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A Fixed Point Approach to Stability of Functional Equations in Modular Spaces

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Abstract. In this paper, we present a fixed point method to prove generalized Hyers–Ulam stability of the generalized Jensen functional equation $f(rx+sy) = rg(x)+sh(x)$ in modular spaces.

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1. Introduction

The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. Recall that the problem of stability of functional equations was motivated by a question of Ulam being asked in 1940 [24] and Hyers answer to it was published in [4]. Hyers's theorem was generalized by Aoki [1] for additive mappings and by Rassias [21] for linear mappings by considering an unbounded Cauchy difference. During the past decades, a number of results concerning the stability of various functional equations have been obtained [3, 5, 7, 8].

The result on the stability of the classical Jensen functional equation was first given by Kominek [11]. The author who presumably investigated the stability problem on a restricted domain for the first time was Skof [22]. The stability of the Jensen equation and its generalizations were studied by a number of mathematicians (cf., e.g., [2, 6, 9, 16, 18]). In this paper, by using some ideas of [9], we investigate the generalized Hyers–Ulame stability of a generalized Jensen functional equation for mappings from linear spaces into modular spaces. The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [19] and were intensively developed by Amemiya, Koshi, Shimogaki, Yamamuro [12, 25] and others. Further and the most complete development of these theories are due to Orlicz, Mazur, Musielak, Luxemburg, Turpin [14, 17, 23] and their collaborators. In the present time the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [20] and interpolation theory [13, 15], which in their turn have broad applications [17]. The importance for

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applications consists in the richness of the structure of modular function spaces, that besides being Banach spaces (or F-spaces in more general setting) are equipped with modular equivalent of norm or metric notions.

Definition 1.1. *Let $\mathscr{X}$ be an arbitrary vector space.*

(a) *A functional $\rho : \mathscr{X} \to [0,\infty]$ is called a modular if for arbitrary $x, y \in \mathscr{X}$,*
  (i) *$\rho(x) = 0$ if and only if $x = 0$,*
  (ii) *$\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,*
  (iii) *$\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,*

(b) *if (iii) is replaced by*
  (iii)$'$ *$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,*

then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, i.e., the vector space $\mathscr{X}_\rho$ given by

$$\mathscr{X}_\rho = \{x \in \mathscr{X} : \rho(\lambda x) \to 0 \ \text{ as } \ \lambda \to 0\}.$$

Let ρ be a convex modular, the modular space $\mathscr{X}_\rho$ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 \,;\, \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

A function modular is said to satisfy the Δ_2-condition if there exists $\kappa > 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in \mathscr{X}_\rho$.

Definition 1.2. *Let $\{x_n\}$ and x be in $\mathscr{X}_\rho$. Then*

(i) *the sequence $\{x_n\}$, with $x_n \in \mathscr{X}_\rho$, is ρ-convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.*
(ii) *The sequence $\{x_n\}$, with $x_n \in \mathscr{X}_\rho$, is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.*
(iii) *A subset $\mathscr{S}$ of $\mathscr{X}_\rho$ is called ρ-complete complete if and only if any ρ-Cauchy sequence is ρ-convergent to an element of $\mathscr{S}$.*

The modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x.

Remark 1.1. Note that ρ is an increasing function. Suppose $0 < a < b$, then property (iii) of Definition 1.1 with $y = 0$ shows that $\rho(ax) = \rho((a/b)bx) \leq \rho(bx)$ for all $x \in \mathscr{X}$. Moreover, if ρ is a convex modular on $\mathscr{X}$ and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$ and also $\rho(x) \leq (1/2)\rho(2x)$ for all $x \in \mathscr{X}$.

A convex function φ defined on the interval $[0,\infty)$, nondecreasing and continuous for $\alpha \geq 0$ and such that $\varphi(0) = 0, \varphi(\alpha) > 0$ for $\alpha > 0$, $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$, is called an Orlicz function. The Orlicz function φ satisfies the Δ_2-condition if there exists $\kappa > 0$ such that $\varphi(2\alpha) \leq \varphi(\alpha)$ for all $\alpha > 0$. Let (Ω, Σ, μ) be a measure space. Let us consider the space $L^0(\mu)$ consisting of all measurable real-valued (or complex-valued) functions on Ω. Define for every $f \in L^0(\mu)$ the Orlicz modular $\rho_\varphi(f)$ by the formula

$$\rho_\varphi(f) = \int_\Omega \varphi(|f|)d\mu.$$

The associated modular function space with respect to this modular is called an Orlicz space, and will be denoted by $L^{\varphi}(\Omega,\mu)$ or briefly L^{φ} . In other words,

$$L^{\varphi} = \{f \in L^0(\mu) \mid \rho_{\varphi}(\lambda f) \to 0 \ \text{as} \ \lambda \to 0\}$$

or equivalently as

$$L^{\varphi} = \{f \in L^0(\mu) \mid \rho_{\varphi}(\lambda f) < \infty \ \text{for some} \ \lambda > 0\}.$$

