t,x(t),\frac{\Phi^{-1}\left(y(t)\right)}{\rho(t)}\right)\right|dt$$

$$\leq L_1t^{1-\beta}\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}s^{k_1}\left[C+B\Phi(||x||)+A||y||\right]ds$$

$$+L_1\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}s^{k_1}\left[C+B\Phi(||x||)+A||y||\right]ds$$

$$+\frac{\left[C_2+B_2\Phi(||x||)+A_2||y||\right]\Gamma(\beta)+L_1\int_0^{t_1}(t_1-s)^{\beta-1}s^{k_1}\left[C+B\Phi(||x||)+A||y||\right]ds}{\Gamma(\beta)t_1^{\beta-1}}$$

$$+\int_0^1|\psi(t)|dt\left[c_2+b_2\Phi(||x||)+a_2||y||\right]$$

$$\leq L_1\frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)}\left[C+B\Phi(||x||)+A||y||\right]+L_1\mathbf{B}(\beta,k_1+1)\left[C+B\Phi(||x||)+A||y||\right]$$

$$+\frac{\left[C_2+B_2\Phi(||x||)+A_2||y||\right]\Gamma(\beta)+L_1\mathbf{B}(\beta,k_1+1)\left[C+B\Phi(||x||)+A||y||\right]}{\Gamma(\beta)t_1^{\beta-1}}$$

$$+||\psi||_1\left[c_2+b_2\Phi(||x||)+a_2||y||\right].$$

It follows that

$$||y||$$

$$\leq L_1\frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)}C+L_1\mathbf{B}(\beta,k_1+1)C+t_1^{1-\beta}C_2+\frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}}C+||\psi||_1c_2$$

$$+\left[L_1\frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)}B+L_1\mathbf{B}(\beta,k_1+1)B+t_1^{1-\beta}B_2+\frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}}B+||\psi||_1b_2\right]\Phi(||x||)$$

$$+\left[L_1\frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)}A+L_1\mathbf{B}(\beta,k_1+1)A+t_1^{1-\beta}A_2+\frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}}A+||\psi||_1a_2\right]||y||$$

$$\leq L_1\frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)}C+L_1\mathbf{B}(\beta,k_1+1)C+t_1^{1-\beta}C_2+\frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}}C+||\psi||_1c_2$$

$$+\left[L_1\frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)}B+L_1\mathbf{B}(\beta,k_1+1)B+t_1^{1-\beta}B_2+\frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}}B+||\psi||_1b_2\right]\times$$

$$\Phi(M_1+N_1\Phi^{-1}(||y||))$$

$$+\left[L_1\frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)}A+L_1\mathbf{B}(\beta,k_1+1)A+t_1^{1-\beta}A_2+\frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}}A+||\psi||_1a_2\right]||y||$$

Without loss of generality, suppose that $||y|| > \Phi(M_1/N_1)$, then we get from (2.1) that

$$||y|| \le L_1 \frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} C + L_1\mathbf{B}(\beta,k_1+1)C + t_1^{1-\beta} C_2 + \frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}} C + ||\psi||_1 c_2$$

$$+ \left[L_1 \frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} B + L_1\mathbf{B}(\beta,k_1+1)B + t_1^{1-\beta} B_2 + \frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}} B + ||\psi||_1 b_2 \right] \times$$

$$\Phi(2N_1\Phi^{-1}(||y||))$$

$$+ \left[L_1 \frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} A + L_1\mathbf{B}(\beta,k_1+1)A + t_1^{1-\beta} A_2 + \frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}} A + ||\psi||_1 a_2 \right] ||y||$$

$$\le L_1 \frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} C + L_1\mathbf{B}(\beta,k_1+1)C + t_1^{1-\beta} C_2 + \frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}} C + ||\psi||_1 c_2$$

$$+ \left[L_1 \frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} B + L_1\mathbf{B}(\beta,k_1+1)B + t_1^{1-\beta} B_2 + \frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}} B + ||\psi||_1 b_2 \right] \times$$

$$\frac{\Phi(\Phi^{-1}(||y||))}{w((2N_1)^{-1})}$$

$$+ \left[L_1 \frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} A + L_1\mathbf{B}(\beta,k_1+1)A + t_1^{1-\beta} A_2 + \frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}} A + ||\psi||_1 a_2 \right] ||y||$$

$$= L_1 \frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} C + L_1\mathbf{B}(\beta,k_1+1)C + t_1^{1-\beta} C_2 + \frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}} C + ||\psi||_1 c_2$$

$$+ \left\{ \left[L_1 \frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} B + L_1\mathbf{B}(\beta,k_1+1)B + t_1^{1-\beta} B_2 + \frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}} B + ||\psi||_1 b_2 \right] \times \right.$$

$$\frac{1}{w((2N_1)^{-1})}$$

$$\left. + L_1 \frac{\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)} A + L_1\mathbf{B}(\beta,k_1+1)A + t_1^{1-\beta} A_2 + \frac{L_1\mathbf{B}(\beta,k_1+1)}{\Gamma(\beta)t_1^{\beta-1}} A + ||\psi||_1 a_2 \right\} ||y||.$$

Form (3.1), there exists a constant $M_2 > \Phi(M_1/N_1)$ such that $||y|| \le M_2$. Hence (3.5) implies that $||x|| \le M_1 + N_1\Phi^{-1}(M_2)$. It follows that Ω_1 is bounded.

