BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37**(2) (2014), 359–367

Convergence Theorems for Multi-Valued Mappings

[1]YEKINI SHEHU AND [2]G. C. UGWUNNADI
[1]Department of Mathematics, University of Nigeria, Nsukka, Nigeria
[2]Department of Mathematics, Micheal Okpara University of Agriculture, Umudike, Abia State, Nigeria
deltanougt2006@yahoo.com, ugwunnadi4u@yahoo.com

Abstract. In this paper, we introduce a new iterative process to approximate a common fixed point of an infinite family of multi-valued generalized nonexpansive maps in a uniformly convex real Banach space and establish strong convergence theorems for the proposed iterative process. Furthermore, strong convergence theorems for infinite family of multi-valued nonexpansive maps are obtained. Our results extend many known recent results in the literature.

2010 Mathematics Subject Classification: 47H06, 47H09, 47J05, 47J25

Keywords and phrases: Strong convergence, multi-valued generalized nonexpansive maps, multi-valued nonexpansive maps, uniformly convex Banach spaces.

1. Introduction

Let D be a nonempty, closed and convex subset of a real Hilbert space H. The set D is called *proximinal* if for each $x \in H$, there exists $y \in D$ such that $||x-y|| = d(x,D)$, where $d(x,D) = \inf\{||x-z|| : z \in D\}$. Let $CB(D)$, $K(D)$ and $CK(D)$ denote the families of nonempty, closed and bounded subsets, nonempty compact subsets and nonempty compact convex subsets of D respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\} \tag{1.1}$$

for $A,B \in CB(D)$. A single-valued map $T : D \to D$ is called *nonexpansive* if $||Tx-Ty|| \le ||x-y||$ for all $x,y \in D$. A multi-valued map $T : D \to CB(D)$ is said to be *nonexpansive* if $H(Tx,Ty) \le ||x-y||$ for all $x,y \in D$. An element $p \in D$ is called a fixed point of $T : D \to D$ (resp., $T : D \to CB(D)$) if $p = Tp$ (resp., $p \in Tp$). The set of fixed points of T is denoted by $F(T)$. A multi-valued map $T : D \to CB(D)$ is said to be *quasi-nonexpansive* if $H(Tx,Tp) \le ||x-p||$ for all $x \in D$ and all $p \in F(T)$. A multi-valued map $T : D \to CB(D)$ is said to be *generalized nonexpansive* [14] if $H(Tx,Ty) \le a||x-y|| + b(d(x,Tx)+d(y,Ty)) + c(d(x,Ty)+d(y,Tx))$ for all $x,y \in D$, where $a+2b+2c \le 1$. It can be shown that if T is

Communicated by Syakila Ahmad.
Received: January 20, 2012; *Revised:* April 27, 2012.

a multi-valued generalized nonexpansive map, then T is a multi-valued quasi-nonexpansive map. Indeed, if $p \in F(T)$ and T is generalized nonexpansive, we obtain for all $x \in D$ that

$$\begin{aligned} H(Tx,Tp) &\leq a||x-p|| + b(d(x,Tx)+d(p,Tp)) + c(d(x,Tp)+d(p,Tx)) \\ &\leq a||x-p|| + b(||x-p|| + d(p,Tx)) + c(d(x,Tp)+d(p,Tx)) \\ &\leq (a+b+c)||x-p|| + (b+c)d(p,Tx) \\ &\leq (a+b+c)||x-p|| + (b+c)H(Tx,Tp). \end{aligned}$$

Hence, $H(Tx,Tp) \leq (a+b+c)/(1-(b+c))||x-p||$. Since $(a+b+c)/(1-(b+c)) \leq 1$, it follows that

$$H(Tx,Tp) \leq ||x-p||.$$

Furthermore, if $T : D \to CB(D)$ is a generalized nonexpansive mapping and for some $p,\ Tp = \{p\}$ then for $x \in F(T) - \{p\}$, we have

$$\begin{aligned} ||x-p|| = H(p,Tx) &= a||p-x|| + b(d(p,Tp)+d(x,Tx)) + c(d(p,Tx)+d(x,Tp)) \\ &= a||p-x|| + c(||p-x|| + ||x-p||) = (a+2c)||x-p||. \end{aligned}$$

Hence $a+2c = 1$ and so $b = 0$. Therefore if $T : D \to CB(D)$ is a generalized nonexpansive mapping and for some $p, Tp = \{p\}$ then $F(T) = \{p\}$ or $b = 0$ and $a+2c = 1$.