It is known that the Orlicz space L^{φ} is ρ_{φ}-complete. Moreover, $(L^{\varphi}, \|.\|_{\rho_{\varphi}})$ is a Banach space, where the Luxemburg norm $\|.\|_{\rho_{\varphi}}$ is defined as follows

$$\|f\|_{\rho_{\varphi}} = \inf\left\{\lambda > 0 \ : \ \int_{\Omega} \varphi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\right\}.$$

Moreover, if $\mathfrak{L}$ is the space of sequences $x = \{x_i\}_{i=1}^{\infty}$ with real or complex terms x_i, $\varphi = \{\varphi_i\}_{i=1}^{\infty}$, φ_i are Orlicz functions and $\rho_{\varphi}(x) = \Sigma_{i=1}^{\infty}\varphi_i(|x_i|)$, we shall write ℓ^{φ} in place of L^{φ}. The space ℓ^{φ} is called the generalized Orlicz sequence space. The motivation for the study of modular spaces (and Orlicz spaces) and many examples are detailed in [19, 17, 20, 15].

2. Stability of a generalized Jensen functional equation

Throughout this paper, we assume that ρ is a convex modular on $\mathscr{X}$ with the Fatou property such that satisfies the Δ_2-condition with $0 < \kappa \leq 2$. In addition, we assume that r,s constant positive integer numbers. In this section, we use some ideas from [9] and we establish the conditional stability of a generalized Jensen functional equation.

Theorem 2.1. *Let $\mathscr{E}$ be a real or complex linear space and let $\mathscr{X}_\rho$ be a ρ-complete modular space. Suppose $f : \mathscr{E} \to \mathscr{X}_\rho$ satisfies the condition $f(0) = 0$ and an inequality of the form*

$$\rho\left(f(x+y) - f(x) - f(y)\right) \leq \phi(x,y) \tag{2.1}$$

for all $x,y \in \mathscr{E}$, where $\phi : \mathscr{E} \times \mathscr{E} \to [0,\infty)$ is a given function such that

$$\phi(2x,2x) \leq 2L\phi(x,x)$$

for all $x \in \mathscr{E}$ and has the property

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{2^n} = 0 \tag{2.2}$$

for all $x,y \in \mathscr{E}$ and a constant $0 < L < 1$. Then there exists a unique additive mapping $j : \mathscr{E} \to \mathscr{X}_\rho$ such that

$$\rho(j(x) - f(x)) \leq \frac{1}{2(1-L)}\phi(x,x) \tag{2.3}$$

for all $x \in \mathscr{E}$.

Proof. We consider the set

$$\mathscr{M} = \{g : \mathscr{E} \to \mathscr{X}_\rho,\ g(0) = 0\}$$

and introduce the convex modular $\widetilde{\rho}$ on $\mathscr{M}$ as follows,

$$\widetilde{\rho}(g) = \inf\{c > 0 : \rho(g(x)) \leq c\phi(x,x)\}.$$

It is sufficient to show that $\widetilde{\rho}$ satisfies the following condition

$$\widetilde{\rho}(\alpha g + \beta h) \leq \alpha\widetilde{\rho}(g) + \beta\widetilde{\rho}(h)$$

if $\alpha+\beta=1$ and $\alpha,\beta\geq 0$. Let $\varepsilon>0$ be given. Then there exist $c_1>0$ and $c_2>0$ such that

$$c_1\leq\widetilde{\rho}(g)+\varepsilon;\ \rho(g(x))\leq c_1\phi(x,x)$$

and

$$c_2\leq\widetilde{\rho}(h)+\varepsilon;\ \rho(h(x))\leq c_2\phi(x,x).$$

If $\alpha+\beta=1$ and $\alpha,\beta\geq 0$, then we get

$$\rho(\alpha g(x)+\beta h(x))\leq\alpha\rho(g(x))+\beta\rho(h(x))\leq(\alpha c_1+\beta c_2)\phi(x,x),$$

whence

$$\widetilde{\rho}(\alpha g+\beta h)\leq\alpha\widetilde{\rho}(g)+\beta\widetilde{\rho}(h)+(\alpha+\beta)\varepsilon.$$

Hence, we have

$$\widetilde{\rho}(\alpha g+\beta h)\leq\alpha\widetilde{\rho}(g)+\beta\widetilde{\rho}(h).$$

Moreover, $\widetilde{\rho}$ satisfies the Δ_2-condition with $0<\kappa\leq 2$.

