BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37**(2) (2014), 369–382

Double Entire Difference Sequence Spaces of Fuzzy Numbers

[1]KULDIP RAJ, [2]SUNIL K. SHARMA AND [3]ANIL KUMAR
School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K, India
[1]kuldeepraj68@rediffmail.com, [2]sunilksharma42@yahoo.co.in, [3]aksharma288@gmail.com

Abstract. In this paper we introduce some double entire difference sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions $\mathscr{M} = (M_{k,l})$. We also make an effort to study some topological properties and some inclusion relations between these spaces.

2010 Mathematics Subject Classification: 40A05, 40D25

Keywords and phrases: Fuzzy numbers, Orlicz function, de La Vallee Poussin means, statistical convergence.

1. Introduction and preliminaries

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool.

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [37] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [17] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. For more details about sequence spaces of fuzzy numbers see [2,7,16,21,22,33] and references therein.

The initial works on double sequences is found in Bromwich [5]. Later on, it was studied by Hardy [11], Moricz [18], Moricz and Rhoades [19], Tripathy [34,35], Başarır and Sonalcan [3] and many others. Hardy [11] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [38] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [23] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between

Communicated by Rosihan M. Ali, Dato'.
Received: December 16, 2011; *Revised:* March 13, 2012.

statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [20] and Mursaleen and Edely [24] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{m,n})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces $\mathscr{BS}$, $\mathscr{BS}(t)$, $\mathscr{CS}_p$, $\mathscr{CS}_{bp}$, $\mathscr{CS}_r$ and $\mathscr{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathscr{M}_u$, $\mathscr{M}_u(t)$, $\mathscr{C}_p$, $\mathscr{C}_{bp}$, $\mathscr{C}_r$ and $\mathscr{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathscr{BS}$, $\mathscr{BV}$, $\mathscr{CS}_{bp}$ and the $\beta(v)$-duals of the spaces $\mathscr{CS}_{bp}$ and $\mathscr{CS}_r$ of double series. Now, recently Başar and Sever [4] have introduced the Banach space $\mathscr{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathscr{L}_q$. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by P-$\lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k,l > n$ see [27]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l.

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [13] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $X : \mathbb{R}^n \to [0,1]$ which satisfies the following four conditions:

(1) X is normal, i.e., there exist an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$;
(2) X is fuzzy convex, i.e., for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1, X(\lambda x + (1-\lambda)y) \geq \min[X(x), X(y)]$;
(3) X is upper semi-continuous;
(4) the closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$, denoted by $[X]^0$, is compact.

Let $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A$ is compact and convex $\}$. The spaces $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A + B = \{a + b,\ a \in A,\ b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a : a \in A\}$$

for $A,B\in C(\mathbb{R}^n)$ and $\lambda\in\mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as

$$\delta_\infty(A,B)=\max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n),\delta_\infty)$ is a complete (non separable) metric space.

For $0<\alpha\leq 1$, the α-level set

$$X^\alpha=\{x\in\mathbb{R}^n : X(x)\geq\alpha\}$$

is a nonempty compact convex, subset of $\mathbb{R}^n$, as is the support X^0. Let $L(\mathbb{R}^n)$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces addition $X+Y$ and scalar multiplication $\lambda X,\lambda\in\mathbb{R}$, in terms of α-level sets, by

$$[X+Y]^\alpha=[X]^\alpha+[Y]^\alpha \quad \text{and} \quad [\lambda X]^\alpha=\lambda[X]^\alpha$$

for each $0\leq\alpha\leq 1$. Define for each $1\leq q<\infty$

$$d_q(X,Y)=\left\{\int_0^1\delta_\infty(X^\alpha,Y^\alpha)^q d\alpha\right\}^{1/q}$$

and $d_\infty(X,Y)=\sup\limits_{0\leq\alpha\leq 1}\delta_\infty(X^\alpha,Y^\alpha)$. Clearly $d_\infty(X,Y)=\lim\limits_{q\to\infty}d_q(X,Y)$ with $d_q\leq d_r$ if $q\leq r$. Moreover $(L(\mathbb{R}^n),d_\infty)$ is a complete, separable and locally compact metric space. Let w denote the set of all fuzzy complex sequences $x=(x_k)_{k=1}^\infty$, and M be an Orlicz function, or a modulus function. Consider

$$\Gamma_M=\left\{x\in w:\lim_{k\to\infty}\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)=0\text{ for some }\rho>0\right\}$$

and

$$\Lambda_M=\left\{x\in w:\sup_k\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)<\infty\text{ for some }\rho>0\right\}.$$

The spaces Γ_M and Λ_M are metric spaces with the metric

$$d(x,y)=\inf\left\{\rho>0:\sup_k\left(M\left(\frac{|x_k-y_k|^{1/k}}{\rho}\right)\right)\leq 1\right\}$$

for all $x=(x_k)$ and $y=(y_k)$ in Γ_M.

