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Regular OS-rpp Semigroups

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Abstract. OS-rpp semigroups are analogies of orthogroups in the range of rpp semigroups. The aim of this paper is to study OS-rpp semigroups whose band of idempotents is a regular band, named regular OS-rpp semigroups. Many characterizations of regular OS-rpp semigroups are obtained. In particular, Yamada's construction of regular OS-rpp semigroups is given.

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1. Introduction

Let S be a semigroup. S is called *right principally projective*, in short, *rpp*, if for any $a \in S$, aS^1, regarded as an S^1-system, is projective. Dually, we can define *left principally projective* (lpp) *semigroup*. In [1], Fountain pointed out that a semigroup S is rpp if and only if each $\mathscr{L}^*$-class of S contains at least one idempotent. Following Fountain [3], we call S an *abundant semigroup* if each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class of S contains at least one idempotent. It is not difficult to see that S is abundant if and only if S is both rpp and lpp. Regular semigroups are abundant semigroups. There are many authors having been investigating such semigroups, for example, you can see [23, 24, 27].

An element a of S is called *completely regular* if there exists an idempotent e of S such that $a\mathscr{H}e$. Moreover, if every element of S is completely regular, then we call S a *completely regular semigroup*. Completely regular semigroups are early known as unions of groups (see [22]). Completely regular semigroups form an important class of semigroups. There are many authors having been investigating such semigroups (for the studies on completely regular semigroups, see the monograph [22]).

As generalizations of completely regular semigroups in the range of rpp semigroups, Y. Q. Guo, K. P. Shum and P. Y. Zhu [18] introduced strongly rpp semigroups. So-called a *strongly rpp semigroup* is an rpp semigroup S in which for any $a \in S$, there exists a unique idempotent $a^\diamond$ such that $a\mathscr{L}^*a^\diamond$ and $a^\diamond a = a$. In [12], Shum and Guo proved that a semigroup is completely regular if and only if it is a regular strongly rpp semigroup.

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Cancellative monoids are strongly rpp semigroups. In 1976, Fountain [1] researched rpp semigroups with central idempotents, called usually *Clifford rpp (C-rpp) semigroups*. It is proved that a semigroup is C-rpp if and only if it is a semilattice of left cancellative monoids. In [18], Y. Q. Guo, Shum and Zhu investigated left C-rpp semigroups and obtained the construction of such semigroups. Guo and Wu [14], and Guo, Zhao and Shum [15] obtained some characterizations of left C-rpp semigroups in terms of C-rpp semigroups. Left C-a semigroups are defined as left C-rpp semigroups that are abundant. In [6], X. J. Guo, Y. Q. Guo and Shum studied left C-a semigroups and established the structure of this kind of semigroups. As a dual of left C-rpp semigroups, Guo [16] defined right C-rpp semigroups. Since then, Guo, C. C. Ren and Shum [11], and Shum and X. M. Ren [26] considered this kind of strongly rpp semigroups. Left GC-lpp semigroups are another generalizations of left C-rpp semigroups in the range of lpp semigroups (for details, see [7]). In [5], Guo researched strongly rpp semigroups satisfying permutation identities and determined the classification of this kind of semigroups. Guo, Shum and Y. Q. Guo [13], and Shum, Guo and X. M. Ren [25] researched perfect rpp semigroups. Perfect rpp semigroups are strongly rpp semigroups whose idempotents form a normal band. Indeed, strongly rpp semigroups satisfying permutation identities are perfect rpp semigroups. It is interesting that Guo, Y. Q. Guo and Shum [8] have proved that any strongly rpp semigroup is indeed a disjoint union of Rees matrix semigroups over a left cancellative monoid. Recently, Guo, Jun and Zhao [10] probed pseudo-C-rpp semigroups. Such a semigroup is a strongly rpp semigroup whose idempotents form a right quasi-normal band.

To study strongly rpp semigroups, Guo, Y. Q. Guo and K. P. Shum in [8] and [9] defined $\overline{\mathscr{R}}$ and $\overline{\mathscr{H}}$ on a strongly rpp semigroup. Let S be a strongly rpp semigroup. On S, define: for any $a,b \in S$,

$$a\overline{\mathscr{R}}b \text{ if and only if } a^{\diamond}\mathscr{R}b^{\diamond},$$

and $\overline{\mathscr{H}} = \mathscr{L}^* \cap \overline{\mathscr{R}}$. In general, $\overline{\mathscr{R}}$ is not a left congruence on S (for details, see [8,9]). We call S a *super rpp semigroup* if $\overline{\mathscr{R}}$ is a left congruence on S. Completely regular semigroups and superabundant semigroups (see [3]) are both super rpp semigroups. Indeed, left C-rpp semigroups, right C-rpp semigroups, perfect rpp semigroups and pseudo-C-rpp semigroups are all super rpp semigroups. In [8], it is proved that a super rpp semigroup is a semilattice of Rees matrix semigroups over a left cancellative monoid. Also, they established a construction of super rpp semigroups whose idempotents constitute a band, called *OS-rpp semigroups*. In addition, He, Y. Q. Guo and Shum [19] investigated OS-rpp semigroups.

A band B is called *regular* if it satisfies the identity: $axya = axaya$ (for bands, see [21]). The aim of this paper is to study OS-rpp semigroups whose band of idempotents is a regular band. For simplicity, we shall call an OS-rpp semigroup whose band of idempotents is regular a *regular OS-rpp semigroup*. Indeed, we have proved that

- Left C-rpp semigroups are super rpp semigroups whose idempotents form a left regular band (see [9, Theorem 4.3]).
- Right C-rpp semigroups are super rpp semigroups whose idempotents form a right regular band (see [9, Theorem 4.5]).
- Perfect rpp semigroups are super rpp semigroups whose idempotents form a normal band (see [13]).
- Pseudo-C-rpp semigroups are super rpp semigroups whose idempotents form a right quasi-normal band (see [10]).

It is well known that left regular bands, right regular bands and normal bands are all regular. So, left C-rpp semigroups, right C-rpp semigroups, perfect rpp semigroups and pseudo-C-rpp semigroups are regular OS-rpp semigroups.

In this paper we study regular OS-rpp semigroups. Any OS-rpp semigroups are semilattices of direct products of a left cancellative monoid and a rectangular band. Indeed, OS-rpp semigroups are just ortho-lc-monoids, in other words, ortho-lc-monoids are OS-rpp semigroups under another viewpoint (see [17]). Of course, regular OS-rpp semigroups are regular ortho-u-monoids under another viewpoint (see [4]). We establish a construction of regular OS-rpp semigroups in terms of left regular bands, C-rpp semigroups and right regular bands (Theorem 3.2). Gong, Guo and Shum [4] also obtained a construction of regular ortho-u-monoids (in fact, regular OS-rpp semigroups). Our construction refines theirs in the following aspects: the construction of Gong, Guo and Shum also depends on a left regular band L, a C-rpp semigroup M and a right regular band R as well as two structure mappings $\varphi : L \times M \to End(L)$ and $\psi : M \times R \to End(R)$; but in our construction, the structure mappings are $\varphi : M \to End(L)$ and $\psi : M \to End(R)$ and the compatible conditions are more natural. In Section 4, we obtain some characterizations of regular OS-rpp semigroups.