Let $\{g_n\}$ be a $\widetilde{\rho}$-Cauchy sequence in $\mathscr{M}_{\widetilde{\rho}}$ and let $\varepsilon>0$ be given. There exists a positive integer $n_0\in\mathbb{N}$ such that $\widetilde{\rho}(g_n-g_m)\leq\varepsilon$ for all $n,m\geq n_0$. Now by considering the definition of the modular $\widetilde{\rho}$, we see that

$$\rho(g_n(x)-g_m(x))\leq\varepsilon\phi(x,x) \tag{2.4}$$

for all $x\in\mathscr{E}$ and $n,m\geq n_0$. If x is any given point of $\mathscr{E}$, (2.4) implies that $\{g_n(x)\}$ is a ρ-Cauchy sequence in $\mathscr{X}_\rho$. Since $\mathscr{X}_\rho$ is ρ-complete, so $\{g_n(x)\}$ is ρ-convergent in $\mathscr{X}_\rho$, for each $x\in\mathscr{E}$. Hence, we can define a function $g:\mathscr{E}\to\mathscr{X}_\rho$ by

$$g(x)=\lim_{n\to\infty}g_n(x)$$

for any $x\in\mathscr{E}$. Let m increase to infinity, then (2.4) implies that

$$\widetilde{\rho}(g_n-g)\leq\varepsilon$$

for all $n\geq n_0$, since ρ has the Fatou property. Thus $\{g_n\}$ is $\widetilde{\rho}$-convergent sequence in $\mathscr{M}_{\widetilde{\rho}}$. Therefore $\mathscr{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete.

Now, we consider the function $\mathscr{T}:\mathscr{M}_{\widetilde{\rho}}\to\mathscr{M}_{\widetilde{\rho}}$ defined by

$$\mathscr{T}g(x):=\frac{1}{2}g(2x)$$

for all $g,h\in\mathscr{M}_{\widetilde{\rho}}$. Let $g,h\in\mathscr{M}_{\widetilde{\rho}}$ and let $c\in[0,\infty]$ be an arbitrary constant with $\widetilde{\rho}(g-h)\leq c$. From the definition of $\widetilde{\rho}$, we have

$$\rho(g(x)-h(x))\leq c\phi(x,x)$$

for all $x\in\mathscr{E}$. By the assumption and the last inequality, we get

$$\rho\left(\frac{g(2x)}{2}-\frac{h(2x)}{2}\right)\leq\frac{1}{2}\rho(g(2x)-h(2x))\leq\frac{1}{2}c\phi(2x,2x)\leq Lc\phi(x,x)$$

for all $x\in\mathscr{E}$. Hence, $\widetilde{\rho}(\mathscr{T}g-\mathscr{T}h)\leq L\widetilde{\rho}(g-h)$, for all $g,h\in\mathscr{M}_{\widetilde{\rho}}$ that is, $\mathscr{T}$ is a $\widetilde{\rho}$-strict contraction. We show that the $\widetilde{\rho}$-strict mapping $\mathscr{T}$ satisfies the conditions of Theorem 3.4 of [10].

Letting $x=y$ in (2.1), we get

$$\rho(f(2x)-2f(x))\leq\phi(x,x) \tag{2.5}$$

for all $x \in \mathscr{E}$. If we replace x by $2x$ in (2.5) we get

$$\rho(f(4x) - 2f(2x)) \leq \phi(2x, 2x)$$

for all $x \in \mathscr{E}$. Since ρ is convex modular and satisfies the Δ_2-condition, we obtain

$$\begin{aligned}\rho\left(\frac{f(4x)}{2} - 2f(x)\right) &\leq \frac{1}{2}\rho(f(4x) - 2f(2x)) + \frac{1}{2}\rho(2f(2x) - 4f(x)) \\ &\leq \frac{1}{2}\phi(2x, 2x) + \frac{\kappa}{2}\phi(x, x)\end{aligned}$$

for all $x \in \mathscr{E}$. Moreover,

$$\rho\left(\frac{f(2^2x)}{2^2} - f(x)\right) \leq \frac{1}{2}\rho\left(2\frac{f(4x)}{2^2} - 2f(x)\right) \leq \frac{1}{2^2}\phi(2x, 2x) + \frac{\kappa}{2^2}\phi(x, x).$$

for all $x \in \mathscr{E}$. By mathematical induction, we can easily see that

$$\rho\left(\frac{f(2^nx)}{2^n} - f(x)\right) \leq \frac{1}{2^n}\sum_{i=1}^{n}\kappa^{n-i}\phi(2^{i-1}x, 2^{i-1}x) \leq \frac{1}{2(1-L)}\phi(x, x) \tag{2.6}$$