In this paper we define Orlicz space of double entire sequence of fuzzy numbers by using regular matrix $A=(a_{mnkl})(m,n,k,l=1,2,3,...)$. By the regularity of A we mean that the matrix which transform convergent sequence into a convergent sequence leaving the limit for detail see [12]; we prove that these spaces are complete paranormed spaces. If E is a linear space over the complex field $\mathbb{C}$, then a paranorm on E is a function $p:E\to\mathbb{R}$ which satisfies the following axioms, for $x,y\in E$,

(1) $p(x)\geq 0$, for all $x\in X$;
(2) $p(-x)=p(x)$, for all $x\in X$;
(3) $p(x+y)\leq p(x)+p(y)$, for all $x,y\in X$;
(4) if (σ_n) is a sequence of scalars with $\sigma_n\to\sigma$ as $n\to\infty$ and (x_n) is a sequence of vectors with $p(x_n-x)\to 0$ as $n\to\infty$, then $p(\sigma_n x_n-\sigma x)\to 0$ as $n\to\infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [36, Theorem 10.4.2, p.183]). For more details about sequence spaces (see [14,15,25,28,29,30,33]) and references therein.

The notion of difference sequence spaces was introduced by Kızmaz [12], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [8] by introducing the spaces $l_\infty(\Delta^m), c(\Delta^m)$ and $c_0(\Delta^m)$.

Let r be non-negative integers. Then for $Z = l_\infty,\ c,\ c_0$, we have sequence spaces

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\},$$

where $\Delta^r x_k = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+v}.$$

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of natural numbers into $L(\mathbb{R}^n)$. The fuzzy number X_n denotes the value of the function $n \in N$ and is called n^{th} term of the sequence.

Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the Orlicz space of entire sequences of fuzzy numbers convergent to zero, written as $\big(M\big((|X_k|^{1/k})/\rho\big)\big) \to 0$ as $k \to \infty$, for some arbitrarily fixed $\rho > 0$ and is defined by

$$\left[\bar{d}\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right) \to 0 \text{ as } k \to \infty\right]$$

is denoted by $\Gamma_M(F)$, with M being a Orlicz function.

Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the space consisting of all those fuzzy sequence $X = (X_k)$ such that $\sup\big(M\big((|X_k|^{1/k})/\rho\big)\big) < \infty$ for some arbitrary $\rho > 0$ is denoted by $\Lambda_M(F)$ and is known as Orlicz space of analytic sequences.

A fuzzy double sequence is a double infinite array of fuzzy numbers. We denote a fuzzy double sequence by $(X_{m,n})$, where $X_{m,n}$'s are fuzzy numbers for each $m, n \in \mathbb{N}$. By $s''(F)$ we denote the set of all double sequences of fuzzy numbers.

A double sequence $X = (X_{k,l})$ of fuzzy numbers is said to be convergent in the Pringsheim's sense or P-convergent to a fuzzy number X_0, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\bar{d}(X_{k,l}, X_0) < \varepsilon \ \text{ for } \ k, l > N,$$

where $\mathbb{N}$ is the set of natural numbers, and we denote by $P - \lim X = X_0$. The number X_0 is called the Pringsheim limit of $X_{k,l}$. More exactly we say that a double sequence $(X_{k,l})$ converges to a finite number X_0 if $X_{k,l}$ tend to X_0 as both k and l tends to ∞ independently of one another.

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} p_{k,l} = H$ and let $D = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_{k,l}\}$ and $\{b_{k,l}\}$ in the complex plane, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq D(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}). \tag{1.1}$$

Let $\mathscr{M} = (M_{k,l})$ be a sequence of Orlicz functions and $A = (a_{mnkl})$ be a regular matrix. Then we define the following classes of sequences in the present paper:

$$\Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r] = \left\{ X = (X_{k,l}) \in s''(F) : \lim_{m,n\to\infty} \sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn} \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} = 0 \right\},$$

and

$$\Lambda^2_{\mathscr{M}}[F,A,p,\Delta^r] = \left\{ X = (X_{k,l}) \in s''(F) : \sup_{m,n} \sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn} \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}$$

and call them respectively the spaces of double sequences of fuzzy numbers which are strongly entire and strongly analytic.

If we take $a_{mnkl} = 1$ for all $m,n,k,l \in \mathbb{N}$ then $\Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$ and $\Lambda^2_{\mathscr{M}}[F,A,p,\Delta^r]$ are reduced to $\Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$ and $\Lambda^2_{\mathscr{M}}[F,p,\Delta^r]$, respectively, defined as

$$\Gamma^2_{\mathscr{M}}[F,p,\Delta^r] = \left\{ X = (X_{k,l}) \in s''(F) : \lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,l=1}^{m,n} \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} = 0 \right\},$$

and

$$\Lambda^2_{\mathscr{M}}[F,p,\Delta^r] = \left\{ X = (X_{k,l}) \in s''(F) : \sup_{m,n} \frac{1}{mn} \sum_{k,l=1}^{m,n} \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}.$$

A metric $\bar{d}$ on $L(\mathbb{R}^n)$ is said to be translation invariant if $\bar{d}(X+Z,Y+Z) = \bar{d}(X,Y)$ for $X,Y,Z \in L(\mathbb{R}^n)$.

The main purpose of this paper is to introduce some double entire difference sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions. We also make an effort to study some topological properties and inclusion relations between the above defined sequence spaces.

2. Main results

Proposition 2.1. *If $\bar{d}$ is a translation invariant metric on $L(\mathbb{R}^n)$, then*

(i) $\bar{d}(\Delta^r X + \Delta^r Y,\bar{0}) \le \bar{d}(\Delta^r X,\bar{0}) + \bar{d}(\Delta^r Y,\bar{0})$,
(ii) $\bar{d}(\lambda\Delta^r X,\bar{0}) \le |\lambda|\bar{d}(\Delta^r X,\bar{0}), |\lambda| > 1$.

