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On the Coefficients of Integrated Expansions and Integrals of Chebyshev Polynomials of Third and Fourth Kinds

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Abstract. Two new analytical closed formulae expressing explicitly third and fourth kinds Chebyshev coefficients of an expansion for an infinitely differentiable function that has been integrated an arbitrary number of times in terms of the original expansion coefficients of the function are stated and proved. Hence, two new formulae expressing explicitly the integrals of third and fourth kinds Chebyshev polynomials of any degree that has been integrated an arbitrary number of times in terms of third and fourth kinds Chebyshev polynomials themselves are also given. New reduction formulae for summing some terminating hypergeometric functions of unit argument are deduced. As an application of how to use Chebyshev polynomials of third and fourth kinds and their shifted polynomials for solving high-order boundary value problems, two numerical solutions of sixth-order boundary value problem are presented and implemented based on applying spectral Galerkin method. Also, two numerical examples are presented, aiming to demonstrate the accuracy and the efficiency of the formulae we have obtained.

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1. Introduction

The Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with Chebyshev polynomials, contain mainly results of Chebyshev polynomials of the first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications, (see for example, [3, 19, 24]). However, there is only a very limited body of literature on Chebyshev polynomials of third and fourth kinds $V_n(x)$ and $W_n(x)$, either from theoretical or practical points of view and their uses in various applications (see, for instance Eslahchi *et al.* [18]). The interested reader in

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Chebyshev polynomials of third and fourth kinds is referred to the excellent book of Mason and Handscomb [26].

If we were asked for "a pecking order" of these four Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$, then we would say that $T_n(x)$ is the most important and versatile. Moreover $T_n(x)$ generally leads to the simplest formulae, whereas results for the other polynomials may involve slight complications. However, all the four kinds of Chebyshev polynomials have their role. For example, $U_n(x)$ is useful in numerical integration (see [25]), while $V_n(x)$ and $W_n(x)$ can be useful in situations in which singularities occur at one end point (+1 or -1) but not at the other (see [26]).

Classical orthogonal polynomials are used successfully and extensively for the numerical solution of linear and nonlinear differential equations (see for instance [2, 4, 11–13, 15–17, 20]).

For spectral and pseudospectral methods; explicit formulae for the expansion coefficients of the derivatives (integrals) in terms of the original expansion coefficients of the function are needed. Formulae for the expansion coefficients of a general order derivative of an infinitely differentiable function in terms of those of the function are available for expansions in Chebyshev [21], Legendre [27], ultraspherical [5,22], Jacobi [7], Laguerre [9], Hermite [10] and Bessel [14] polynomials.

As an alternative approach to differentiating solution expansions is to integrate the differential equation q times, where q is the order of the equation. An advantage of this approach is that the general equation in the algebraic system contains a finite number of terms. Phillips and Karageorghis [28] have followed this approach to obtain a formula for the coefficients of an expansion of ultraspherical polynomials that has been integrated an arbitrary number of times in terms of the coefficients of the original expansion. Doha [6] proved the same formula but in a simpler way than the formula suggested by Phillips and Karageorghis [28]. Also Doha proved a more general formula for Jacobi polynomials in [8], in which the q times repeated integrals for Jacobi polynomials are given in terms of hypergeometric series of type ${}_3F_2(1)$ which can not be summed in closed form except for certain special values of its parameters. In [15], Doha and Bhrawy used the expressions for the q repeated integrals of Jacobi polynomials, for solving the integrated forms of fourth-order differential equations by using the Galerkin method, and they showed that the resulted systems are cheaper than those obtained from applying the Galerkin method to solve the differentiated ones. This motivates our interest in deriving the qth repeated integration for like Chebyshev polynomials of third and fourth kinds.

Up to now, and to the best of our knowledge, no closed analytical formulae for the coefficients of integrated expansions and integrals of Chebyshev polynomials of third and fourth kinds are known yet and are traceless in the literature. This also motivates our interest in such polynomials.