To apply Lemma 2.1, let Ω be a non-empty open bounded subset of X such that $\Omega \supset \overline{\Omega_1}$ centered at zero.

It is easy to see from Lemma 2.2 that L is a Fredholm operator of index zero with $\text{Ker}\, L = \{0 \in X\}$, $N: X \to Y$ L-compact. Suppose Ω is a nonempty open subset of X. One can see that

$$L(x,y) \ne \lambda N(x,y) \text{ for all } (x,y) \in E \cap \partial\Omega \text{ and } \lambda \in (0,1).$$

Thus, from Lemma 2.1,

$$L(x,y) = N(x,y)$$

has at least one solution $(x,y) \in E \cap \overline{\Omega}$. So x is a solution of BVP(1.3). The proof of Theorem 3.1 is complete. ∎

4. An example

Now, we present an example, which can not be covered by known results, to illustrate Theorem 2.1.

Example 4.1. Consider the boundary value problem for fractional differential equation

$$(4.1)\quad \begin{cases} D_{0^+}^{\frac{1}{2}}\left[t^{\frac{1}{12}}D_{0^+}^{\frac{2}{3}}u(t)\right]^3+t^{-\frac{1}{4}}f(t,u(t),D_{0^+}^{\alpha}u(t))=0, \quad t\in(0,1), t\neq\frac{1}{2},\\ \lim_{t\to 1}t^{\frac{1}{3}}u(t)-\lim_{t\to 0}t^{\frac{1}{3}}u(t)=b_1\int_0^1 s^{\frac{1}{3}}u(s)ds,\\ \lim_{t\to 1}t^{\frac{1}{2}}\left[t^{\frac{1}{12}}D_{0^+}^{\frac{2}{3}}u(t)\right]^3-\lim_{t\to 0}t^{\frac{1}{2}}\left[t^{\frac{1}{12}}D_{0^+}^{\frac{2}{3}}u(t)\right]^3=b_2\int_0^1 s[u(s)]^3ds,\\ \lim_{t\to\frac{1}{2}^+}u(t)=\frac{1}{\sqrt[3]{2}}B_1u(1/2),\\ \lim_{t\to\frac{1}{2}^+}\Phi(\rho(t)D_{0^+}^{\alpha}u(t))=\frac{1}{2}B_2[u(1/2)]^3. \end{cases}$$

where b_1, b_2, B_1, B_2 are nonnegative numbers and

$$f(t,x,y)=C+Btx^3+At^{\frac{1}{4}}y^3,$$

Then BVP(4.1) has at least one solution if $b_1+\frac{B_1}{\sqrt[3]{2}}<1$, and

$$(4.2)\quad \begin{aligned} &8(2+1/\sqrt[3]{3})^3\left(\frac{\mathbf{B}(2/3,3/4)}{\Gamma(2/3)(1-b_1-B_1/\sqrt[3]{2})}\right)^3\left[\frac{\mathbf{B}(1/2,3/4)}{\Gamma(1/2)}+\frac{\mathbf{B}(1/2,3/4)}{\Gamma(1/2)t_1^{\beta-1}}+\mathbf{B}(1/2,3/4)\right]B\\ &+8(2+1/\sqrt[3]{3})^3\left(\frac{\mathbf{B}(2/3,3/4)}{\Gamma(2/3)(1-b_1-B_1/\sqrt[3]{2})}\right)^3 t_1^{1-\beta}B_2\\ &+8(2+1/\sqrt[3]{3})^3\left(\frac{\mathbf{B}(2/3,3/4)}{\Gamma(2/3)(1-b_1-B_1/\sqrt[3]{2})}\right)^3 b_2\\ &+\left[\frac{\mathbf{B}(1/2,3/4)}{\Gamma(1/2)}+\mathbf{B}(1/2,3/4)+\frac{\mathbf{B}(1/2,3/4)}{\sqrt{2}\Gamma(1/2)}\right]A<1. \end{aligned}$$

Proof. Corresponding to BVP(1.3), we see that

$\alpha=\frac{2}{3}$, $\beta=\frac{1}{2}$,

$\rho(t)=t^{\frac{1}{12}}$,

$\Phi(x)=x^3$ with $\Phi^{-1}(x)=x^{\frac{1}{3}}$, the supporting function of Φ is $\omega(x)=x^3$

and the supporting function of Φ^{-1} is $\nu(x)=x^{\frac{1}{3}}$,

$q(t)=t^{-\frac{1}{4}}$, $\phi(t)=\psi(t)=1$,

$G(t,x,y)=H(t,x,y)=t^{\frac{1}{3}}x$, $I(t,x,y)=J(t,x,y)=t^{\frac{1}{3}}x$.