Proof. It is easy to prove so we omit the details. ∎

Theorem 2.1. *$\Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$ is a complete metric space with the metric is given by*

$$\rho(X,Y) = \sup_{mn} \left[\frac{1}{mn} \left(M_{k,l}\left(\frac{\bar{d}(\Delta^r(X_{mn}+Y_{mn}),\bar{0})^{1/m+n}}{\rho} \right) \right) \right]^p,$$

where $X = (\Delta^r X_{mn})$ and $Y = (\Delta^r Y_{mn})$ are the double difference sequences of fuzzy numbers.

Proof. It is easy to show that this is a metric space. We will show that it is complete. Let $\{X^{(i)}\}$ be a Cauchy sequence in $\Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$. Then

$$\left(M_{k,l}\left(\frac{\bar{d}(\Delta^r X^{(i)}_{mn},\bar{0})^{1/m+n}}{\rho}\right)\right)$$

will be a Cauchy sequence in $\Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$. Therefore for each m and n, we have

$$\left(M_{k,l}\left(\frac{\bar{d}(\Delta^r X^{(i)}_{mn},\bar{0})^{1/m+n}}{\rho}\right)\right)$$

is a Cauchy sequence in $L(\mathbb{R}^n)$. Since $L(\mathbb{R}^n)$ is complete, thus

$$\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}(\Delta^r X^{(i)}_{mn},\bar{0})^{1/m+n}}{\rho}\right)\right)\right)\to 0 \quad \text{as} \quad i\to\infty.$$

Put $\Delta^r X=(\Delta^r X_{mn})$, since

$$\left(M_{k,l}\left(\frac{\bar{d}(\Delta^r X^{(i)}_{mn},\bar{0})^{1/m+n}}{\rho}\right)\right)$$

is a Cauchy sequence in $\Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$, there exists $n_0\in N$ such that for all i.

$$\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X^{(i)}_{mn}-\Delta^r X^{(j)}_{mn}),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p\to 0 \quad \text{as} \quad m,n\to\infty.$$

Let $j\to\infty$, we get

$$\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X^{(i)}_{mn}-\Delta^r X_{mn}),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p\to 0 \quad \text{as} \quad m,n\to\infty.$$

Therefore $\left(M_{k,l}\left((\bar{d}(\Delta^r X^{(i)}_{mn},\bar{0})^{1/m+n})/\rho\right)\right)\to 0$ as $i\to\infty$.

Now we have to show that $X\in\Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$. Since $\Delta^r X^{(i)}\in\Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$, there exists $\Delta^r X^{(i)}_0\in L(\mathbb{R}^n)$ such that

$$\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X^{(i)}_{mn}-\Delta^r X^{(i)}_0),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p\to 0 \quad \text{as} \quad m,n\to\infty.$$

Hence

$$\begin{aligned}
&\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X^{(i)}_{mn}-\Delta^r X^{(j)}_0),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p\\
&\leq\left(\frac{1}{mn}\bar{d}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X^{(i)}_{mn}-\Delta^r X^{(j)}_0),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p\\
&\quad+\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X^{(i)}_{mn}-\Delta^r X^{(i)}_0),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p\\
&\quad+\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X^{i}_{mn}-\Delta^r X^{(j)}_0),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p
\end{aligned}$$

$\to 0$ as $m,n\to\infty$. Thus $\big(M_{k,l}\big((\bar{d}(\Delta^r X_{mn}^{(i)},\bar{0})^{1/m+n})/\rho\big)\big)$ is a Cauchy sequence in $L(\mathbb{R}^n)$. Since $L(\mathbb{R}^n)$ is complete, there exists $\Delta^r X_0 \in L(\mathbb{R}^n)$ such that

$$\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X_0^{(i)}-\Delta^r X_0),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p \to 0 \quad \text{as} \quad m,n\to\infty.$$

Therefore

$$\begin{aligned}
&\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X_{mn}-\Delta^r X_0),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p\\
&\leq \left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X_{mn}^{(i)}-\Delta^r X_{mn}),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p\\
&\quad+\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X_{mn}^{(i)}-\Delta^r X_0^{(i)}),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p\\
&\quad+\left(\frac{1}{mn}\left(M_{k,l}\left(\frac{\bar{d}((\Delta^r X_0^{(i)}-\Delta^r X_0),\bar{0})^{1/m+n}}{\rho}\right)\right)\right)^p
\end{aligned}$$

$\to 0$ as $m,n\to\infty$. This implies that $X=(X_{k,l})\in\Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$. This completes the proof of the theorem. ∎

Theorem 2.2. $\Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$ *and* $\Lambda^2_{\mathscr{M}}[F,A,p,\Delta^r](\inf p_k>0)$ *are complete with respect to the topology generated by the paranorm h is defined by*

$$h(X)=\sup_{m,n}\left(\sum_{k,l=1}^{m,n}\frac{a_{mnkl}}{mn}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}\right),\ \left|\sum_{k,l=1}^{m,n}\frac{a_{mnkl}}{mn}\right|^{p_{k,l}}\to 0 \quad \textit{as} \quad k,l\to\infty, \tag{2.1}$$

where $\bar{d}$ *is a translation invariant and* $X=(X_{k,l})$ *be a double sequence of fuzzy numbers.*

Proof. Let $(X^{(s)})$ be a Cauchy sequence in $\Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$. Then

$$\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r(X^{(s)}-X^{(t)}),\bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}\to 0 \quad \text{as} \quad s,t\to\infty,$$

that is

$$\lim_{s,t\to\infty}\sum_{k,l=1}^{m,n}\frac{a_{mnkl}}{mn}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r(X_{k,l}^{(s)}-X_{k,l}^{(t)}),\bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}=0 \quad \text{for all } m,n. \tag{2.2}$$