2. Preliminaries

Throughout this paper we will use the terminologies and notation of [2] and [20]. We recall some known results which are used in the sequel. We begin by giving some elementary facts about $\mathscr{L}^*$; dual for $\mathscr{R}^*$.

Lemma 2.1. *Let S be a semigroup and $a, b \in S$. Then the following conditions are equivalent:*

(1) $a\mathscr{L}^*b$.
(2) *For all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.*

The following is an easy consequence of Lemma 2.1, due to [1].

Lemma 2.2. *Let S be a semigroup and $a, e^2 = e \in S$. Then the following conditions are equivalent:*

(1) $a\mathscr{L}^*e$.
(2) *$ae = a$ and for all $x, y \in S^1$, $ax = ay$ implies that $ex = ey$.*

It is well known that $\mathscr{L}^*$ is a right congruence while $\mathscr{R}^*$ is a left congruence. In general, $\mathscr{L} \subseteq \mathscr{L}^*$ and $\mathscr{R} \subseteq \mathscr{R}^*$. But when a and b are regular elements of S, $a\mathscr{R}\ (\mathscr{L})\ b$ if and only if $a\mathscr{R}^*\ (\mathscr{L}^*)\ b$. In particular, when S is regular, in this case $\mathscr{L} = \mathscr{L}^*$ and $\mathscr{R} = \mathscr{R}^*$. For convenience, we use $E(S)$ to denote the set of idempotents of S; a^* to denote an idempotent $\mathscr{L}^*$-related to a and $a^\dagger$ to denote an idempotent $\mathscr{R}^*$-related to a.

In order to research rpp semigroups, Guo, Shum and Zhu [18] introduced the so-called (ℓ)-Green's relations:

$$a\mathscr{L}^{(\ell)}b \Leftrightarrow Kera_l = Kerb_l,\ i.e.,\ \mathscr{L}^{(\ell)} = \mathscr{L}^*;$$
$$a\mathscr{R}^{(\ell)}b \Leftrightarrow Ima_r = Imb_r,\ i.e.,\ \mathscr{R}^{(\ell)} = \mathscr{R};$$
$$\mathscr{D}^{(\ell)} = \mathscr{L}^{(\ell)} \vee \mathscr{R}^{(\ell)};$$
$$\mathscr{H}^{(\ell)} = \mathscr{L}^{(\ell)} \cap \mathscr{R}^{(\ell)};$$

$$a\mathscr{J}^{(\ell)}b \Leftrightarrow J^{(\ell)}(a) = J^{(\ell)}(b),$$

where $a_l(a_r)$ is the inner left(right) translation on S^1; $J^{(\ell)}(a)$ is the smallest ideal of S containing $a \in S$ and that is a union of some $\mathscr{L}^{(\ell)}$-classes. In fact, $\mathscr{D}^{(\ell)} = \mathscr{L}^{(\ell)} \circ \mathscr{R}^{(\ell)} = \mathscr{R}^{(\ell)} \circ \mathscr{L}^{(\ell)}$ (see [8]); moreover, in a strongly rpp semigroup, $\mathscr{D}^{(\ell)} = \overline{\mathscr{R}} \circ \mathscr{L}^{(\ell)}$ (see [9]).

The following lemma gives some properties of super rpp semigroups.

Lemma 2.3. [9] *The following statements are equivalent for a strongly rpp semigroup S:*

(1) *S is a super rpp semigroup.*
(2) *For every $a \in S$, $J^{(\ell)}(a) = Sa^{\diamond}S$.*
(3) $\mathscr{J}^{(\ell)}|_{E(S)\times E(S)} = \mathscr{J}|_{E(S)\times E(S)}$.
(4) $\mathscr{J}^{(\ell)} = \mathscr{D}^{(\ell)}$.
(5) $\mathscr{D}^{(\ell)}$ *is a semilattice congruence.*

Let I and Λ be nonempty sets, and M a monoid. Assume $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix whose entries are units of M. Form the set $T = I \times M \times \Lambda$. On T, define a multiplication by

$$(i,x,\lambda)(j,y,\mu) = (i,xp_{\lambda j}y,\mu).$$

With the above multiplication, T is a semigroup. We call the semigroup T the *Rees matrix semigroup over the monoid M*, in notation, $\mathscr{M}(M,I,\Lambda;P)$. Moreover, when M is a left cancellative monoid, $\mathscr{M}(M,I,\Lambda;P)$ is a $\mathscr{D}^{(\ell)}$-simple strongly rpp semigroup; and vice versa (see [8]). It is easy to check that $Reg(\mathscr{M}(M,I,\Lambda;P))$ (the set of regular elements of $\mathscr{M}(M,I,\Lambda;P)) = \mathscr{M}(Reg(M),I,\Lambda;P)$ is a completely simple semigroup. Note that $\mathscr{M}(M,I,\Lambda;P)$ is OS-rpp if and only if $Reg(\mathscr{M}(M,I,\Lambda;P))$ is an orthodox semigroup. Also, by [22, Theorem III.5.2], $Reg(\mathscr{M}(M,I,\Lambda;P))$ is orthodox if and only if it isomorphic to $Reg(M) \times R$, where R is a rectangular band isomorphic to $I \times \Lambda$. So, $\mathscr{M}(M,I,\Lambda;P)$ is an OS-rpp semigroup if and only if it is isomorphic to $M \times R$.

Recall from [13], a semigroup is called a *left (right) cancellative plank* if it is isomorphic to the direct product of a left (right) cancellative monoid and a rectangular band. By [9, Theorem 2.4], we have that a strongly rpp semigroup is super rpp if and only if it is a semilattice of Rees matrix semigroups over a left cancellative monoid. By the arguments in the last paragraph, this easily derives the following lemma.