It is easy to see that

• $\rho(t) = t^{\frac{1}{12}}$ satisfies that $\rho|_{(0,t_1]} \in C^0(0,t_1]$ and $\rho|_{(t_1,1]} \in C^0(t_1,1)$ and $\rho(t) \geq \frac{t^{-k}\nu(t^{\beta-1})}{L}$ for all $t \in (0,1), t \neq t_1$ with $L = 1, k = -\frac{1}{4}$.

• $q(t) = t^{-\frac{1}{4}}$ satisfies that $q|_{(0,t_1]} \in C^0(0,t_1]$ and $q|_{(t_1,1]} \in C^0(t_1,1)$ and $|q(t)| \leq L_1 t^{k_1}$ for all $t \in (0,1)$ with $L_1 = 1, k_1 = -\frac{1}{4}$.

Furthermore, we have

$$f\left(t, t^{\alpha-1}x, \frac{\Phi^{-1}(t^{\beta-1}y)}{\rho(t)}\right) = C + Bx^3 + Ay,$$

$$G\left(t, t^{\alpha-1}x, \frac{\Phi^{-1}(t^{\beta-1}y)}{\rho(t)}\right) = b_1 x,$$

$$H\left(t, t^{\alpha-1}x, \frac{\Phi^{-1}(t^{\beta-1}y)}{\rho(t)}\right) = b_2 x^3,$$

$$I\left(t_1, t_1^{\alpha-1}x, \frac{\Phi^{-1}(t_1^{\beta-1}y)}{\rho(t_1)}\right) = B_1 x,$$

$$J\left(t_1, t_1^{\alpha-1}x, \frac{\Phi^{-1}(t_1^{\beta-1}y)}{\rho(t_1)}\right) = B_2 x^3.$$

It is easy to see that (B) holds. Then Theorem 3.1 implies that BVP(4.1) has at least one solution if $b_1 + \frac{B_1}{\sqrt[3]{2}} < 1$, and (4.2) holds. ∎

References

[1] A. Arara, M. Benchohra, N. Hamidi and J. J. Nieto, Fractional order differential equations on an unbounded domain, *Nonlinear Anal.* **72** (2010), no. 2, 580–586.

[2] M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems with non-linear fractional differential equations, *Appl. Anal.* **87** (2008), no. 7, 851–863.

[3] M. Belmekki, J. J. Nieto and R. Rodríguez-López, Existence of periodic solution for a nonlinear fractional differential equation, *Bound. Value Probl.* **2009**, Art. ID 324561, 18 pp.

[4] R. Dehghani and K. Ghanbari, Triple positive solutions for boundary value problem of a nonlinear fractional differential equation, *Bull. Iranian Math. Soc.* **33** (2007), no. 2, 1–14, 91.

[5] G. L. Karakostas, Positive solutions for the Φ-Laplacian when Φ is a sup-multiplicative-like function, *Electron. J. Differential Equations* **2004**, No. 68, 12 pp. (electronic).

[6] E. R. Kaufmann and E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.* **2008**, No. 3, 11 pp.

[7] A. A. Kilbas and J. J. Trujillo, Differential equations of fractional order: Methods, results and problems. I, *Appl. Anal.* **78** (2001), no. 1–2, 153–192.

[8] Y. Liu, Positive solutions for singular FDEs, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **73** (2011), no. 1, 89–100.

[9] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Regional Conference Series in Mathematics, 40, Amer. Math. Soc., Providence, RI, 1979.

[10] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, Wiley, New York, 1993.

[11] A. M. Nahušev, The Sturm-Liouville problem for a second order ordinary differential equation with fractional derivatives in the lower terms, *Dokl. Akad. Nauk SSSR* **234** (1977), no. 2, 308–311.

[12] J. J. Nieto, Maximum principles for fractional differential equations derived from Mittag-Leffler functions, *Appl. Math. Lett.* **23** (2010), no. 10, 1248–1251.

[13] J. J. Nieto, Comparison results for periodic boundary value problems of fractional differential equations, *Fractional Differential Equations*, **1** (2011), 99–104.

[14] S. Z. Rida, H. M. El-Sherbiny and A. A. M. Arafa, On the solution of the fractional nonlinear Schrödinger equation, *Phys. Lett. A* **372** (2008), no. 5, 553–558.
[15] X. Wang and C. Bai, Periodic boundary value problems for nonlinear impulsive fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 3, 15 pp.
[16] Z. Wei, W. Dong and J. Che, Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative, *Nonlinear Anal.* **73** (2010), no. 10, 3232–3238.
[17] Z. Wei and W. Dong, Periodic boundary value problems for Riemann-Liouville sequential fractional differential equations, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 87, 13 pp.
[18] S. Zhang, The existence of a positive solution for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **252** (2000), no. 2, 804–812.
[19] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electron. J. Differential Equations* **2006**, No. 36, 12 pp. (electronic).
[20] Y. Zhao, S. Sun, Z. Han and M. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, *Appl. Math. Comput.* **217** (2011), no. 16, 6950–6958.