Hence

$$\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r(X_{k,l}^{(s)}-X_{k,l}^{(t)}),\bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}\to 0 \quad \text{as} \quad s,t\to\infty \quad \text{for all } k,l,$$

which implies that $\big[M_{k,l}\big((\bar{d}(\Delta^r X_{k,l}^{(s)},\bar{0})^{1/k+l})/\rho\big)\big]^{p_{k,l}}$ is a Cauchy sequence in $\mathbb{R}$ for each k,l and so there exists $Y=(\Delta^r Y_{k,l})$ such that $\big[M_{k,l}\big((\bar{d}(\Delta^r X_{k,l}^{(s)},\bar{0})^{1/k+l})/\rho\big)\big]^{p_{k,l}}\to(\Delta^r Y_{k,l})$ as

$s \to \infty$ for each k,l. Now, from (2.2) we have, for $\varepsilon > 0$, there exists natural number N such that

$$\left(\sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r(X_{k,l}^{(s)} - X_{k,l}^{(t)}), \bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}\right) < \varepsilon \tag{2.3}$$

for $s,t > N$ and for all m,n. Hence, for any natural number g, we have from (2.3)

$$\left(\sum_{k,l\le g} \frac{a_{mnkl}}{mn}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r(X_{k,l}^{(s)} - Y_{k,l}^{(t)}), \bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}\right) < \varepsilon \quad \text{for } s,t > N \text{ for all } n. \tag{2.4}$$

Now fix $s > N$ and let $t \to \infty$. Then from (2.3), we have

$$\left(\sum_{k,l\le g} \frac{a_{mnkl}}{mn}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r(X_{k,l}^{(s)} - Y_{k,l}), \bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}\right) < \varepsilon \quad \text{for } s > N \text{ for all } n.$$

Since this is valid for any natural number g, we have

$$\left(\sum_{k,l\le g} \frac{a_{mnkl}}{mn}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r(X_{k,l}^{(s)} - Y_{k,l}), \bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}\right) < \varepsilon \quad \text{for } s > N \text{ for all } m,n,$$

that is,

$$\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r(X^{(s)} - Y), \bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}} \to 0 \quad \text{as} \quad s \to \infty,$$

and thus $\left[M_{k,l}\left((\bar{d}(\Delta^r X^{(s)}, \bar{0})^{1/k+l})/\rho\right)\right]^{p_{k,l}} \to (\Delta^r Y)$ as $s \to \infty$, and therefore

$$\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r(X^{(s)} - Y), \bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}} \in \Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r].$$

Hence $\Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$ is complete. Similarly we can prove the completeness of $\Lambda^2_{\mathscr{M}}[F,A,p,\Delta^r]$. ∎

Theorem 2.3. *Let $X = (X_{k,l})$ be a double sequence of fuzzy numbers and $\bar{d}$ be a translation invariant. Let $A = (a_{mnkl}), (m,n,k,l = 1,2,3,...)$ be an infinite matrix. Then $\Gamma^2_{\mathscr{M}}[F,A,p] \subset \Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$ if and only if given $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that*

$$|\Delta^{r-1} a_{mnkl} - \Delta^{r-1} a_{m+1,n+1,k,l}| < \varepsilon^{m,n} N^{k,l} \quad (m,n,k,l = 1,2,3,...).$$

Proof. Let $X = (X_{k,l}) \in \Gamma^2_{\mathscr{M}}[F,A,p]$ and let

$$Y_{m,n} = \left(\sum_{k,l=1}^{\infty,\infty} a_{mnkl}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}\right) \quad (m,n = 1,2,3,...),$$

so that $\Delta^r Y_{m,n} = \left(\sum_{k,l=1}^{\infty,\infty}(\Delta^{r-1}a_{mnkl} - \Delta^{r-1}a_{m+1,n+1,k,l})\left[M_{k,l}\left((\bar{d}(\Delta^r X_{k,l}, \bar{0})^{1/k+l})/\rho\right)\right]^{p_{k,l}}\right)$. Then $(\Delta^r Y_{m,n}) \in \Gamma^2_{\mathscr{M}}[F,A,p]$ if and only if given any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that $|\Delta^{r-1}a_{m,n,k,l} - \Delta^{r-1}a_{m+1,n+1,k,l}| < \varepsilon^{m,n}N^{k,l}$. Now $(\Delta^r Y_{m,n} \in \Gamma^2_{\mathscr{M}}[F,A,p])$ if and only if $(Y_{m,n}) \in \Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$. Thus $\Gamma^2_{\mathscr{M}}[F,A,p] \subset \Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$ if and only if the condition holds. This completes the proof. ∎

Theorem 2.4. *Let $X = (X_{k,l})$ be a double sequence of fuzzy numbers and $\bar{d}$ be a translation invariant. If $A = (a_{mnkl})$ transform $\Gamma^2_{\mathscr{M}}[F,A,p]$ into $\Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$. Then $\lim_{n\to\infty}(\Delta^{r-1}a_{mnkl} - \Delta^{r-1}a_{m+1,n+1,k,l})q^{m,n} = 0$ for all integers $q > 0$ and each fixed $k,l = 1,2,3,\ldots$.*