A map $T : D \to CB(D)$ is said to satisfy *Condition (I)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0,\ f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x,Tx) \geq f(d(x,F(T)))$$

for all $x \in D$.

A family $\{T_i : D \to CB(D),\ i = 1,2,3,\ldots\}$ is said to satisfy *Condition (II)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0,\ f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x,T_ix) \geq f(d(x,\cap_{i=1}^{\infty}F(T_i)))$$

for all $i = 1,2,3,\ldots$ and $x \in D$.

The mapping $T : D \to CB(D)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in D such that $d(x_n,Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in D$. We note that if D is compact, then every multi-valued mapping $T : D \to CB(D)$ is hemicompact.

The fixed point theory of multi-valued nonexpansive mappings is much more complicated and difficult than the corresponding theory of single-valued nonexpansive mappings. However, some classical fixed point theorems for single-valued nonexpansive mappings have already been extended to multi-valued mappings. The first results in this direction were established by Markin [10] in Hilbert spaces and by Browder [2] for spaces having weakly continuous duality mapping. Lami Dozo [7] generalized these results to a Banach space satisfying Opial's condition.

In 1974, by using Edelstein's method of asymptotic centers, Lim [8] obtained a fixed point theorem for a multi-valued nonexpansive self-mapping in a uniformly convex Banach space.

Theorem 1.1. [8] *Let D be a nonempty, closed convex and bounded subset of a uniformly convex Banach space E and $T : D \to K(E)$ a multi-valued nonexpansive mapping. Then T has a fixed point.*

In 1990, Kirk-Massa [6] gave an extension of Lim's theorem proving the existence of a fixed point in a Banach space for which the asymptotic center of a bounded sequence in a closed bounded convex subset is nonempty and compact.

Theorem 1.2. [6] *Let D be a nonempty, closed convex and bounded subset of a Banach space E and $T : D \to CK(E)$ a multi-valued nonexpansive mapping. Suppose that the asymptotic center in E of each bounded sequence of E is nonempty and compact. Then T has a fixed point.*

Banach contraction mapping principle was extended nicely multi-valued mappings by Nadler [11] in 1969. (Below is stated in a Banach space setting).

Theorem 1.3. [11] *Let D be a nonempty closed subset of a Banach space E and $T : D \to CB(D)$ a multi-valued contraction. Then T has a fixed point.*

In 1953, Mann [9] introduced the following iterative scheme to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 1$$

where the initial point x_0 is taken arbitrarily in D and $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$. However, we note that Mann's iteration yields only weak convergence, see, for example [5].

In 2005, Sastry and Babu [14] proved that the Mann and Ishikawa iteration schemes for a multi-valued nonexpansive mapping T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point p may be different from q.

In 2007, Panyanak [12] extended the results of Sastry and Babu to uniformly convex Banach spaces and proved the following theorems.

Theorem 1.4. [12] *Let E be a uniformly convex Banach space, D a nonempty closed bounded convex subset of E and $T : D \to P(D)$ a multi-valued nonexpansive mapping that satisfies condition (I). Assume that (i) $0 \leq \alpha_n < 1$ and (ii) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$. Suppose that $F(T)$ a nonempty proximinal subset of D. Then the Mann iterates $\{x_n\}$ defined by $x_0 \in D$,*

$$x_{n+1} = \alpha_n y_n + (1-\alpha_n)x_n, \quad \alpha_n \in [a,b],\ 0 < a < b < 1,\ n \geq 0, \tag{1.2}$$

where $y_n \in Tx_n$ such that $||y_n - u_n|| = d(u_n, Tx_n)$ and $u_n \in F(T)$ such that $||x_n - u_n|| = d(x_n, F(T))$, converges strongly to a fixed point of T.

Theorem 1.5. *(Panyanak, [12]) Let E be a uniformly convex Banach space, D a nonempty compact convex subset of E and $T : D \to P(D)$ a multi-valued nonexpansive mapping with a fixed point p. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\Sigma_{n=1}^{\infty}\alpha_n\beta_n = \infty$. Then the Ishikawa iterates $\{x_n\}$ defined by $x_0 \in D$,*

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \quad \beta_n \in [0,1],\ n \geq 0$$

$z_n \in Tx_n$ such that $||z_n - p|| = d(p, Tx_n)$, and

$$x_{n+1} = \alpha_n z_n' + (1-\alpha_n)x_n, \quad \alpha_n \in [0,1],\ n \neq 0$$

$z_n' \in Ty_n$ such that $||z_n' - p|| = d(p, Ty_n)$, converges strongly to a fixed point of T.