for all $x \in \mathscr{E}$. Next, we assert that $\delta_{\widetilde{\rho}}(f) = \sup\{\widetilde{\rho}\,(\mathscr{T}^n(f) - \mathscr{T}^m(f))\,; n, m \in \mathbb{N})\} < \infty$. It follows from inequality (2.6) that

$$\begin{aligned}\rho\left(\frac{f(2^nx)}{2^n} - \frac{f(2^mx)}{2^m}\right) &\leq \frac{1}{2}\rho\left(2\frac{f(2^nx)}{2^n} - 2f(x)\right) + \frac{1}{2}\rho\left(2\frac{f(2^mx)}{2^m} - 2f(x)\right) \\ &\leq \frac{\kappa}{2}\rho\left(\frac{f(2^nx)}{2^n} - f(x)\right) + \frac{\kappa}{2}\rho\left(\frac{f(2^mx)}{2^m} - f(x)\right) \\ &\leq \frac{1}{1-L}\phi(x, x),\end{aligned}$$

for every $x \in \mathscr{E}$ and $n, m \in \mathbb{N}$, which implies that

$$\widetilde{\rho}\,(\mathscr{T}^n(f) - \mathscr{T}^m(f)) \leq \frac{1}{1-L},$$

for all $n, m \in \mathbb{N}$. By the definition of $\delta_{\widetilde{\rho}}(f)$, we have $\delta_{\widetilde{\rho}}(f) < \infty$. Lemma 3.3 of [10] shows that $\{\mathscr{T}^n(f)\}$ is $\widetilde{\rho}$-converges to $j \in \mathscr{M}_{\widetilde{\rho}}$. Since ρ has the Fatou property inequality (2.6), gives $\widetilde{\rho}(\mathscr{T}j - f) < \infty$.

If we replace x by 2^nx in inequality (2.5), then we obtain

$$\widetilde{\rho}\left(f(2^{n+1}x) - 2f(2^nx)\right) \leq \phi(2^nx, 2^nx),$$

for all $x \in \mathscr{E}$. Whence

$$\begin{aligned}\rho\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^nx)}{2^n}\right) &\leq \frac{1}{2^{n+1}}\rho\left(f(2^{n+1}x) - 2f(2^nx)\right) \leq \frac{1}{2^{n+1}}\phi(2^n, 2^nx) \\ &\leq \frac{1}{2^{n+1}}2^nL^n\phi(x, x) \leq \frac{L^n}{2}\varphi(x, x) \leq \phi(x, x)\end{aligned}$$

for all $x \in \mathscr{E}$. Therefore $\widetilde{\rho}(\mathscr{T}(j) - j) < \infty$. It follows from [10, Theorem 3.4] that $\widetilde{\rho}$-limit of $\{\mathscr{T}^n(f)\}$ i.e., $j \in \mathscr{M}_{\widetilde{\rho}}$ is fixed point of map $\mathscr{T}$. If we replace x by 2^nx and y by 2^ny in inequality (2.1), then we obtain

$$\rho\left(f(2^n(x+y)) - f(2^nx) - f(2^ny)\right) \leq \phi(2^x, 2^ny)$$

for all $x, y \in \mathscr{E}$. Hence,

$$\rho\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}\right) \leq \frac{1}{2^n}\rho\left(f(2^n(x+y)) - f(2^n x) - f(2^n y)\right)$$
$$\leq \frac{\phi(2^x, 2^n y)}{2^n}$$

for all $x, y \in \mathscr{E}$. Taking the limit, we deduce that $j(x+y) = j(x) + j(y)$ for all $x, y \in \mathscr{E}$. It follows from inequality (2.6) that

$$\widetilde{\rho}(j - f) \leq \frac{1}{2(1-L)}.$$

If j^* is another fixed point of $\mathscr{T}$, then

$$\widetilde{\rho}(j - j^*) \leq \frac{1}{2}\widetilde{\rho}(2\mathscr{T}(j) - 2f) + \frac{1}{2}\widetilde{\rho}(2\mathscr{T}(j^*) - 2f)$$
$$\leq \frac{\kappa}{2}\widetilde{\rho}(\mathscr{T}(j) - f) + \frac{\kappa}{2}\widetilde{\rho}(\mathscr{T}(j^*) - f) \leq \frac{\kappa}{2(1-L)} < \infty.$$

Since $\mathscr{T}$ is $\widetilde{\rho}$-strict contraction, we get

$$\widetilde{\rho}(j - j^*) = \widetilde{\rho}(\mathscr{T}(j) - \mathscr{T}(j^*)) \leq L\widetilde{\rho}(j - j^*),$$

which implies that $\widetilde{\rho}(j - j^*) = 0$ or $j = j^*$, since $\widetilde{\rho}(j - j^*) < \infty$. This prove the uniqueness of j. ∎

Corollary 2.1. *Let $\mathscr{E}$ be a normed space and let $\mathfrak{F}$ be a Banach space. Suppose $f : \mathscr{E} \to \mathfrak{F}$ is a mapping with $f(0) = 0$ and there exist constants $\varepsilon, \theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon + \theta(\|x\|^p + \|y\|^p),$$

for all $x, y \in \mathscr{E}$. Then there exists a unique additive mapping $j : \mathscr{E} \to \mathfrak{F}$ such that

$$\|f(x) - j(x)\| \leq \frac{\varepsilon}{2 - 2^p} + \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x, y \in \mathscr{E}$.