Proof. Let $Y_{m,n} = \left(\sum_{k,l=1}^{\infty,\infty} a_{mnkl}\left[M_{k,l}\left((\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l})/\rho\right)\right]^{p_{k,l}}\right)$ $(m,n = 1,2,3,\ldots)$ formally. Let $(X_{k,l}) \in \Gamma^2_{\mathscr{M}}[F,A,p]$ and $(Y_{m,n}) \in \Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$. But then $(\Delta^r Y_{m,n}) \in \Gamma^2_{\mathscr{M}}[F,A,p]$,

$$\Delta^r Y_{m,n} = \left(\sum_{k,l=1}^{\infty,\infty}(\Delta^{r-1}a_{m,n,k,l} - \Delta^{r-1}a_{m+1,n+1,k,l})\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}}\right),$$

$(m,n = 1,2,3,\ldots)$. Take $(X_{k,l}) = \delta^{k,l} = (0,0,0,\ldots,1,0,0,\ldots), 1$ in the k^{th} place and zero's elsewhere. Then $(X_{k,l}) \in \Gamma^2_{\mathscr{M}}[F,A,p]$. We have $\Delta^r Y_{m,n} = \Delta^{r-1}a_{m,n,k,l} - \Delta^{r-1}a_{m+1,n+1,k,l}$. But $(\Delta^r Y_{m,n}) \in \Gamma^2_{\mathscr{M}}[F,A,p]$. Hence $\sum_{k,l=1}^{\infty,\infty}(\Delta^{r-1}a_{m,n,k,l} - \Delta^{r-1}a_{m+1,n+1,k,l})q^{m,n} < \infty$ for all integers q and each fixed $k,l = 1,2,3,\ldots$. This completes the proof. ∎

Theorem 2.5. *Let $X = (X_{k,l})$ be a double sequence of fuzzy numbers and $\bar{d}$ be a translation invariant. If $A = (a_{mnkl})$ transform $\Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$ into $\Gamma^2_{\mathscr{M}}[F,A,p]$. Then $\lim_{m,n\to\infty} a_{mnkl}q^{m,n} = 0$ for all integers q.*

Proof. Let

$$t_{m,n} = \left(\sum_{k,l=1}^{\infty,\infty}\frac{a_{m,n,k,l}}{mn}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^p\right)$$

with $(X_{k,l}) \in \Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r], (t_{m,n}) \in \Gamma^2_{\mathscr{M}}[F,A,p]$ and

$$s_{m,n} = \left(\sum_{k,l=1}^{\infty,\infty}\frac{a_{m,n,k,l}}{mn}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k+1,l+1},\bar{0})^{1/k+l}}{\rho}\right)\right]^p\right),$$

$(s_{m,n}) \in \Gamma^2_{\mathscr{M}}[F,A,p]$. Then

$$\begin{aligned} Y_{m,n} = (t_{m,n} - s_{m,n}) &= \left(\sum_{k,l=1}^{\infty,\infty}\frac{a_{m,n,k,l}}{mn}\left[M_{k,l}\left(\frac{\bar{d}((\Delta^r X_{k,l} - \Delta^r X_{k+1,l+1}),\bar{0})^{1/k+l}}{\rho}\right)\right]^p\right) \\ &= \left(\sum_{k,l=1}^{\infty,\infty}\frac{a_{m,n,k,l}}{mn}\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^p\right) \end{aligned}$$

and $\Delta^r X_{k,l} \in \Gamma^2_{\mathscr{M}}[F,A,p]$ and $(Y_{m,n}) \in \Gamma^2_{\mathscr{M}}[F,A,p]$. Hence $(a_{m,n,k,l})q^{m,n} \to 0$ as $n \to \infty$ for all k,l. This completes the proof of the theorem. ∎

Theorem 2.6. *Let $X = (X_{k,l})$ be a double sequence of fuzzy numbers and $\bar{d}$ be a translation invariant. If $A = (a_{mnkl})$ transforms $\Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$ into $\Gamma^2_{\mathscr{M}}[F,A,p,\Delta^r]$, then $(a_{mnkl})q^{m,n} \to 0$ and $(a_{m+1,n+1,k,l})q^{m,n} \to 0$ as $m,n \to \infty$.*

Proof. From Theorems 2.3 and 2.4 we have $a_{mnkl}q^{m,n} \to 0$ and

$$(\Delta^{r-1}a_{m,n,k,l} - \Delta^{r-1}a_{m+1,n+1,k,l})q^{m,n} \to 0 \quad \text{as} \quad m,n \to \infty$$

for all positive integers q and for all k,l.

$$\begin{aligned} &\Rightarrow (a_{mnkl})q^{m,n} \to 0 \quad \text{and} \quad (\Delta^{r-1}a_{mnkl} - \Delta^{r-1}a_{m+1,n+1,k,l})q^{m,n} \to 0 \\ &\Rightarrow (a_{m+1,n+1,k,l})q^{m,n} \to 0 \quad \text{and} \quad (a_{mnkl})q^{m,n} \to 0 \quad \text{as} \quad m,n \to \infty \quad \text{for all } k,l. \end{aligned}$$

This completes the proof. ∎

3. Δ^r-statistical convergence

The idea of statistical convergence of a sequence was introduced by Fast [9]. Statistical convergence was generalized by Buck [6] and studied by many other authors, using a regular nonnegative summability matrix A in place of Cesaro matrix. The existing literature on statistical convergence have been restricted to real or complex analysis, but at the first time Nuray and Savaş [26] extended the idea to apply the sequences of fuzzy numbers. For more details on fuzzy sequence spaces and statistical convergence see [10,31,32] and references therein. The generalized de la Valee-Pousin mean is defined by $t_n(X) = (1/\lambda_n)\sum_{k\in I_n} X_k$, where $\lambda = (\lambda_n)$ is non-decreasing sequence of positive numbers such that $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$.