Lemma 2.4. *An rpp semigroup is an OS-rpp semigroup if and only if it is a semilattice of left cancellative planks.*

A band is called *left (right) regular* if it satisfies the identity: $xy = xyx$ ($yz = zyz$). Obviously, left regular bands and right regular bands are regular bands. Moreover, we have

Lemma 2.5. [22] *The following statements are equivalent for a band B:*

(1) *B is regular.*
(2) $\mathscr{L}$ *and* $\mathscr{R}$ *are congruences on B.*
(3) *B is a spined product of a left regular band and a right regular band.*

Lemma 2.6. [8] *If S is a strongly rpp semigroup and x is a regular element of S, then $x^{\diamond}\mathscr{H}x$. Moreover, all regular elements of S are completely regular.*

Y. Q. Guo, Shum and Zhu [18] proved that a strongly rpp semigroup is a left C-rpp semigroup if and only if $\mathscr{L}^{(\ell)}$ is a semilattice congruence on S. In the literature, right C-rpp semigroup is the dual of a left C-rpp semigroup. In fact, the former is neither dual to the latter in sense of Green's $*$-relations nor in sense of Green's (ℓ)-relations. But it is pointed out, by Guo, Y. Q. Guo and Shum in [9], that a strongly rpp semigroup S is a right C-rpp semigroup if and only if $\overline{\mathscr{R}}$ is a semilattice congruence on S.

Lemma 2.7. [18] *Let S be an rpp semigroup. Then the following statements are equivalent:*

(1) *S is a left C-rpp semigroup.*
(2) *S is a strongly rpp semigroup in which $\mathscr{L}^{(\ell)}$ is a semilattice congruence on S.*
(3) *S is a semilattice of direct products of left zero bands I_α and left cancellative monoids M_α.*

Lemma 2.8. [9, 11] *Let S be an rpp semigroup.Then the following statements are equivalent:*

(1) *S is a right C-rpp semigroup.*
(2) *S is a strongly rpp semigroup in which $\overline{\mathscr{R}}$ is a semilattice congruence on S.*
(3) *S is a semilattice of direct products of left cancellative monoids M_α and right zero bands I_α.*

3. Yamada's structure

The aim of this section is to establish the structure of regular OS-rpp semigroups. We shall construct a regular OS-rpp semigroup by means of a left regular band, a C-rpp semigroup and a right regular band, and then prove that any regular OS-rpp semigroup is isomorphic to some semigroup constructed in this way.

Consider

Y	a semilattice
$L=(Y;L_\alpha)$	the semilattice decomposition of the left regular band L into left zero bands L_α with $\alpha \in Y$.
$R=(Y;R_\alpha)$	the semilattice decomposition of the right regular band R into right zero bands R_α with $\alpha \in Y$.
$M=(Y;M_\alpha)$	the semilattice decomposition of the C-rpp semigroup M into left cancellative monoids M_α with $\alpha \in Y$.

Denote by $End_\ell(L)$ the semigroup of endomorphisms (on the left) of L and by $End_r(R)$ the semigroup of endomorphisms (on the right) of R. Let

$$\varphi : M \to End_\ell(L); \quad m \mapsto \varphi_m$$

and

$$\psi : M \to End_r(R); \quad m \mapsto \psi_m.$$

If the following three conditions hold:

$(RO1)$: $\varphi_m L_\beta \subseteq L_{\alpha\beta}$ and $R_\beta \psi_m \subseteq R_{\alpha\beta}$ if $m \in M_\alpha$;
$(RO2)$: for all $u \in M_\alpha, x \in L_\alpha, y \in L_\beta$ and $z \in L_\gamma$, if $x(\varphi_u y) = x(\varphi_u z)$, then $x(\varphi_{1_\alpha} y) = x(\varphi_{1_\alpha} z)$, where 1_α is the identity of the monoid M_α;
$(RO3)$: $\lambda_a \varphi_m \varphi_n = \lambda_a \varphi_{mn}$ and $\psi_m \psi_n \rho_u = \psi_{mn} \rho_u$ if $m \in M_\alpha, n \in M_\beta, a \in L_{\alpha\beta}, u \in R_{\alpha\beta}$,

then we call $(M,L,R;\varphi,\psi)$ an *RO-system*, in notation, $RO(M,L,R;\varphi,\psi)$.

Given an $RO(M,L,R;\varphi,\psi)$. We form the set

$$RO(M,L,R;\varphi,\psi) = \{(a,m,u) : a \in L_\alpha, u \in R_\alpha, m \in M_\alpha \text{ for some } \alpha \in Y\}$$

and define a multiplication by

$$(a,m,u)\circ(b,n,v)=(a(\varphi_m b),mn,(u\psi_n)v).$$

Now let $m\in M_\alpha$ and $n\in M_\beta$. Then $a\in L_\alpha, u\in R_\alpha$ and $b\in L_\beta, v\in R_\beta$. By Condition ($RO1$), $\varphi_m b\in L_{\alpha\beta}$ and $a(\varphi_m b)\in L_{\alpha\beta}$; similarly, $(u\psi_n)v\in R_{\alpha\beta}$. Obviously, $mn\in M_{\alpha\beta}$. Thus

$$(a(\varphi_m b),mn,(u\psi_n)v)\in RO(M,L,R;\varphi,\psi).$$

Therefore $\circ$ is well defined. Moreover, we may prove

Lemma 3.1. *$(RO(M,L,R;\varphi,\psi),\circ)$ is a semigroup.*

Proof. Let $(a,m,u)\in L_\alpha\times M_\alpha\times R_\alpha$, $(b,n,v)\in L_\beta\times M_\beta\times R_\beta$, $(c,k,w)\in L_\gamma\times M_\gamma\times R_\gamma$, then

$$\begin{aligned}
[(a,m,u)\circ(b,n,v)]\circ(c,k,w) &= (a(\varphi_m b),mn,(u\psi_n)v)\circ(c,k,w)\\
&= (a(\varphi_m b)(\varphi_{mn}c),mnk,[((u\psi_n)v)\psi_k]w)\\
&= (a(\varphi_m b)(\varphi_m\varphi_n c),mnk,(u\psi_n\psi_k)(v\psi_k)w)\\
&= (a(\varphi_m b)(\varphi_m\varphi_n c),mnk,(u\psi_n\psi_k)(v\psi_k)w)\\
&= (a[\varphi_m(b(\varphi_n c))],mnk,(u\psi_{nk})(v\psi_k)w)\\
&= (a[\varphi_m(b(\varphi_n c))],mnk,(u\psi_{nk})(v\psi_k)w)\\
&= (a,m,u)\circ(b(\varphi_n c),nk,(v\psi_k)w)\\
&= (a,m,u)\circ[(b,n,v)\circ(c,k,w)]
\end{aligned}$$

therefore, we have $(RO(M,L,R;\varphi,\psi),\circ)$ is a semigroup. ∎

Lemma 3.2.