The structure of the paper is as follows. In Section 2, we give some relevant properties of Chebyshev polynomials of third and fourth kinds and their shifted polynomials. In Section 3, we state and prove two theorems, in the first one, third kind Chebyshev coefficients of an expansion for an infinitely differentiable function that has been integrated an arbitrary number of times is given in terms of third kind Chebyshev coefficients of the original expansion of the function, and in the second, we give explicitly the q repeated integration of Chebyshev polynomial of third kind of any degree in terms of the Chebyshev polynomials of third kind themselves. The corresponding formulae to those obtained in Section 3, for Chebyshev polynomials of the fourth kind are presented in Section 4. Two new reduction formulae for summing some terminating hypergeometric functions of the type ${}_3F_2(1)$ are given in Section 5. In Section 6, we present and implement two numerical spectral solutions of sixth-order two point boundary value problems using shifted Chebyshev third kind-Galerkin method (SC3GM) and shifted Chebyshev fourth kind-Galerkin method (SC4GM). In Section 7, two numerical examples are presented to show the accuracy and the efficiency of the two proposed algorithms of Section 6. Some concluding remarks are given in Section 8.

2. Some properties of Chebyshev polynomials of third and fourth kinds

The Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of third and fourth kinds are polynomials in x defined respectively by [26]

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}} \quad \text{and} \quad W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}},$$

where $x = \cos\theta$. The polynomials $V_n(x)$ and $W_n(x)$ are, in fact, rescalings of two particular Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for the two nonsymmetric special cases $\beta = -\alpha = \pm 1/2$. These are given explicitly by

$$V_n(x) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{(-\frac{1}{2},\frac{1}{2})}(x) \tag{2.1}$$

and

$$W_n(x) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{(\frac{1}{2},-\frac{1}{2})}(x). \tag{2.2}$$

It is readily seen that

$$W_n(x) = (-1)^n V_n(-x)$$

and therefore, it is sufficient to establish properties for $V_n(x)$, and hence deduce analogous properties for $W_n(x)$(replacing x by $-x$).

The polynomials $V_n(x)$ and $W_n(x)$ are orthogonal on $(-1,1)$ with respect to the weight functions $\sqrt{(1+x)/(1-x)}$ and $\sqrt{(1-x)/(1+x)}$, respectively, i.e.,

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\, V_m(x)\, V_n(x)\, dx = \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\, W_m(x)\, W_n(x)\, dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n, \end{cases}$$

and they are may be generated by using the two recurrence relations

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \quad n = 2,3,\ldots, \quad \text{with} \quad V_0(x) = 1, \quad V_1(x) = 2x-1,$$

and

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x), \quad n = 2,3,\dots, \quad \text{with} \quad W_0(x) = 1, \quad W_1(x) = 2x+1.$$

The following two structure formulae are useful in the sequel,

$$V_n(x) = \frac{1}{2n(n+1)}\left[n\,DV_{n+1}(x) - DV_n(x) - (n+1)\,DV_{n-1}(x)\right], \quad n \geq 1, \tag{2.3}$$

$$W_n(x) = \frac{1}{2n(n+1)}\left[n\,DW_{n+1}(x) + DW_n(x) - (n+1)\,DW_{n-1}(x)\right], \quad n \geq 1,$$

with $D \equiv d/(dx)$.

2.1. Shifted Chebyshev polynomials of third and fourth kinds

The shifted Chebyshev polynomials of third and fourth kinds are defined on $[a,b]$, respectively as

$$V_n^*(x) = V_n\left(\frac{2x-a-b}{b-a}\right), \quad W_n^*(x) = W_n\left(\frac{2x-a-b}{b-a}\right).$$

All properties of Chebyshev polynomials of third and fourth kinds, can be easily transformed to give the corresponding properties for their shifted polynomials.

The orthogonality relations of $V_k^*(x)$ and $W_k^*(x)$ on $[a,b]$ with respect to the weight functions $\sqrt{(x-a)/(b-x)}$ and $\sqrt{(b-x)/(x-a)}$, are given by

$$\int_a^b \sqrt{\frac{x-a}{b-x}}\, V_k^*(x)\, V_j^*(x)\, dx = \int_a^b \sqrt{\frac{b-x}{x-a}}\, W_k^*(x)\, W_j^*(x)\, dx = \begin{cases} (b-a)\dfrac{\pi}{2}, & k = j, \\ 0, & k \neq j. \end{cases}$$