The double sequence $\lambda_2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \le \lambda_{m,n} + 1, \quad \lambda_{m,n+1} \le \lambda_{m,n} + 1,$$

$$\lambda_{m,n} - \lambda_{m+1,n} \le \lambda_{m,n+1} - \lambda_{m+1,n+1}, \quad \lambda_{1,1} = 1,$$

and

$$I_{m,n} = \{(k,l) : m - \lambda_{m,n} + 1 \le k \le m,\ n - \lambda_{m,n} + 1 \le l \le n\}.$$

The generalized double de la Vallee-Poussin mean is defined by

$$t_{m,n} = t_{m,n}(x_{k,l}) = \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} x_{k,l}.$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be Δ^r-statistical convergent to fuzzy number zero if

$$\lim_{m,n\to\infty} \frac{1}{mn} \left| \left\{ k,l \le m,n : \left[M_{k,l} \left(\frac{\bar{d}(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \ge \varepsilon \right\} \right| = 0,$$

i.e.,$\left[M_{k,l}\left((\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l})/\rho\right)\right] < \varepsilon$. In this case we write

$$St(\Delta^r) - \lim_{k,l\to\infty} \left[M_{k,l} \left(\frac{(X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] = 0.$$

Let $X = (X_{k,l})$ be a sequence of fuzzy numbers and $p = (p_{k,l})$ be a sequence of strictly positive real numbers. Then the sequence $X = (X_{k,l})$ is said to be strongly Δ^r-convergent if there is a fuzzy number zero such that

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k\le n} \left[M_{k,l} \left(\frac{\bar{d}(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} = 0.$$

Let $X = (X_{k,l})$ be a sequence of fuzzy numbers. Then the space consisting of all those fuzzy sequences $X = (X_{k,l})$such that $\sup(M_{k,l}((X_{k,l},\bar{0})/\rho)) < \infty$ for some arbitrary $\rho > 0$ is denoted by $\Lambda_M(\Delta^r)$ and is known as Δ^r-Orlicz space of analytic sequences of fuzzy numbers.

Theorem 3.1. *If $(X_{k,l}), (Y_{k,l}) \in St(\Delta^r)$ and $c \in L(\mathbb{R}^n)$, then*

(i) $St(\Delta^r) - \lim \left(c \left[M_{k,l} \left(\frac{(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \right) = cSt(\Delta^r) - \lim \left[M_{k,l} \left(\frac{(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right],$

(ii) $St(\Delta^r) - \lim \left[M_{k,l} \left(\frac{(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right]$
$= St(\Delta^r) - \lim \left[M_{k,l} \left(\frac{(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] + St(\Delta^r) - \lim \left[M_{k,l} \left(\frac{(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right]$,
where $\bar{d}$ is a translation invariant.

Proof. (i) Let $St(\Delta^r) - \lim \left[M_{k,l} \left(((\Delta^r X_{k,l}, \bar{0})^{1/k+l})/\rho \right) \right] c \in L(\mathbb{R})$ and $\varepsilon > 0$ be given. Then the proof follows from the following inequality

$$\frac{1}{mn} \left| \left\{ k,l \leq m,n : \left[M_{k,l} \left(\frac{\bar{d}(c\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \varepsilon \right\} \right|$$
$$\leq \frac{1}{mn} \left| \left\{ k,l \leq m,n : \left[M_{k,l} \left(\frac{\bar{d}(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \frac{\varepsilon}{|c|} \right\} \right|.$$

(ii) Suppose that

$$St(\Delta^r) - \lim \left[M_{k,l} \left(\frac{(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] = 0$$

and

$$St(\Delta^r) - \lim \left[M_{k,l} \left(\frac{\bar{d}(\Delta^r Y_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] = 0.$$

By Minkowski's inequality we get

$$\left[M_{k,l} \left(\frac{\bar{d}((\Delta^r X_{k,l} + \Delta^r Y_{k,l}), \bar{0})^{1/k+l}}{\rho} \right) \right]$$
$$= \left[M_{k,l} \left(\frac{\bar{d}(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] + \left[M_{k,l} \left(\frac{\bar{d}(\Delta^r Y_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right].$$

Therefore given $\varepsilon > 0$ we have

$$\frac{1}{mn} \left| \left\{ k,l \leq m,n : \left[M_{k,l} \left(\frac{\bar{d}((\Delta^r X_{k,l} + \Delta^r Y_{k,l}), \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \varepsilon \right\} \right|$$
$$\leq \frac{1}{mn} \left| \left\{ k,l \leq m,n : \left[M_{k,l} \left(\frac{\bar{d}(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \frac{\varepsilon}{2} \right\} \right|$$
$$+ \frac{1}{mn} \left| \left\{ k,l \leq m,n : \left[M_{k,l} \left(\frac{\bar{d}(\Delta^r Y_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \frac{\varepsilon}{2} \right\} \right|.$$

Hence $St(\Delta^r) - \lim \left[M_{k,l}((((\Delta^r X_{k,l} + \Delta^r Y_{k,l}), \bar{0})^{1/k+l})/\rho) \right] = 0$. This completes the proof. ∎

Theorem 3.2. *If a sequence $X = (X_{k,l})$ is Δ^r-statistically convergent to the fuzzy number zero and $\liminf_{mn}(\lambda_{mn}/mn) > 0$, then it is Δ^r-statistically convergent to zero.*