(1) $E(RO(M,L,R;\varphi,\psi))=\{(a,1_\alpha,u)\mid a\in L_\alpha,u\in R_\alpha,1_\alpha\in M_\alpha,\alpha\in Y\}$.
(2) *$(RO(M,L,R;\varphi,\psi),\circ)$ is an rpp semigroup.*

Proof. (1) Let $(a,m,u)\in L_\alpha\times M_\alpha\times R_\alpha$. If

$$(a,m,u)=(a,m,u)(a,m,u)=(a(\varphi_m a),m^2,(u\psi_m)u)=(a,m^2,u),$$

then we get $m^2=m$, so that $m=1_\alpha$ since M_α is a left cancellative monoid. Thus

$$E(S)=\{(a,1_\alpha,u)\mid a\in L_\alpha,u\in R_\alpha,1_\alpha\in M_\alpha,\alpha\in Y\}.$$

(2) For any $(b,n,v)\in L_\beta\times M_\beta\times R_\beta$, $(c,k,w)\in L_\gamma\times M_\gamma\times R_\gamma$, if $(a,m,u)(b,n,v)=(a,m,u)(c,k,w)$, then $(a(\varphi_m b),mn,(u\psi_n)v)=(a(\varphi_m c),mk,(u\psi_k)w)$, so we have $a(\varphi_m b)=a(\varphi_m c), mn=mk$ and $(u\psi_n)v=(u\psi_k)w$. For $mn=mk$, we have $mn\in M_{\alpha\beta}, mk\in M_{\alpha\gamma}$ and $\alpha\beta=\alpha\gamma$. On the other hand, since $M_{\alpha\beta}$ is a left cancellative monoid, $mn=mk$ implies that $1_\alpha n=1_\alpha k$. For $a(\varphi_m b)=a(\varphi_m c)$, by ($RO2$), we have $a(\varphi_{1_\alpha}b)=a(\varphi_{1_\alpha}c)$. Hence we obtain $(a(\varphi_{1_\alpha}b),1_\alpha n,(u\psi_n)v)=(a(\varphi_{1_\alpha}c),1_\alpha k,(u\psi_k)w)$, that is, $(a,1_\alpha,u)(b,n,v)=(a,1_\alpha,u)(c,k,w)$. In addition, $(a,m,u)(a,1_\alpha,u)=(a,m,u)$, therefore by Lemma 2.2, $(a,m,u)\mathscr{L}^*(a,1_\alpha,u)$, consequently, $(RO(M,L,R;\varphi,\psi),\circ)$ is an rpp semigroup. ∎

Lemma 3.3. *Let $S=(RO(M,L,R;\varphi,\psi),\circ)$ and $(a,m,u)\in L_\alpha\times M_\alpha\times R_\alpha$, $(b,n,v)\in L_\beta\times M_\beta\times R_\beta$. Then $(a,m,u)\mathscr{L}^*(b,n,v)$ if and only if $u=v$.*

Proof. ($\Rightarrow$) Suppose that $(a,m,u)\mathscr{L}^*(b,n,v)$. By the proof of Lemma 3.2, we know $(a,1_\alpha,u)$ $\mathscr{L}^*(b,1_\beta,v)$, so

$$(a,1_\alpha,u) = (a,1_\alpha,u)(b,1_\beta,v) = (a(\varphi_{1_\alpha}b),1_{\alpha\beta},(u\psi_{1_\beta})v)$$

and

$$(b,1_\beta,v) = (b,1_\beta,v)(a,1_\alpha,u) = (b(\varphi_{1_\beta}a),1_{\beta\alpha},(v\psi_{1_\alpha})u).$$

It follows that $1_\alpha = 1_{\alpha\beta} = 1_{\beta\alpha} = 1_\beta$ which implies that $\alpha=\beta$ and $u=(u\psi_{1_\beta})v$, thus $u=v$ since R_α is a right zero band.

($\Leftarrow$) Assume that $u=v$. Obviously, $\alpha=\beta$. Compute

$$(a,1_\alpha,u) = (a,1_\alpha,u)(b,1_\beta,v) \text{ and } (b,1_\beta,v) = (b,1_\beta,v)(a,1_\alpha,u).$$

Now, $(a,1_\alpha,u)\mathscr{L}^*(b,1_\beta,v)$. But, by the proof of Lemma 3.2, $(a,m,u)\mathscr{L}^*(a,1_\alpha,u)$ and $(b,1_\beta,v)\mathscr{L}^*(b,n,v)$, thus $(a,m,u)\mathscr{L}^*(b,n,v)$. ∎

Lemma 3.4. *$(RO(M,L,R;\varphi,\psi),\circ)$ is a strongly rpp semigroup and $(a,m,u)^\diamond = (a,1_\alpha,u)$ for any $(a,m,u)\in L_\alpha\times M_\alpha\times R_\alpha$ with $\alpha\in Y$.*

Proof. By Lemma 3.2, $(a,m,u)\mathscr{L}^*(a,1_\alpha,u)$ and $(a,1_\alpha,u)(a,m,u) = (a(\varphi_{1_\alpha}a),1_\alpha m,(u\psi_m)$ $u) = (a,m,u)$. If $(a',1_\beta,u')\in E(S)$, and $(a,m,u)\mathscr{L}^*(a',1_\beta,u')$ and

$$(a',1_\beta,u')(a,m,u) = (a,m,u),$$

then by Lemma 3.3, $\alpha=\beta$ and $u'=u$. Now,

$$(a,m,u) = (a',1_\beta,u')(a,m,u) = (a'(\varphi_{1_\alpha}a),1_\alpha m,(u\psi_m)u) = (a',m,u),$$

so $a=a'$. Thus $(a',1_\beta,u') = (a,1_\alpha,u)$. Consequently, $(RO(M,L,R;\varphi,\psi),\circ)$ is a strongly rpp semigroup. Obviously, $(a,m,u)^\diamond = (a,1_\alpha,u)$. ∎

Lemma 3.5. *Let $S=(RO(M,L,R;\varphi,\psi),\circ)$. For any $(a,m,u)\in L_\alpha\times M_\alpha\times R_\alpha$ and $(b,n,v)\in L_\beta\times M_\beta\times R_\beta$, we have*

1. *(1) $(a,m,u)\overline{\mathscr{R}}(b,n,v)$ if and only if $a=b$;*
2. *(2) $(a,m,u)\overline{\mathscr{H}}(b,n,v)$ if and only if $a=b$ and $u=v$.*

Proof. (1) Suppose $(a,m,u)\overline{\mathscr{R}}(b,n,v)$. Then by Lemma 3.4,

$$(a,1_\alpha,u) = (a,m,u)^\diamond\mathscr{R}(b,n,v)^\diamond = (b,1_\beta,v),$$

it follows that

$$(a,1_\alpha,u) = (b,1_\beta,v)(a,1_\alpha,u) = (b(\varphi_{1_\beta}a),1_{\beta\alpha},(v\psi_{1_\alpha})u)$$

and

$$(b,1_\beta,v) = (a,1_\alpha,u)(b,1_\beta,v) = (a(\varphi_{1_\alpha}b),1_{\alpha\beta},(u\psi_{1_\beta})v),$$

hence $1_\alpha = 1_{\beta\alpha} = 1_{\alpha\beta} = 1_\beta$, thereby $\alpha=\beta$ and $a=b(\varphi_{1_\beta}a)$, thus $a=b$ since L_α is a left zero band.