Later, Song and Wang [17] noted there was a gap in the proofs of Theorem 1.5 above and [14, Theorem 5]. They further solved/revised the gap and also gave the affirmative answer Panyanak [12] question using the Ishikawa iterative scheme. In the main results, the domain of T is still compact, which is a strong condition (see [17, Theorem 1]) and T satisfies condition (I) (see [17, Theorem 1]).

Recently, Abbas *et al.* [1] introduced the following one-step iterative process to compute common fixed points of two multi-valued nonexpansive mappings.

$$(1.3) \qquad \begin{cases} x_1 \in D \\ x_{n+1} = a_n x_n + b_n y_n + c_n z_n, \quad \forall n \geq 1. \end{cases}$$

Using (1.3), Abbas *et al.* [1] proved weak and strong convergence theorems for approximation of common fixed point of two multi-valued nonexpansive mappings in Banach spaces.

Motivated by the ongoing research and the above mentioned results, we introduce a new iterative scheme for approximation of common fixed points of infinite family of multi-valued generalized nonexpansive maps in a real Banach space. Furthermore, we prove strong convergence theorems for approximation of common fixed points of infinite family of multi-valued generalized nonexpansive maps in a uniformly convex real Banach space. Next, we prove a necessary and sufficient condition for strong convergence of our new iterative process to a common fixed point of infinite family of multi-valued nonexpansive maps. Our results extend the results of Sastry and Babu [14], Panyanak [12], Song and Wang [17], Abbas *et al.* [1], Qin *et al.* [13], and Gang and Shangquan [3].

2. Preliminaries

Let E be Banach space and dim$E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \left|\left|\frac{x+y}{2}\right|\right| : ||x|| = ||y|| = 1; \varepsilon = ||x-y||\right\}.$$

E is *uniformly convex* if for any $\varepsilon \in (0,2]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in E$ with $||x|| \leq 1$, $||y|| \leq 1$ and $||x-y|| \geq \varepsilon$, then $||1/2(x+y)|| \leq 1-\delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Lemma 2.1. [15] *Suppose E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that* $\limsup_{n\to\infty} ||x_n|| \leq r$, $\limsup_{n\to\infty} ||y_n|| \leq r$ *and* $\lim_{n\to\infty} ||t_n x_n + (1-t_n) y_n|| = r$ *hold for some $r > 0$. Then* $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 2.2. [4] *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0,2r] \to \mathbb{R}, \quad g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\left|\left|\sum_{n=1}^{\infty} \lambda_n x_n\right|\right|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

3. Main results

Lemma 3.1. *Let E be a uniformly convex real Banach space and D a nonempty, closed and convex subset of E. Let $T_1, T_2, T_3, \dots$ be multi-valued generalized nonexpansive maps of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ for which $T_i p = \{p\}$, $\forall p \in \cap_{i=1}^{\infty} F(T_i)$. Let $\{\alpha_{ni}\}_{n=1}^{\infty}$, $i = 0,1,2,\dots$ be a sequence in $[\varepsilon, 1-\varepsilon]$, $\varepsilon \in (0,1)$ such that $\sum_{i=0}^{\infty} \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_1 \in D$,*

$$x_{n+1} = \alpha_{n0} x_n + \sum_{i=1}^{\infty} \alpha_{ni} y_n^{(i)}, \tag{3.1}$$

where $y_n^{(i)} \in T_i x_n$, $i = 1,2,3,\cdots$. Then

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad \forall i = 1,2,3,\cdots.$$

Proof. Let $x^* \in \cap_{i=1}^{\infty} F(T_i)$. Since T_i, $i = 1,2,3,\dots$ is generalized nonexpansive, we obtain

$$\begin{aligned} H(T_i x_n, T_i x^*) &\leq a||x_n - x^*|| + b(d(x_n, T_i x_n) + d(x^*, T_i x^*)) + c(d(x_n, T_i x^*) + d(x^*, T_i x_n)) \\ &\leq a||x_n - x^*|| + b(||x_n - x^*|| + d(x^*, T_i x_n)) + c(d(x_n, T_i x^*) + d(x^*, T_i x_n)) \\ &\leq (a+b+c)||x_n - x^*|| + (b+c) d(x^*, T_i x_n) \\ &\leq (a+b+c)||x_n - x^*|| + (b+c) H(T_i x^*, T_i x_n). \end{aligned}$$