Proof. Given $\varepsilon > 0$ we have

$$\frac{1}{mn} \left| \left\{ k,l \leq m,n : \left[M_{k,l} \left(\frac{\bar{d}(\Delta^r X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \varepsilon \right\} \right|$$

$$\supset \frac{1}{mn}\left|\left\{k,l \le I_{m,n} : \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right] \ge \varepsilon\right\}\right|$$

Therefore,

$$\begin{aligned}
&\frac{1}{mn}\left|\left\{k,l \le m,n : \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right] \ge \varepsilon\right\}\right| \\
&\ge \frac{1}{mn}\left|\left\{k,l \le I_{m,n} : \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right] \ge \varepsilon\right\}\right| \\
&\ge \frac{\lambda_{mn}}{mn}\frac{1}{\lambda_{mn}}\left|\left\{k,l \le I_{m,n} : \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right] \ge \varepsilon\right\}\right|.
\end{aligned}$$

Taking limit as $m,n \to \infty$ and using $\liminf_{mn}(\lambda_{mn}/mn) > 0$, we get X is Δ^r-statistically convergent to zero. This completes the proof. ∎

Theorem 3.3. *Let* $0 \le p_{k,l} \le q_{k,l}$ *and let* $\{q_{k,l}/p_{k,l}\}$ *be bounded. Then* $\Gamma^2_{\mathscr{M}}[F,q,\Delta^r] \subset \Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$.

Proof. Let

$$X \in \Gamma^2_{\mathscr{M}}[F,q,\Delta^r] \quad \text{and given} \quad \varepsilon > 0, \quad \text{we have} \tag{3.1}$$

$$\frac{1}{mn}\left|\left\{k,l \le m,n : \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^{q_{k,l}} \ge \varepsilon\right\}\right| \to 0 \quad \text{as} \quad m,n \to \infty. \tag{3.2}$$

Let $t_{k,l} = \left[M_{k,l}\left((\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l})/\rho\right)\right]^{q_{k,l}}$ and $\lambda_{k,l} = p_{k,l}/q_{k,l}$. Since $p_{k,l} \le q_{k,l}$, we have $0 \le \lambda_{k,l} \le 1$. Take $0 < \lambda < \lambda_{k,l}$. Define $u_{k,l} = t_{k,l}(t_{k,l} \ge 1); u_{k,l} = 0(t_{k,l} < 1)$ and $v_{k,l} = 0(t_{k,l} \ge 1); v_{k,l} = t_{k,l}(t_{k,l} < 1), t_{k,l} = u_{k,l} + v_{k,l}, i.e., t_{k,l}^{\lambda_{k,l}} = u_{k,l}^{\lambda_{k,l}} + v_{k,l}^{\lambda_{k,l}}$. Now it follows that

$$u_{k,l}^{\lambda_{k,l}} \le u_{k,l} \le t_{k,l} \quad \text{and} \quad v_{k,l}^{\lambda_{k,l}} \le v_{k,l}^{\lambda}. \tag{3.3}$$

Since $t_{k,l}^{\lambda_{k,l}} = u_{k,l}^{\lambda_{k,l}} + v_{k,l}^{\lambda_{k,l}}$, then $t_{k,l}^{\lambda_{k,l}} \le t_{k,l} + v_{k,l}^{\lambda}$,

$$\begin{aligned}
&\frac{1}{mn}\left|\left\{k,l \le m,n : \left(\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^{q_{k,l}}\right)^{\lambda_{k,l}} \ge \varepsilon\right\}\right| \\
&\le \frac{1}{mn}\left|\left\{k,l \le m,n : \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^{q_{k,l}} \ge \varepsilon\right\}\right|
\end{aligned}$$

$$\begin{aligned}
\Rightarrow &\frac{1}{mn}\left|\left\{k,l \le m,n : \left(\left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^{q_{k,l}}\right)^{p_{k,l}/q_{k,l}} \ge \varepsilon\right\}\right| \\
&\le \frac{1}{mn}\left|\left\{k,l \le m,n : \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^{q_{k,l}} \ge \varepsilon\right\}\right|
\end{aligned}$$

$$\Rightarrow \frac{1}{mn}\left|\left\{k,l \le m,n : \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^{p_{k,l}} \ge \varepsilon\right\}\right|$$
$$\le \frac{1}{mn}\left|\left\{k,l \le m,n : \left[M_{k,l}\left(\frac{\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l}}{\rho}\right)\right]^{q_{k,l}} \ge \varepsilon\right\}\right|.$$

But $1/(mn)\left|\left\{k,l \le m,n : \left[M_{k,l}\left((\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l})/\rho\right)\right]^{q_{k,l}} \ge \varepsilon\right\}\right| \to 0$ as $m,n \to \infty$ by (3.2). Therefore $1/(mn)\left|\left\{k,l \le m,n : \left[M_{k,l}\left((\bar{d}(\Delta^r X_{k,l},\bar{0})^{1/k+l})/\rho\right)\right]^{p_{k,l}} \ge \varepsilon\right\}\right| \to 0$ as $m,n \to \infty$. Hence

$$X \in \Gamma^2_{\mathscr{M}}[F,p,\Delta^r]. \tag{3.4}$$

From (3.1) and (3.4) we get $\Gamma^2_{\mathscr{M}}[F,q,\Delta^r] \subset \Gamma^2_{\mathscr{M}}[F,p,\Delta^r]$. This completes the proof of the theorem. ∎

Acknowledgement. The authors thank the referees for their valuable suggestions that improved the presentation of the paper.