Proof. It is known that every normed space is modular space with the modular $\rho(x) = \|x\|$ and $\kappa = 2$. Define $\phi(x, y) = \varepsilon + \theta(\|x\|^p + \|y\|^p)$ and apply Theorem 2.1. ∎

Now, we are ready to prove stability the functional equation $f(rx + sy) = rg(x) + sh(y)$.

Theorem 2.2. *Let $f, g, h : \mathscr{E} \to \mathscr{X}_\rho$ be mappings with $f(0) = g(0) = h(0) = 0$ satisfying*

$$\rho(f(rx + sy) - rg(x) - sh(y)) \leq \phi(x, y) \tag{2.7}$$

for all $x, y \in \mathscr{E}$, where $\phi : \mathscr{E} \times \mathscr{E} \to [0, \infty)$ is given function. If there exists $0 < L < 1$ such that

$$\phi(2x, 2x) \leq 2L\phi(x, x)$$

and has the property

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{2^n} = 0 \tag{2.8}$$

for all $x,y \in \mathscr{E}$. Then there exists a unique additive mapping $\mathscr{A}:\mathscr{E}\to\mathscr{X}_\rho$ such that

$$\rho(f(x)-\mathscr{A}(x)) \le \psi(x,x)$$
$$\rho(g(x)-\mathscr{A}(x)) \le \frac{\kappa}{2r}\left(\phi(x,0)+\psi(rx,rx)\right)$$
$$\rho(h(x)-\mathscr{A}(x)) \le \frac{\kappa}{2s}\left(\phi(0,x)+\psi(sx,sx)\right)$$

for all $x \in \mathscr{E}$, where $\psi(x,x) = 1/(2(1-L))\big[(\kappa/2)\phi(x/r,x/r)+(\kappa^2/4)(\phi(x/r,0)+\phi(0,x/s))\big]$.

Proof. Letting $y=0$ in (2.7) we get

$$\rho(f(rx)-rg(x) \le \phi(x,0)$$

for all $x\in\mathscr{E}$. Letting $x=0$ in (2.7) we get

$$\rho(f(sy)-sh(y)) \le \phi(0,y)$$

for all $y\in\mathscr{E}$. Then

$$\begin{aligned}
&\rho(f(rx+sy)-f(rx)-f(sy))\\
&\le \frac{1}{2}\rho\left(2(f(rx+sy)-rg(x)-sh(y)\right)+\frac{1}{2}\rho\left(2(rg(x)-f(rx)-f(sy)+sh(y)\right)\\
&\le \frac{\kappa}{2}\rho\left(f(rx+sy)-rg(x)-sh(y)\right)+\frac{\kappa}{2}\rho\left(rg(x)-f(rx)-f(sy)+sh(y)\right)\\
&\le \frac{\kappa}{2}\phi(x,y)+\frac{\kappa^2}{4}(\phi(x,0)+\phi(0,y)).
\end{aligned}$$

Replacing x by $(1/r)x$ and y by $(1/s)y$ in the above inequality, we obtain

$$\rho(f(x+y)-f(x)-f(y)) \le \frac{\kappa}{2}\phi\left(\frac{x}{r},\frac{y}{s}\right)+\frac{\kappa^2}{4}\left[\phi\left(\frac{x}{r},0\right)+\phi\left(0,\frac{y}{s}\right)\right]$$

for all $x,y\in\mathscr{E}$. By Theorem 2.1, there exists a unique additive mapping $\mathscr{A}:\mathscr{E}\to\mathscr{X}_\rho$ given by $\mathscr{A}(x)=\lim_{n\to\infty}(f(2^n x)/2^n)$ such that

$$\rho(f(x)-\mathscr{A}(x)) \le \psi(x,x) \tag{2.9}$$

for all $x\in\mathscr{E}$. Since $\mathscr{A}$ is a additive, we have $\mathscr{A}(qx)=q\mathscr{A}(x)$ for all rational numbers q and $x\in\mathscr{E}$. It follows from inequalities (2.7) and (2.9) that