Hence,

$$H(T_i x_n, T_i x^*) \leq \left(\frac{a+b+c}{1-(b+c)} \right) ||x_n - x^*||.$$

Since $(a+b+c)/(1-(b+c)) \leq 1$, it follows that

$$H(T_i x_n, T_i x^*) \leq ||x_n - x^*||. \tag{3.2}$$

Then from (3.1) and (3.2), we have the following estimates,

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \alpha_{n0}||x_n - x^*|| + \sum_{i=1}^{\infty} \alpha_{ni} ||y_n^{(i)} - x^*|| \leq \alpha_{n0}||x_n - x^*|| + \sum_{i=1}^{\infty} \alpha_{ni} d(y_n^{(i)}, T_i x^*) \\ &\leq \alpha_{n0}||x_n - x^*|| + \sum_{i=1}^{\infty} \alpha_{ni} H(T_i x_n, T_i x^*) \leq \alpha_{n0}||x_n - x^*|| + \sum_{i=1}^{\infty} \alpha_{ni} ||x_n - x^*|| \\ &= ||x_n - x^*||. \end{aligned} \tag{3.3}$$

Thus, $\lim_{n\to\infty} ||x_n - x^*||$ exists. Now using Lemma 2.2, we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= \left|\left| \alpha_{n0} x_n + \sum_{i=1}^{\infty} \alpha_{ni} y_n^{(i)} - x^* \right|\right|^2 = \left|\left| \alpha_{n0}(x_n - x^*) + \sum_{i=1}^{\infty} \alpha_{ni}(y_n^{(i)} - x^*) \right|\right|^2 \\ &\leq \alpha_{n0}||x_n - x^*||^2 + \sum_{i=1}^{\infty} \alpha_{ni} ||y_n^{(i)} - x^*||^2 - \alpha_{n0}\alpha_{ni} g(||x_n - y_n^{(i)}||) \\ &\leq \alpha_{n0}||x_n - x^*||^2 + \sum_{i=1}^{\infty} \alpha_{ni} (d(y_n^{(i)}, T_i x^*))^2 - \alpha_{n0}\alpha_{ni} g(||x_n - y_n^{(i)}||) \\ &\leq \alpha_{n0}||x_n - x^*||^2 + \sum_{i=1}^{\infty} \alpha_{ni} (H(T_i x_n, T_i x^*))^2 - \alpha_{n0}\alpha_{ni} g(||x_n - y_n^{(i)}||) \\ &\leq \alpha_{n0}||x_n - x^*||^2 + \sum_{i=1}^{\infty} \alpha_{ni} ||x_n - x^*||^2 - \alpha_{n0}\alpha_{ni} g(||x_n - y_n^{(i)}||) \end{aligned}$$

$$= ||x_n - x^*||^2 - \alpha_{n0}\alpha_{ni} g(||x_n - y_n^{(i)}||). \tag{3.4}$$

This implies that

$$0 \le \varepsilon^2 g(||x_n - y_n^{(i)}||) \le \alpha_{n0}\alpha_{ni} g(||x_n - y_n^{(i)}||) \le ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2.$$

Hence $\lim_{n\to\infty} g(||x_n - y_n^{(i)}||) = 0$. By property of g, we have $\lim_{n\to\infty} ||x_n - y_n^{(i)}|| = 0$. Then

$$d(x_n, T_i x_n) \le ||x_n - y_n^{(i)}|| \to 0, \quad n \to \infty, \ i = 1,2,3,\cdots.$$

This completes the proof. ∎

Theorem 3.1. *Let E be a uniformly convex real Banach space and D a nonempty, closed and convex subset of E. Let $T_1, T_2, T_3, \ldots$ be multi-valued generalized nonexpansive maps of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ for which $T_i p = \{p\}$, $\forall p \in \cap_{i=1}^{\infty} F(T_i)$ and $\{T_i\}_{i=1}^{\infty}$ satisfying condition (II). Let $\{\alpha_{ni}\}_{n=1}^{\infty}$, $i = 0,1,2,\ldots$ be a sequence in $[\varepsilon, 1-\varepsilon]$, $\varepsilon \in (0,1)$ such that $\sum_{i=0}^{\infty} \alpha_{ni} = 1$ for all $n \ge 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by* (3.1). *Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$.*