Conversely, if $a=b$, then

$$(a,1_\alpha,u) = (b,1_\beta,v)(a,1_\alpha,u) \text{ and } (b,1_\beta,v) = (a,1_\alpha,u)(b,1_\beta,v),$$

hence $(a,m,u)^\diamond = (a,1_\alpha,u)\mathscr{R}(b,1_\beta,v) = (b,n,v)^\diamond$, thus $(a,m,u)\overline{\mathscr{R}}(b,n,v)$.

(2) It is immediate from (1) and Lemma 3.3. ∎

Lemma 3.6. *$(RO(M,L,R;\varphi,\psi),\circ)$ is a super rpp semigroup.*

Proof. By Lemma 3.4, it remains to show that $\overline{\mathscr{R}}$ is a left congruence on S. To the end, let $(a,m,u) \in L_\alpha \times M_\alpha \times R_\alpha$, $(b,n,v) \in L_\beta \times M_\beta \times R_\beta$ and $(a,m,u)\overline{\mathscr{R}}(b,n,v)$, then by Lemma 3.5, $a = b$ and $\alpha = \beta$. For any $(c,k,w) \in L_\gamma \times M_\gamma \times R_\gamma$, compute

$$(c,k,w)(a,m,u) = (c(\varphi_k a), km, (w\psi_m)u)$$

and

$$(c,k,w)(b,n,v) = (c(\varphi_k b), kn, (w\psi_n)v) = (c(\varphi_k a), kn, (w\psi_n)v).$$

Now, by Lemma 3.5(1), $(c,k,w)(a,m,u)\overline{\mathscr{R}}(c,k,w)(b,n,v)$ and whence $\overline{\mathscr{R}}$ is a left congruence on S, as required. ∎

Theorem 3.1. *$(RO(M,L,R;\varphi,\psi),\circ)$ is a regular OS- rpp semigroup.*

Proof. It suffices to prove that $E(RO(M,L,R;\varphi,\psi))$ is a regular band. By Lemma 3.2(1), it is easy to check that $E(RO(M,L,R;\varphi,\psi))$ is a band. If $(a,1_\alpha,u)\mathscr{L}(b,1_\alpha,u)$, then $(a,1_\alpha,u)$, $(b,1_\alpha,u) \in L_\alpha \times M_\alpha \times R_\alpha$. For any $(c,1_\beta,w) \in L_\beta \times M_\beta \times R_\beta$, since

$$(c,1_\beta,w)(a,1_\alpha,u) = (c(\varphi_{1_\beta} a), 1_{\beta\alpha}, (w\psi_{1_\alpha})u),$$

and

$$(c,1_\beta,w)(b,1_\alpha,u) = (c(\varphi_{1_\beta} b), 1_{\beta\alpha}, (w\psi_{1_\alpha})u),$$

and by Lemma 3.3, we have $(c,1_\beta,w)(a,1_\alpha,u)\mathscr{L}(c,1_\beta,w)(b,1_\alpha,u)$, whence $\mathscr{L}$ is a left congruence on S. Note that the restriction of $\overline{\mathscr{R}}$ to idempotents is just $\mathscr{R}$ and by a similar argument as those on $\mathscr{L}$, we can prove $\mathscr{R}$ is a congruence on $E(RO(M,L,R;\varphi,\psi))$. Thus, by Lemma 2.5, $E(RO(M,L,R;\varphi,\psi))$ is a regular band. The proof is completed. ∎

Now we are devoted to prove the converse of Theorem 3.1. In the rest part of this section we assume S is always a regular OS-rpp semigroup. Then by Lemma 2.4, we may suppose that S is a semilattice Y of left cancellative planks $S_\alpha = I_\alpha \times M_\alpha \times \Lambda_\alpha$ with $\alpha \in Y$, left zero band I_α and right zero band Λ_α. Then $E(S) = \cup_{\alpha\in Y} E(S_\alpha)$, where $E(S_\alpha) = I_\alpha \times \{1_\alpha\} \times \Lambda_\alpha$ and 1_α is the identity of M_α.

Lemma 3.7. *Let $\alpha,\beta \in Y$, $\alpha \le \beta$. Then $E(S_\alpha)$ is a rectangular band and for any $a,b \in S_\alpha$, $e \in E(S_\beta)$, we have $ab = aeb$.*

Proof. Note that $E(S_\alpha) \cong I_\alpha \times \Lambda_\alpha$. It is obvious that $E(S_\alpha)$ is a rectangular band. For any $a,b \in S_\alpha$, $e \in E(S_\beta)$, then by hypothesis, $a^\diamond e \in E(S_\alpha)$, hence $ab = a(a^\diamond b^\diamond)b = a(a^\diamond(a^\diamond e)b^\diamond)b$ $= aeb$ since $E(S_\alpha)$ is a rectangular band. ∎

Now, we fix the elements $c_\alpha = (k_\alpha, 1_\alpha, \xi_\alpha)$ in every S_α. For any $\alpha,\beta \in Y$, $\alpha \ge \beta$, define a mapping $\phi_{\alpha,\beta}$ by the requirement

$$c_\beta(k_\alpha, m, \xi_\alpha)c_\beta = (k_\beta, m\phi_{\alpha,\beta}, \xi_\beta).$$

By the similar arguments as in the proof of [22, Lemma V.2.2, p.210–211], we can define a strong semilattice $M = [Y; M_\alpha; \phi_{\alpha,\beta}]$ such that for any $\alpha,\beta \in Y$, $(i,m,\lambda) \in S_\alpha$ and $(j,n,\mu) \in S_\beta$, we have

$$(i,m,\lambda)(j,n,\mu) = (k, m\phi_{\alpha,\beta} n\phi_{\alpha,\beta}, v). \tag{3.1}$$

Consider the bijection

$$\eta : (i,1_\alpha,\lambda) \mapsto (i,\lambda)$$

from the regular band $E(S) = \cup_{\alpha\in Y}(I_\alpha \times \{1_\alpha\} \times \Lambda_\alpha)$ into the set $E = \cup_{\alpha\in Y}(I_\alpha \times \Lambda_\alpha)$. We now translate the operation of $E(S)$ onto E by

$$(i,\lambda)(j,\mu) = (l,\kappa) \Leftrightarrow (i,1_\alpha,\lambda)(j,1_\beta,\mu) = (l,1_{\alpha\beta},\kappa) \tag{3.2}$$

such that η becomes a semigroup isomorphism and so E is a regular band. However, by Lemma 2.5, we can easily verify that $l[\kappa]$ in (3.2) depends only on i and j [λ and μ]. Now, by the same method as in the proof of [22, Lemma V.2.3, p.211-212], $I = \cup_{\alpha\in Y} I_\alpha$ becomes a left regular band under the operation

$$ij = l \Leftrightarrow (i,\lambda)(j,\mu) = (l,\kappa)$$

while $\Lambda = \cup_{\alpha\in Y}\Lambda_\alpha$ a right regular band under the operation

$$\lambda\mu = \kappa \Leftrightarrow (i,\lambda)(j,\mu) = (l,\kappa).$$

It is easy to see that E is the spined product of I and Λ with respect to Y.