$$\begin{aligned}
\rho(g(x)-\mathscr{A}(x)) &\le \frac{1}{2}\rho\left(2\left[g(x)-\frac{1}{r}f(rx)\right]\right)+\frac{1}{2}\rho\left(2\left[\frac{1}{r}f(rx)-\mathscr{A}(x)\right]\right)\\
&\le \frac{\kappa}{2r}(\rho(rg(x)-f(rx)))+\frac{\kappa}{2r}(\rho(f(rx)-\mathscr{A}(rx))\\
&\le \frac{\kappa}{2r}\phi(x,0)+\frac{\kappa}{2r}\psi(rx,rx)
\end{aligned}$$

for all $x\in\mathscr{E}$. Similarly, we obtain the following inequality

$$\rho(h(x)-\mathscr{A}(x)) \le \frac{\kappa}{2s}\phi(x,0)+\frac{\kappa}{2s}\psi(sx,sx)$$

for all $x\in\mathscr{E}$. ∎

Corollary 2.2. *Let $\mathscr{E}$ be a normed space and let $\mathscr{X}_\rho$ be a ρ-complete. Suppose $f : \mathscr{E} \to \mathscr{X}_\rho$ is a mapping with $f(0) = 0$ and there exist constants $\varepsilon, \theta \geq 0$ and $p \in [0,1)$ such that*

$$\rho(f(rx+sy) - rf(x) - sf(y)) \leq \varepsilon + \theta(\|x\|^p + \|y\|^p),$$

for all $x, y \in \mathscr{E}$. Then there exists a unique additive mapping $\mathscr{A} : \mathscr{E} \to \mathscr{X}_\rho$ such that

$$\rho(f(x) - \mathscr{A}(x)) \leq \frac{\kappa+\kappa^2}{2(2-2^p)}\varepsilon + \left[\frac{\kappa}{r^p} + \frac{\kappa^2}{2r^p} + \frac{\kappa^2}{2s^p}\right]\frac{\theta\|x\|^p}{2-2^p}.$$

Proof. Define $\phi(x,y) = \varepsilon + \theta(\|x\|^p + \|y\|^p)$ and apply Theorem 2.2. ∎

Corollary 2.3. *Let $\mathscr{E}$ be a normed space and let $\mathfrak{F}$ be a Banach space. Suppose $f : \mathscr{E} \to \mathfrak{F}$ is a mapping with $f(0) = 0$ and there exist constants $\varepsilon, \theta \geq 0$ and $p \in [0,1)$ such that*

$$\|f(rx+sy) - rf(x) - sf(y)\| \leq \varepsilon + \theta(\|x\|^p + \|y\|^p),$$

for all $x, y \in \mathscr{E}$. Then there exists a unique additive mapping $\mathscr{A} : \mathscr{E} \to \mathfrak{F}$ such that

$$\|f(x) - \mathscr{A}(x)\| \leq \frac{3}{2-2^p}\varepsilon + \left[\frac{2}{r^p} + \frac{1}{s^p}\right]\frac{\theta\|x\|^p}{2-2^p}.$$

The following example shows that our results in this paper differ form some results of [9].

Example 2.1. Let φ be an Orlicz function and satisfy the Δ_2-condition with $0 < \kappa \leq 2$. Let $f, g, h : \mathscr{E} \to L^\varphi$ be mappings with $f(0) = g(0) = h(0) = 0$ satisfying

$$\int_\Omega \phi(|f(rx+sy) - rg(x) - sh(y)|)d\mu \leq \phi(x,y) \tag{2.10}$$

for all $x, y \in \mathscr{E}$, where $\phi : \mathscr{E} \times \mathscr{E} \to [0,\infty)$ is given function. If there exists $0 < L < 1$ such that

$$\phi(2x,2x) \leq 2L\phi(x,x)$$

and has the property

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{2^n} = 0 \tag{2.11}$$

for all $x, y \in \mathscr{E}$. Then there exists a unique additive mapping $\mathscr{A} : \mathscr{E} \to \mathscr{X}_\rho$ such that

$$\int_\Omega \varphi(|f(x) - \mathscr{A}(x)|)d\mu \leq \psi(x,x)$$
$$\int_\Omega \varphi(|g(x) - \mathscr{A}(x)|)d\mu \leq \frac{\kappa}{2r}(\phi(x,0) + \psi(rx,rx))$$
$$\int_\Omega \varphi(|h(x) - \mathscr{A}(x)|)d\mu \leq \frac{\kappa}{2s}(\phi(0,x) + \psi(sx,sx))$$

for all $x \in \mathscr{E}$, where

$$\psi(x,x) = \frac{1}{2(1-L)}\left[\left(\frac{\kappa}{2}\right)\phi\left(\frac{x}{r},\frac{x}{r}\right) + \left(\frac{\kappa^2}{4}\right)\left(\phi\left(\frac{x}{r},0\right) + \phi\left(0,\frac{x}{s}\right)\right)\right].$$

3. Stability of generalized Jensen functional equation on restricted domains

In this section, we investigate the stability of our generalized Jensen equation on restricted domains. The idea and methods used in this section is taken from the paper by Jung *et al.* [9].