Proof. Since $\{T_i\}_{i=1}^{\infty}$ satisfies condition (II), we have that $d(x_n, \cap_{i=1}^{\infty} F(T_i)) \to 0$ as $n \to \infty$. Thus, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset \cap_{i=1}^{\infty} F(T_i)$ such that

$$||x_{n_k} - p_k|| < \frac{1}{2^k}$$

for all k. By Lemma 3.1, we obtain that

$$||x_{n_{k+1}} - p_k|| \le ||x_{n_k} - p_k|| < \frac{1}{2^k}.$$

We now show that $\{p_k\}$ is a Cauchy sequence in D. Observe that

$$||p_{k+1} - p_k|| \le ||p_{k+1} - x_{n_{k+1}}|| + ||x_{n_{k+1}} - p_k|| < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

This shows that $\{p_k\}$ is a Cauchy sequence in D and thus converges to $p \in D$. Since

$$d(p_k, T_i p)|| \le H(T_i p_k, T_i p) \le ||p_k - p||$$

and $p_k \to p$ as $k \to \infty$, it follows that $d(p, T_i p) = 0$ and thus $p \in \cap_{i=1}^{\infty} F(T_i)$ and $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} ||x_n - p||$ exists, it follows that $\{x_n\}$ converges strongly to p. This completes the proof. ∎

Theorem 3.2. *Let E be a uniformly convex real Banach space and D a nonempty, closed and convex subset of E. Let $T_1, T_2, T_3, \ldots$ be multi-valued generalized nonexpansive maps of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ for which $T_i p = \{p\}$, $\forall p \in \cap_{i=1}^{\infty} F(T_i)$ and T_i is hemicompact for some $i \in \mathbb{N}$ and T_i is continuous for each $i = 1,2,3,\cdots$. Let $\{\alpha_{ni}\}_{n=1}^{\infty}$, $i = 0,1,2,\ldots$ be a sequence in $[\varepsilon, 1-\varepsilon]$, $\varepsilon \in (0,1)$ such that $\sum_{i=0}^{\infty} \alpha_{ni} = 1$ for all $n \ge 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by* (3.1). *Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$.*

Proof. Since $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, $\forall i = 1,2,3,\ldots$ and T_i is hemicompact for each $i = 1,2,3,\ldots$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$ for some $p \in D$. Since T_i is continuous for each $i = 1,2,3,\ldots$, we have $d(x_{n_k}, T_i x_{n_k}) \to d(p, T_i p)$. As a result, we have that $d(p, T_i p) = 0$, $\forall i = 1,2,3,\ldots$ and so $p \in \cap_{i=1}^{\infty} F(T_i)$. Since $\lim_{n\to\infty} ||x_n - p||$ exists, it follows that $\{x_n\}$ converges strongly to p. This completes the proof. ∎

Theorem 3.3. *Let E be a uniformly convex real Banach space and D a nonempty compact convex subset of E. Let $T_1, T_2, T_3, \dots$ be multi-valued nonexpansive maps of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ for which $T_i p = \{p\}$, $\forall p \in \cap_{i=1}^{\infty} F(T_i)$. Let $\{\alpha_{ni}\}_{n=1}^{\infty}$, $i = 0, 1, 2, \dots$ be a sequence in $[\varepsilon, 1-\varepsilon]$, $\varepsilon \in (0,1)$ such that $\sum_{i=0}^{\infty} \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by (3.1). Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$.*

Proof. From the compactness of D, there exists a subsequence $\{x_{n_k}\}_{n=k}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} ||x_{n_k} - q|| = 0$ for some $q \in D$. Thus,

$$\begin{aligned} d(q, T_i q) &\leq ||x_{n_k} - q|| + d(x_{n_k}, T_i x_{n_k}) + H(T_i x_{n_k}, T_i q) \\ &\leq 2||x_{n_k} - q|| + d(x_{n_k}, T_i x_{n_k}) \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Hence, $q \in \cap_{i=1}^{\infty} F(T_i)$. Now, on taking q in place of x^*, we get that $\lim_{n\to\infty} ||x_n - q||$ exists. This completes the proof. ∎

The following result gives a necessary and sufficient condition for strong convergence of the sequence in (3.1) to a common fixed point of an infinite family of multi-valued nonexpansive maps $\{T_i\}_{i=1}^{\infty}$.