References

[1] B. Altay and F. Başar, Some new spaces of double sequences, *J. Math. Anal. Appl.* **309** (2005), no. 1, 70–90.

[2] H. Altınok and M. Mursaleen, Delta-statistically boundedness for sequences of fuzzy numbers, *Taiwanese J. Math.* **15** (2011), no. 5, 2081–2093.

[3] M. Basarir and O. Sonalcan, On some double sequence spaces, *J. Indian Acad. Math.* **21** (1999), no. 2, 193–200.

[4] F. Başar and Y. Sever, The space rL_q of double sequences, *Math. J. Okayama Univ.* **51** (2009), 149–157.

[5] T. J. Bromwich, *An Introduction to the Theory of Infinite Series*, Macmillan and co. Ltd., New York, 1965.

[6] R. C. Buck, Generalized asymptotic density, *Amer. J. Math.* **75** (1953), 335–346.

[7] R. Çolak, Y. Altın and M. Mursaleen, On some sets of difference sequences of fuzzy numbers, *Soft Computing* **15** (2011), 787-793.

[8] R. Çolak and M. Et, On some generalized difference sequence spaces and related matrix transformations, *Hokkaido Math. J.* **26** (1997), no. 3, 483–492.

[9] H. Fast, Sur la convergence statistique, *Colloquium Math.* **2** (1951), 241–244 (1952).

[10] A. Gökhan, M. Et and M. Mursaleen, Almost lacunary statistical and strongly almost lacunary convergence of sequences of fuzzy numbers, *Math. Comput. Modelling* **49** (2009), no. 3-4, 548–555.

[11] G. H. Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil., Soc.* **19** (1917), 86–95.

[12] H. Kızmaz, On certain sequence spaces, *Canad. Math. Bull.* **24** (1981), no. 2, 169–176.

[13] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.* **10** (1971), 379–390.

[14] I. J. Maddox, *Elements of Functional Analysis*, Cambridge Univ. Press, London, 1970.

[15] I. J. Maddox, Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc.* **100** (1986), no. 1, 161–166.

[16] E. Malkowsky, M. Mursaleen and S. Suantai, The dual spaces of sets of difference sequences of order m and matrix transformations, *Acta Math. Sin. (Engl. Ser.)* **23** (2007), no. 3, 521–532.

[17] M. Matloka, Sequences of fuzzy numbers, *BUSEFAL* **28** (1986), 28–37.

[18] F. Móricz, Extensions of the spaces c and c_0 from single to double sequences, *Acta Math. Hungar.* **57** (1991), no. 1-2, 129–136.

[19] F. Móricz and B. E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Cambridge Philos. Soc.* **104** (1988), no. 2, 283–294.

[20] M. Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.* **293** (2004), no. 2, 523–531.

[21] M. Mursaleen, Generalized spaces of difference sequences, *J. Math. Anal. Appl.* **203** (1996), no. 3, 738–745.

[22] M. Mursaleen and M. Başarir, On some new sequence spaces of fuzzy numbers, *Indian J. Pure Appl. Math.* **34** (2003), no. 9, 1351–1357.

[23] M. Mursaleen and O. H. H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.* **288** (2003), no. 1, 223–231.

[24] M. Mursaleen and O. H. H. Edely, Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.* **293** (2004), no. 2, 532–540.

[25] H. Nakano, Concave modulars, *J. Math. Soc. Japan* **5** (1953), 29–49.

[26] F. Nuray and E. Savaş, Statistical convergence of sequences of fuzzy numbers, *Math. Slovaca* **45** (1995), no. 3, 269–273.

[27] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, *Math. Ann.* **53** (1900), no. 3, 289–321.

[28] K. Raj, A. K. Sharma and S. K. Sharma, A sequence space defined by Musielak-Orlicz function, *Int. J. Pure Appl. Math.* **67** (2011), no. 4, 475–484.

[29] K. Raj, S. K. Sharma and A. K. Sharma, Difference sequence spaces in n-normed spaces defined by Musielak-Orlicz function, *Armen. J. Math.* **3** (2010), no. 3, 127–141.

[30] K. Raj and S. K. Sharma, Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function, *Acta Univ. Sapientiae Math.* **3** (2011), no. 1, 97–109.

[31] E. Savaş, New double sequence spaces of fuzzy numbers, *Quaest. Math.* **33** (2010), no. 4, 449–456.

[32] E. Savaş and M. Mursaleen, On statistically convergent double sequences of fuzzy numbers, *Inform. Sci.* **162** (2004), no. 3-4, 183–192.

[33] Ö. Talo and F. Başar, Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations, *Comput. Math. Appl.* **58** (2009), no. 4, 717–733.

[34] B. C. Tripathy, Generalized difference paranormed statistically convergent sequences defined by Orlicz function in a locally convex space, *Soochow J. Math.* **30** (2004), no. 4, 431–446.

[35] B. C. Tripathy, Statistically convergent double sequences, *Tamkang J. Math.* **34** (2003), no. 3, 231–237.

[36] A. Wilansky, *Summability Through Functional Analysis*, North-Holland Mathematics Studies, 85, North-Holland, Amsterdam, 1984.

[37] L. A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.

[38] M. Zeltser, *Investigation of Double Sequence Spaces by Soft and Hard Analytical Methods*, Dissertationes Mathematicae Universitatis Tartuensis, 25, Tartu Univ. Press, Tartu, 2001.