Lemma 3.8. *For any $\alpha \in Y$ and $m \in M_\alpha$, define a mapping $\varphi_m : I \to I$ by the requirement*

$$(k_\alpha, m, \xi_\alpha)(b, 1_\beta, \xi_\beta) = (\varphi_m b, -, -) \quad (\beta \in Y, b \in I_\beta).$$

Then $\varphi_m \in End_l(I)$ such that

(1) *$\varphi_m I_\beta \subseteq I_{\alpha\beta}$ if $m \in M_\alpha$;*
(2) *for all $u \in M_\alpha, x \in I_\alpha, y \in I_\beta$ and $z \in I_\gamma$, if $x(\varphi_u y) = x(\varphi_u z)$, then $x(\varphi_{1_\alpha} y) = x(\varphi_{1_\alpha} z)$, where 1_α is the identity of the monoid M_α;*
(3) *$\lambda_a\varphi_m\varphi_n = \lambda_a\varphi_{mn}$ if $m \in M_\alpha, n \in M_\beta, a \in I_{\alpha\beta}$;*
(4) *$(a,m,u)(b,n,v) = (a(\varphi_m b), -, -)$ if $(a,m,u),(b,n,v) \in S$.*

Proof. (1), (3) and (4) can be obtained by similar arguments as in [22, Lemma V.2.4, p.212–214]. Here we omit the detail.

(2) Let $u \in M_\alpha, x \in I_\alpha, y \in I_\beta$ and $z \in I_\gamma$ such that $x(\varphi_u y) = x(\varphi_u z)$. It follows that $\alpha\gamma = \alpha\beta$ since $x(\varphi_u y) \in I_{\alpha\beta}, x(\varphi_u z) \in I_{\alpha\gamma}$. Note that $u1_\beta = (u1_\beta)1_{\alpha\beta} = u(1_\beta 1_{\alpha\beta}) = u1_{\alpha\beta}$ and similarly, $u1_\gamma = u1_{\alpha\gamma}$. We observe that $u1_\beta = u1_\gamma$. Compute

$$(x,u,\xi_\alpha)(y,1_\beta,\xi_\beta) = (x(\varphi_u y), u1_\beta, a) \quad \text{and}$$
$$(x,u,\xi_\alpha)(z,1_\gamma,\xi_\gamma) = (x(\varphi_u z), u1_\gamma, b) \quad (a,b \in \Lambda_{\alpha\beta}).$$

Thus

$$\begin{aligned}
&(x,u,\xi_\alpha)(y,1_\beta,\xi_\beta)(x(\varphi_u y),1_{\alpha\beta},ab)\\
&= (x(\varphi_u y),u1_\beta,a)(x(\varphi_u y),1_{\alpha\beta},ab)\\
&= ((x(\varphi_u y))^2,u1_\beta 1_{\alpha\beta},aab)\\
&= (x(\varphi_u y),u1_\beta,ab)\\
&= (x(\varphi_u z),u1_\gamma,ab)\\
&= ((x(\varphi_u z))^2,u1_\gamma 1_{\alpha\gamma},bab) \quad (\text{since } \Lambda_{\alpha\gamma} \text{ is a right zero band})\\
&= (x(\varphi_u z),u1_\gamma,b)(x(\varphi_u z),1_{\alpha\gamma},ab)\\
&= (x,u,\xi_\alpha)(z,1_\gamma,\xi_\gamma)(x(\varphi_u y),1_{\alpha\gamma},ab),
\end{aligned}$$

thereby

$$(x,1_\alpha,\xi_\alpha)(y,1_\beta,\xi_\beta)(x(\varphi_u y),1_{\alpha\beta},ab) = (x,1_\alpha,\xi_\alpha)(z,1_\gamma,\xi_\gamma)(x(\varphi_u y),1_{\alpha\gamma},ab)$$

since $(x,u,\xi_\alpha)\mathscr{L}^*(x,1_\alpha,\xi_\alpha)$, thus

$$\begin{aligned}(x(\varphi_{1_\alpha}y),1_{\alpha\beta},ab) &= (x(\varphi_{1_\alpha}y),1_{\alpha\beta},a)(x(\varphi_u y),1_{\alpha\beta},ab)\\ &= (x,1_\alpha,\xi_\alpha)(y,1_\beta,\xi_\beta)(x(\varphi_u y),1_{\alpha\beta},ab)\\ &= (x,1_\alpha,\xi_\alpha)(z,1_\gamma,\xi_\gamma)(x(\varphi_u y),1_{\alpha\gamma},ab)\\ &= (x(\varphi_{1_\alpha}z),1_{\alpha\gamma},ab),\end{aligned}$$

therefore by comparing the components, $x(\varphi_{1_\alpha}y) = x(\varphi_{1_\alpha}z)$. ∎

Dually, we have the following lemma:

Lemma 3.9. *For any $\alpha \in Y$ and $m \in M_\alpha$, define a mapping $\psi_m : \Lambda \to \Lambda$ by the requirement*

$$(k_\beta,1_\beta,v)(k_\alpha,m,\xi_\alpha) = (-,-,v\psi_m)(\beta \in Y, v \in \Lambda_\beta).$$

Then $\psi_m \in End_r(\Lambda)$ such that

(1) $\psi_m\Lambda_\beta \subseteq \Lambda_{\alpha\beta}$ *if* $m \in M_\alpha$*;*
(2) $\psi_m\psi_n\rho_u = \psi_{mn}\rho_u$ *if* $m \in M_\alpha, n \in M_\beta, u \in \Lambda_{\alpha\beta}$*;*
(3) $(b,n,v)(a,m,u) = (-,-,(v\psi_m)u)$ *if* $(b,n,v),(a,m,u) \in S$.