Theorem 3.1. *Let $(\mathscr{E},\rho)$ be a modular space and let $\mathscr{X}_\rho$ be a ρ-complete modular space. Let $d>0$, $\varepsilon>0$, and $f:\mathscr{E}\to\mathscr{X}_\rho$ with $f(0)=0$ such that*

$$\rho(f(rx+sy)-rf(x)-sf(y))\le\varepsilon \tag{3.1}$$

for all $x,y\in\mathscr{E}$ with $\rho(x)+\rho(y)\ge d$. Then there exists a unique additive mapping $\mathscr{A}:\mathscr{E}\to\mathscr{X}_\rho$ such that

$$\rho(f(x)-\mathscr{A}(x))\le\frac{\kappa+\kappa^2}{2}\left(\frac{\kappa}{2}+\frac{\kappa^2}{2^2}+\frac{\kappa^3}{2^3}+\frac{\kappa^4}{2^4}+\frac{\kappa^4}{2^4}\right)\varepsilon$$

for all $x\in\mathscr{E}$.

Proof. Let $x,y\in\mathscr{E}$ with $\rho(x)+\rho(y)<d$. Suppose that, for $x=y=0$, z is an element of $\mathscr{E}$ with $\rho(z)\ge d$. Furthermore, for $x\neq 0$ or $y\neq 0$, let

$$z:=\begin{cases}\left(1+\frac{d}{\rho(x)}\right)x & \text{if } \rho(x)\ge\rho(y)\\ \left(1+\frac{d}{\rho(y)}\right)y & \text{if } \rho(x)\le\rho(y).\end{cases}$$

Clearly, we see that

$$\begin{aligned}\rho\left(\left[2+\frac{s}{r}\right]z+\frac{s}{r}y\right)+\rho\left(\left[1+\frac{2r}{s}\right]z-\frac{r}{s}x\right)&\ge d,\\ \rho(x)+\rho(z)&\ge d,\\ \rho\left(2\left[1+\frac{s}{r}\right]z\right)+\rho(y)&\ge d,\\ \rho\left(2\left[1+\frac{s}{r}\right]z\right)+\rho\left(\left[1+\frac{2r}{s}\right]z-\frac{a}{b}x\right)&\ge d,\\ \rho\left(\left[2+\frac{s}{r}\right]z+\frac{s}{r}y\right)+\rho(z)&\ge d.\end{aligned}$$

Next, we show that the first inequality holds and other inequalities are trivial. To this end, let $\rho(x)\ge\rho(y)$, we put $\alpha=[2+s/r]z+(s/r)y$, $\beta=[1+(2r/s)]z-(r/s)x$, $\gamma=-(s/r)y$ and $\eta=-(r/s)x$. Then we obtain

$$\begin{aligned}\rho(\alpha)+\rho(\beta)&\ge 2\rho\left(\frac{\alpha+\gamma}{2}\right)-\rho(\gamma)+2\rho\left(\frac{\beta+\eta}{2}\right)-\rho(\eta)\\ &=2\rho\left(\frac{1}{2}\left[2+\frac{s}{r}\right]z\right)-\rho\left(\frac{s}{r}y\right)+2\rho\left(\frac{1}{2}\left[1+\frac{2r}{s}\right]z\right)-\rho\left(\frac{r}{s}x\right)\\ &\ge\rho(z)+\rho\left(\frac{s}{2r}z\right)-\rho\left(\frac{s}{r}y\right)+\rho\left(\frac{z}{2}\right)+\rho\left(\frac{r}{s}z\right)-\rho\left(\frac{r}{s}x\right)\\ &\ge\rho(z)+\rho\left(\frac{s}{r}x\right)-\rho\left(\frac{s}{r}y\right)+\rho\left(\frac{z}{2}\right)+\rho\left(\frac{r}{s}x\right)-\rho\left(\frac{r}{s}x\right)\\ &\ge d.\end{aligned}\tag{3.2}$$