Theorem 3.4. *Let D be a nonempty, closed and convex subset of a real Banach space E. Let $T_1, T_2, T_3, \dots,$ be multi-valued nonexpansive maps of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ for which $T_i p = \{p\}$, $\forall p \in \cap_{i=1}^{\infty} F(T_i)$. Let $\{\alpha_{ni}\}_{n=1}^{\infty}$, $i = 0, 1, \dots, m$ be a sequence in $[\varepsilon, 1-\varepsilon]$, $\varepsilon \in (0,1)$ such that $\sum_{i=0}^{\infty} \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by (3.1). Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. By (3.3), we have

$$||x_{n+1} - x^*|| \leq ||x_n - x^*||.$$

This gives

$$d(x_{n+1}, F) \leq d(x_n, F).$$

Hence, $\lim_{n\to\infty} d(x_n, F)$ exists. By hypothesis, $\liminf_{n\to\infty} d(x_n, F) = 0$ so we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in D. Let $\varepsilon > 0$ be given and since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists n_0 such that for all $n \geq n_0$, we have

$$d(x_n, F) < \frac{\varepsilon}{4}.$$

In particular, $\inf\{||x_{n_0} - p|| : p \in F\} < \frac{\varepsilon}{4}$, so that there must exist a $p^* \in F$ such that

$$||x_{n_0} - p^*|| < \frac{\varepsilon}{2}.$$

Now, for $m, n \geq n_0$, we have

$$||x_{n+m} - x_n|| \leq |||x_{n+m} - p^*|| + ||x_n - p^*|| \leq 2||x_{n_0} - p^*|| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

Hence, $\{x_n\}$ is a Cauchy sequence in a closed subset D of a Banach space E and therefore, it must converge in D. Let $\lim_{n\to\infty} x_n = p$. Now, for each $i = 1, 2, 3, \dots$, we obtain

$$\begin{aligned} d(p, T_i p) &\leq d(p, x_n) + d(x_n, T_i x_n) + H(T_i x_n, T_i p) \\ &\leq d(p, x_n) + d(x_n, T_i x_n) + d(x_n, p) \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

gives that $d(p,T_ip)=0,\ i=1,2,3,\dots$ which implies that $p\in T_ip$. Consequently, $p\in F=\cap_{i=1}^{\infty}F(T_i)\neq\emptyset$. ∎

Corollary 3.1. [1] *Let E be a uniformly convex real Banach space satisfying Opial's condition. Let D be a nonempty, closed and convex of E. Let T,S be multi-valued nonexpansive mappings of D into $K(D)$ such that $F(T)\cap F(S)\neq\emptyset$. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequence in $(0,1)$ satisfying $a_n+b_n+c_n\leq 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by*

$$\begin{cases} x_1\in D \\ x_{n+1}=a_nx_n+b_ny_n+c_nz_n, \quad \forall n\geq 1 \end{cases} \tag{3.5}$$

where $y_n\in Tx_n$, $z_n\in Sx_n$ such that $||y_n-p||\leq d(p,Tx_n)$ and $||z_n-p||\leq d(p,Sx_n)$ whenever p is a fixed point of any one of mappings T and S. Then, $\{x_n\}_{n=1}^{\infty}$ converges weakly to a common fixed point of $F(T)\cap F(S)$.

Corollary 3.2. [1] *Let E be a real Banach space and D a nonempty, closed and convex subset of E. Let T,S be multi-valued nonexpansive mappings of D into $K(D)$ such that $F(T)\cap F(S)\neq\emptyset$. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequence in $(0,1)$ satisfying $a_n+b_n+c_n\leq 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by*

$$\begin{cases} x_1\in D \\ x_{n+1}=a_nx_n+b_ny_n+c_nz_n, \quad \forall n\geq 1 \end{cases} \tag{3.6}$$

where $y_n\in Tx_n$, $z_n\in Sx_n$ such that $||y_n-p||\leq d(p,Tx_n)$ and $||z_n-p||\leq d(p,Sx_n)$ whenever p is a fixed point of any one of mappings T and S. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $F(T)\cap F(S)$ if and only if $\liminf_{n\to\infty}d(x_n,F)=0$.