Define

$$\varphi:\ M \to End(I);\ m \mapsto \varphi_m$$

and

$$\psi:\ M \to End(\Lambda);\ m \mapsto \psi_m.$$

By Lemmas 3.8 and 3.9, $(M,I,\Lambda;\varphi,\psi)$ is an RO-system. By the same reason, it is not difficult to check that the identity mapping is an isomorphism of S onto $RO(M,I,\Lambda;\varphi,\psi)$.

We arrive at the main result of this section.

Theorem 3.2. (**Yamada structure theorem**) *If $(M,L,R;\varphi,\psi)$ is an RO-system, then the semigroup $(RO(M,L,R;\varphi,\psi),\circ)$ is a regular OS-rpp semigroup. Conversely, any regular OS-rpp semigroup can be constructed in this manner.*

Left C-rpp semigroups are regular OS-rpp semigroups whose band of idempotents is a left regular band. So, for the semigroup $(RO(M,L,R;\varphi,\psi),\circ)$, it is a left C-rpp semigroup if and only if R is a semilattice. It is not difficult to see that $R \cong Y$. In this case, if we identify R with Y, then $\alpha\psi_m = \alpha\beta$ for $m \in M_\beta$. So, we have

- any element of the semigroup $(RO(M,L,R;\varphi,\psi),\circ)$ is of the form: (x,m,α) with $x \in L_\alpha, m \in M_\alpha$;
- $(x,m,\alpha)(y,n,\beta) = (x(\varphi_m y),mn,\alpha\beta)$.

It is not difficult to see that for the element (x,m,α) the third component α can be determined by the second component m. Based on these arguments, we can obtain the structure of left C-rpp semigroups as follows, which follows form Theorem 3.2 and is essential [15, Theorem 3.1].

Corollary 3.1. *Let Y be a semilattice, $L = (Y;L_\alpha)$ the semilattice decomposition of the left regular band L into left zero bands L_α with $\alpha \in Y$ and $M = (Y;M_\alpha)$ the semilattice decomposition of the C-rpp semigroup M into left cancellative monoids M_α with $\alpha \in Y$. Let*

$$\varphi : M \to End_\ell(L);\ m \mapsto \varphi_m$$

satisfy the following conditions:

$(LC1)$: $\varphi_m L_\beta \subseteq L_{\alpha\beta}$ *if* $m \in M_\alpha$;
$(LC2)$: *for all* $u \in M_\alpha, x \in L_\alpha, y \in L_\beta$ *and* $z \in L_\gamma$, *if* $x(\varphi_u y) = x(\varphi_u z)$, *then* $x(\varphi_{1_\alpha} y) = x(\varphi_{1_\alpha} z)$, *where* 1_α *is the identity of the monoid* M_α;
$(LC3)$: $\lambda_a \varphi_m \varphi_n = \lambda_a \varphi_{mn}$ *if* $m \in M_\alpha, n \in M_\beta, a \in L_{\alpha\beta}$.
On $LC = \cup_{\alpha\in Y} L_\alpha \times M_\alpha$ *define a multiplication by*

$$(a,m) \circ (b,n) = (a(\varphi_m b), mn).$$

Then S is a left C-rpp semigroup. Conversely, any left C-rpp semigroup can be obtained in this way.

On the other hand, it is well known that right C-rpp semigroups are regular OS-rpp semigroups whose band of idempotents is a right regular band. So, for the semigroup $(RO(M,L,R;\varphi,\psi),\circ)$, it is a right C-rpp semigroup if and only if L is a semilattice. It is not difficult to see that $L \cong Y$. In this case, if we identify L with Y, then $\varphi_m(\alpha) = \alpha\beta$ for $m \in M_\beta$. So, we have

- any element of the semigroup $(RO(M,L,R;\varphi,\psi),\circ)$ is of the form: (α,m,u) with $u \in R_\alpha, m \in M_\alpha$;
- $(\alpha,m,u)(\beta,n,v) = (\alpha\beta, mn, (u\psi_n)v)$.

It is not difficult to see that for the element (α,m,u) the first component can be determined by the second component m. Based on these arguments and by Theorem 3.2, we can obtain the structure of right C-rpp semigroups as follows, which is essential [11, Theorem 4.7].

Corollary 3.2. *Let Y be a semilattice,* $R = (Y;R_\alpha)$ *the semilattice decomposition of the right regular band R into right zero bands* R_α *with* $\alpha \in Y$, *and* $M = (Y;M_\alpha)$ *the semilattice decomposition of the C-rpp semigroup M into left cancellative monoids* M_α *with* $\alpha \in Y$. *Let*

$$\psi : M \to End_r(R);\ m \mapsto \psi_m.$$

such that
$(RC1)$: $R_\beta \psi_m \subseteq R_{\alpha\beta}$ *if* $m \in M_\alpha$;
$(RC2)$: $\psi_m \psi_n \rho_u = \psi_{mn} \rho_u$ *if* $m \in M_\alpha, n \in M_\beta, u \in R_{\alpha\beta}$.
On $T = \cup_{\alpha\in Y} M_\alpha \times R_\alpha$ *define a multiplication by*

$$(m,u) \circ (n,v) = (mn, (u\psi_n)v).$$

Then T is a right C-rpp semigroup. Conversely, any right C-rpp semigroup can be obtained in this way.

4. Characterizations

In this section, we manage the properties of regular OS-rpp semigroups.

To begin with, we have

Theorem 4.1. *A semigroup is a regular OS-rpp semigroup if and only if it is isomorphic to the spined product [21] of a left C-rpp semigroup and a right C-rpp semigroup.*

Proof. With notation of Theorem 3.2, it is clear that the mapping

$$\chi : (a,m,u) \mapsto ((a,m),(m,u)) \qquad ((a,m,u) \in S)$$

is an isomorphism. The rest follows from Corollaries 3.1 and 3.2. ∎

Lemma 4.1. *Let S be a super rpp semigroup. Then S is a left C-rpp semigroup if and only if S satisfies the identity:* $ax = axa^\diamond$.

Proof. ($\Rightarrow$) Assume S is a left C-rpp semigroup. For any $a,x \in S$, we have $a\mathscr{L}^{(\ell)}a^\diamond$ and $x\mathscr{L}^{(\ell)}x^\diamond$. Since S is a left C-rpp semigroup, we obtain that $\mathscr{L}^{(\ell)}$ is a semilattice congruence on S, hence $ax\mathscr{L}^{(\ell)}a^\diamond x^\diamond \mathscr{L}^{(\ell)}x^\diamond a^\diamond$, thereby $ax = axx^\diamond a^\diamond = axa^\diamond$.