Now, we set

$$\theta = f(rx+sy) - rf\left(\left[2+\frac{s}{r}\right]z+\frac{s}{r}y\right) - sf\left(\left[1+\frac{2r}{s}\right]z-\frac{r}{s}x\right),$$

$$\lambda = f(rx+sz) - rf(x) - sf(z),$$

$$\mu = f(2(r+s)z+sy) - rf\left(2\left[1+\frac{s}{r}\right]z\right) - sf(y),$$

$$\nu = -f(rx+sz) + rf\left(2\left[1+\frac{s}{r}\right]z\right) + sf\left(\left[1+\frac{2r}{s}\right]z-\frac{r}{s}x\right)$$

and

$$\vartheta = -f(2(r+s)z+sy) + rf\left(\left[2+\frac{s}{r}\right]z+\frac{s}{r}y\right) + sf(z).$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}
\rho(f(rx+sy)-rf(x)-sf(y)) &= \rho(\theta+\lambda+\mu+\nu+\vartheta) \le \frac{1}{2}\rho(2\theta)+\frac{1}{2}\rho(2(\lambda+\mu+\nu+\vartheta)) \\
&\le \frac{\kappa}{2}\rho(\theta)+\frac{\kappa}{2}\rho(\lambda+\mu+\nu+\vartheta) \\
&\;\;\vdots \\
&\le \frac{\kappa}{2}\rho(\theta)+\frac{\kappa^2}{2^2}\rho(\lambda)+\frac{\kappa^3}{2^3}\rho(\mu)+\frac{\kappa^4}{2^4}\rho(\nu)+\frac{\kappa^4}{2^4}\rho(\theta) \\
&\le \left(\frac{\kappa}{2}+\frac{\kappa^2}{2^2}+\frac{\kappa^3}{2^3}+\frac{\kappa^4}{2^4}+\frac{\kappa^4}{2^4}\right)\varepsilon
\end{aligned}$$

for all $x, y \in \mathscr{E}$. We thus obtain

$$\rho(f(rx+sy)-rf(x)-sf(y)) \le \left(\frac{\kappa}{2}+\frac{\kappa^2}{2^2}+\frac{\kappa^3}{2^3}+\frac{\kappa^4}{2^4}+\frac{\kappa^4}{2^4}\right)\varepsilon$$

for all $x, y \in \mathscr{E}$. Now the result asserted by the above Theorem can be deduced fairly easily from Corollary 2.2 with $\theta = p = 0$. ∎

Corollary 3.1. *Let $\mathscr{E}$ be a normed space and let $\mathfrak{F}$ be a Banach space. Let $d > 0$, $\varepsilon > 0$, and $f : \mathscr{E} \to \mathfrak{F}$ be a mapping with $f(0) = 0$ such that*

$$\|f(rx+sy)-rf(x)-sf(y)\| \le \varepsilon$$

for all $x, y \in \mathscr{E}$ with $\|x\|+\|y\| \ge d$. Then there exists a unique additive mapping $\mathscr{A} : \mathscr{E} \to \mathfrak{F}$ such that

$$\|f(x)-\mathscr{A}(x)\| \le 15\varepsilon$$

for all $x \in \mathscr{E}$.

Corollary 3.2. *Let $(\mathscr{E}, \rho)$ be a modular space and let $\mathscr{X}_\rho$ be a ρ-complete modular space. Let $f : \mathscr{E} \to \mathscr{X}_\rho$ be a mapping with $f(0) = 0$. Then f is additive if and only if*

$$\rho(f(rx+sy)-rf(x)-sf(y)) \to 0 \quad as \quad \rho(x)+\rho(y) \to \infty. \tag{3.3}$$

Proof. The proof of this corollary is similar to [9, Corollary 3.2]. ∎

Example 3.1. Let $\hat{\varphi} = \{\varphi_i\}$ be a sequence of Orlicz functions and let $(\ell^{\hat{\varphi}}, \rho_{\hat{\varphi}})$ be a generalized Orlicz sequence space associated to $\hat{\varphi} = \{\varphi_i\}$. Let $(L^{\varphi}, \rho_{\varphi})$ be an Orlicz space and φ satisfy the Δ_2-condition with $0 < \kappa \leq 2$. Suppose $d > 0$, $\varepsilon > 0$ and $f : \ell^{\hat{\varphi}} \to L^{\varphi}$ with $f(0) = 0$ such that

$$\int_{\Omega} \varphi\left(|f(rx+sy) - rf(x) - sf(y)|\right) \leq \varepsilon$$

for all $x, y \in \ell^{\hat{\varphi}}$ with $\rho_{\varphi}(x) + \rho_{\varphi}(y) = \Sigma_{i=1}^{\infty} \varphi_i(|x_i|) + \Sigma_{i=1}^{\infty} \varphi_i(|y_i|) \geq d$. Then there exists a unique additive mapping $\mathscr{A} : \ell^{\hat{\varphi}} \to L^{\varphi}$ such that

$$\int_{\Omega} \varphi\left(|f(x) - \mathscr{A}(x)|\right) \leq \frac{\kappa + \kappa^2}{2} \left(\frac{\kappa}{2} + \frac{\kappa^2}{2^2} + \frac{\kappa^3}{2^3} + \frac{\kappa^4}{2^4} + \frac{\kappa^4}{2^4} \right) \varepsilon$$

for all $x \in \ell^{\hat{\varphi}}$.

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