Remark 3.1. We observe that the iterative scheme (1.3) can be re-written as

$$x_{n+1}=\alpha_{n0}x_n+\sum_{i=1}^{2}\alpha_{ni}y_n^{(i)}, \tag{3.7}$$

where $\alpha_{n0}=\alpha_n,\ \alpha_{n1}=b_n,\ \alpha_{n2}=c_n,\ y_n^{(1)}=y_n,\ y_n^{(2)}=z_n$. Furthermore, this is an iterative scheme for approximation of common fixed point of two multivalued nonexpansive mappings. Motivated by (3.7), we introduced our iterative scheme (3.1) for approximation of common fixed point of an infinite family of multi-valued generalized nonexpansive mappings. This is the proper justification of introducing our iteration scheme (3.1).

Remark 3.2. Our iterative scheme (3.1) reduces to the iterative scheme (1.2) when considering approximation of fixed point of a multivalued nonexpansive mapping. Furthermore, our iterative scheme (3.1) is more general than the Mann iterative scheme considered by Sastry and Babu [14] and Song and Wang [17].

Remark 3.3. Our iterative scheme (3.1) reduces to the iterative schemes (3.5) and (3.6) when considering approximation of common fixed point of two multivalued nonexpansive mappings.

Remark 3.4. Our results extend the results of Sastry and Babu [14], Panyanak [12] and Song and Wang [17] from approximation of a fixed point of a *single multi-valued nonexpansive mapping* to approximation of common fixed point of *an infinite family of multi-valued generalized nonexpansive mappings* and Abbas *et al.* [1] from approximation of a common fixed point of *two multi-valued nonexpansive mappings* to approximation of common fixed point of *an infinite family of multi-valued generalized nonexpansive mappings*.

Acknowledgement. The authors would like to express their sincere thanks to the anonymous referees for their valuable suggestions and comments which improved the original version of the manuscript greatly.

References

[1] A. Abbas, S. H. Khan, A. R. Khan and R. P. Agarwal, Common fixed points of two multivalued nonexpansive mappings by one-step iterative scheme, *Appl. Math. Lett.* **24** (2011), no. 2, 97–102.

[2] F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, in *Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968)*, 1–308, Amer. Math. Soc., Providence, RI.

[3] G. Cai and B. Shangquan, Strong convergence theorems for variational inequality problems and fixed point problems in banach spaces, *Bull. Malays. Math. Sci. Soc. (2)* **36** (2013), no. 2, 525–540.

[4] S. Chang, J. K. Kim and X. R. Wang, Modified block iterative algorithm for solving convex feasibility problems in Banach spaces, *J. Inequal. Appl.* **2010**, Art. ID 869684, 14 pp.

[5] A. Genel and J. Lindenstrauss, An example concerning fixed points, *Israel J. Math.* **22** (1975), no. 1, 81–86.

[6] W. A. Kirk and S. Massa, Remarks on asymptotic and Chebyshev centers, *Houston J. Math.* **16** (1990), no. 3, 357–364.

[7] E. Lami Dozo, Multivalued nonexpansive mappings and Opial's condition, *Proc. Amer. Math. Soc.* **38** (1973), 286–292.

[8] T. C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space, *Bull. Amer. Math. Soc.* **80** (1974), 1123–1126.

[9] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953), 506–510.

[10] J. T. Markin, A fixed point theorem for set valued mappings, *Bull. Amer. Math. Soc.* **74** (1968), 639–640.

[11] S. B. Nadler, Jr., Multi-valued contraction mappings, *Pacific J. Math.* **30** (1969), 475–488.

[12] B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, *Comput. Math. Appl.* **54** (2007), no. 6, 872–877.

[13] X. Qin, Y. J. Cho and S. M. Kang, An iterative method for an infinite family of nonexpansive mappings in Hilbert spaces, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 2, 161–171.

[14] K. P. R. Sastry and G. V. R. Babu, Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point, *Czechoslovak Math. J.* **55(130)** (2005), no. 4, 817–826.

[15] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Austral. Math. Soc.* **43** (1991), no. 1, 153–159.

[16] N. Shahzad and H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, *Nonlinear Anal.* **71** (2009), no. 3-4, 838–844.

[17] Y. Song and H. Wang, Erratum to: "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces" [Comput. Math. Appl. **54** (2007), no. 6, 872–877] by B. Panyanak, *Comput. Math. Appl.* **55** (2008), no. 12, 2999–3002.