($\Leftarrow$) Suppose that S satisfies the identity: $ax = axa^\diamond$. Then for any $e, f \in E(S)$, we have $ef = efe$ by $e = e^\diamond$, whence $ef = (ef)^2$, thereby $E(S)$ is a left regular band. We have now proved that S is an OS-rpp semigroup whose band of idempotents is a left regular band, hence by [9, Theorem 4.3], S is a left C-rpp semigroup. ∎

Lemma 4.2. *Let S be a super rpp semigroup. Then S is a right C-rpp semigroup if and only if S satisfies the identity:* $ax = x^\diamond ax$.

Proof. ($\Rightarrow$) Assume S is a right C-rpp semigroup. Then by Lemma 2.8, $\overline{\mathscr{R}}$ is a semilattice congruence. Now, for any $a,x \in S$, since $a\overline{\mathscr{R}}a^\diamond$ and $x\overline{\mathscr{R}}x^\diamond$, we have $ax\overline{\mathscr{R}}a^\diamond x^\diamond \overline{\mathscr{R}}x^\diamond a^\diamond$, hence by definition of $\overline{\mathscr{R}}$, $(ax)^\diamond \mathscr{R} x^\diamond a^\diamond$. Thus

$$ax = (ax)^\diamond ax = x^\diamond a^\diamond (ax)^\diamond ax = x^\diamond a^\diamond ax = x^\diamond ax.$$

($\Leftarrow$) Now let S satisfy the identity: $ax = x^\diamond ax$. Then for any $e, f \in E(S)$, since $f = f^\diamond$, we have $ef = fef$, whence $ef = (ef)^2$, that is, $E(S)$ is a right regular band. Now, S is an OS-rpp semigroup whose band of idempotents is a right regular band. It follows from [9, Theorem 4.5] that S is a right C-rpp semigroup. ∎

Let ρ be an equivalence on S. It is easy to verify that the equivalence $\rho \cap (E(S) \times E(S))$ is an equivalence on $E(S)$. In [22], we call $\rho \cap (E(S) \times E(S))$ the *trace* of ρ and in notation, $tr\rho$ and for an endomorphism φ of S, if φ fixes every element of $S\varphi$, then we say φ is a *retraction* and $S\varphi$ is a *retract* of S.

Theorem 4.2. *The following conditions on an OS-rpp semigroup S are equivalent:*

(1) *S is a regular OS-rpp semigroup.*
(2) $tr\mathscr{L}^{(\ell)}$ *and* $tr\overline{\mathscr{R}}$ *are congruences on* $E(S)$.
(3) *S satisfies the identity* $abca = aba^\diamond ca$.
(4) *For every* $e \in E(S)$, *the mapping* ψ_e *defined by*

$$\psi_e : a \to eae \qquad (a \in S)$$

is an endomorphism of S.
(5) *For every* $e \in E(S)$, *the semigroup eSe is a retract of S.*

Proof. (1) $\Rightarrow$ (2) If S is a regular OS-rpp semigroup, then $E(S)$ is a regular band. By Lemma 2.5, $\mathscr{L}^{E(S)}$ and $\mathscr{R}^{E(S)}$ are congruences on $E(S)$. Note that $tr\mathscr{L}^{(\ell)} = \mathscr{L}^{E(S)}$ and $tr\overline{\mathscr{R}} = \mathscr{R}^{E(S)}$, whence the assertion.

(2) $\Rightarrow$ (1) If (2) holds, then by $tr\mathscr{L}^{(\ell)} = \mathscr{L}^{E(S)}$ and $tr\overline{\mathscr{R}} = \mathscr{R}^{E(S)}$, $\mathscr{L}$ and $\mathscr{R}$ are congruence on the band $E(S)$, hence by Lemma 2.5, $E(S)$ is a regular band. Therefore, S is a regular OS-rpp semigroup.

(1) $\Rightarrow$ (3) By Theorem 4.1, we assume S is a spined product of a left C-rpp semigroup S_1 and a right C-rpp semigroup S_2. We first prove that for any $(x,m) \in S$ with $x \in S_1$ and $m \in S_2$, $(x,m)^\diamond = (x^\diamond, m^\diamond)$. To see the end, let $(y,n),(z,k) \in S$ and $(x,m)(y,n) = (x,m)(z,k)$, then $(xy, mn) = (xz, mk)$, hence by comparing the components, $xy = xz$ and $mn = mk$, thereby $x^\diamond y = x^\diamond z$ and $m^\diamond n = m^\diamond k$ since $x\mathscr{L}^{(\ell)}x^\diamond$ and $m\mathscr{L}^{(\ell)}m^\diamond$, thus $(x^\diamond, m^\diamond)(y,n) = (x^\diamond, m^\diamond)(z,k)$,

now associating with $(x,m)(x^\diamond,m^\diamond)=(x,m)$ and by Lemma 2.2, $(x^\diamond,m^\diamond)\mathscr{L}^{(\ell)}(x,m)$; on the other hand, since $(x^\diamond,m^\diamond)(x,m)=(x,m)$, consequently, $(x,m)^\diamond=(x^\diamond,m^\diamond)$ since S is strongly rpp.

If $a=(x,m),b=(y,n),c=(z,k)\in S$ with $x,y,z\in S_1$ and $m,n,k\in S_2$, then

$$\begin{aligned} abca &= (x,m)(y,n)(z,k)(x,m)=(xyzx,mnkm)\\ &= (xyx^\diamond zx,mnm^\diamond km)=(x,m)(y,n)(x^\diamond,m^\diamond)(z,k)(x,m) \quad \text{(by Lemmas 4.1 and 4.2)}\\ &= (x,m)(y,n)(x,m)^\diamond(z,k)(x,m)\\ &= aba^\diamond ca. \end{aligned}$$

(3) $\Rightarrow$ (4) For any $a,b\in S$, since $e=e^\diamond$ and by (3), we have

$$(ab)\psi_e=eabe=eaebe=eaeebe=(a\psi_e)(b\psi_e),$$

hence ψ_e is an endomorphism of S.

(4) $\Rightarrow$ (5) Clearly, for every $e\in E(S)$, ψ_e is a retraction of S onto eSe. Therefore, the semigroup eSe is a retract of S.

(5) $\Rightarrow$ (1) Let $e\in E(S)$ and ψ_e a retraction of S onto eSe. For any $e,g\in E(S)$, we obtain

$$\begin{aligned} efge &= (efge)\psi_e\\ &= (e\psi_e)(f\psi_e)(g\psi_e)(e\psi_e)\\ &= (e\psi_e)(efe)(eee)(g\psi_e)(e\psi_e)\\ &= (e\psi_e)(f\psi_e)(e\psi_e)(g\psi_e)(e\psi_e)\\ &= (efege)\psi_e\\ &= efege. \end{aligned}$$

Therefore S is a regular OS-rpp semigroup. ∎

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