3. The coefficients of integrated expansion and integrals of Chebyshev polynomials of third kind

3.1. The coefficients of integrated expansion of $V_n(x)$

Following Phillips and Karageorghis [28] and Doha [6], and let $b_n^{(q)}$, $q \geq 1$, denotes the third kind Chebyshev expansion coefficients of $f(x)$, $x \in [-1,1]$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} b_n^{(q)} V_n(x), \tag{3.1}$$

and let $f(x)$ be an infinitely differentiable function, then we may express the ℓth derivative of $f(x)$ in the form

$$f^{(\ell)}(x) = \sum_{n=0}^{\infty} b_n^{(q-\ell)} V_n(x), \quad \ell \geq 0, \tag{3.2}$$

and in particular

$$f^{(q)}(x) = \sum_{n=0}^{\infty} b_n V_n(x), \quad b_n = b_n^{(0)}. \tag{3.3}$$

It is clear from Equations (3.1) and (3.2) that

$$\sum_{n=0}^{\infty} b_n^{(q)} \frac{dV_n(x)}{dx} = \sum_{n=0}^{\infty} b_n^{(q-1)} V_n(x), \tag{3.4}$$

then, if we substitute the identity (2.3) into Equation (3.4), we get the following difference equation

$$b_n^{(q)} = \frac{1}{2n} b_{n-1}^{(q-1)} - \frac{1}{2n(n+1)} b_n^{(q-1)} - \frac{1}{2(n+1)} b_{n+1}^{(q-1)}, \quad \forall\, n \geq 1,\ q \geq 1. \tag{3.5}$$

Now, we prove the following theorem.

Theorem 3.1. *Let $f(x)$ be an infinitely differentiable function defined on $[-1,1]$. Then the third kind Chebyshev coefficients $b_n^{(q)}$ of $f(x)$ are related to third kind Chebyshev coefficients b_n of the qth derivative of $f(x)$ by*

$$b_n^{(q)} = \sum_{m=0}^{q} A_{m,n,q}\, b_{2m+n-q} + \sum_{m=1}^{q} B_{m,n,q}\, b_{2m+n-q-1}, \tag{3.6}$$

where

$$A_{m,n,q} = \frac{(-1)^m\, q!\,(m+n-q)!}{2^q\, m!\,(m+n)!\,(q-m)!} \quad \text{and} \quad B_{m,n,q} = \frac{(-1)^m\, q!\,(m+n-q-1)!}{2^q\,(m-1)!\,(m+n)!\,(q-m)!}.$$

Proof. We proceed by induction on q. For $q=1$, the application of formula (3.5) yields the required result. Assuming that the theorem is valid for q, we have to show that it is true for $q+1$, i.e.,

$$b_n^{(q+1)} = \sum_{m=0}^{q+1} A_{m,n,q+1}\, b_{2m+n-q-1} + \sum_{m=1}^{q+1} B_{m,n,q+1}\, b_{2m+n-q-2}. \tag{3.7}$$

Replacing q by $q+1$ in (3.5) leads to

$$b_n^{(q+1)} = \frac{1}{2n} b_{n-1}^{(q)} - \frac{1}{2n(n+1)} b_n^{(q)} - \frac{1}{2(n+1)} b_{n+1}^{(q)}, \quad n = 1,2,\ldots. \tag{3.8}$$

Thus after the application of the induction hypothesis, the right hand side of (3.8) becomes

$$\begin{aligned} b_n^{(q+1)} =& \frac{1}{2n}\left\{\sum_{m=0}^{q} A_{m,n-1,q}\, b_{2m+n-q-1} + \sum_{m=1}^{q} B_{m,n-1,q}\, b_{2m+n-q-2}\right\} \\ &- \frac{1}{2n(n+1)}\left\{\sum_{m=0}^{q} A_{m,n,q}\, b_{2m+n-q} + \sum_{m=1}^{q} B_{m,n,q}\, b_{2m+n-q-1}\right\} \\ &- \frac{1}{2(n+1)}\left\{\sum_{m=0}^{q} A_{m,n+1,q}\, b_{2m+n-q+1} + \sum_{m=1}^{q} B_{m,n+1,q}\, b_{2m+n-q}\right\}. \end{aligned}$$

The last equation can be written in the form

$$b_n^{(q+1)} = \sum_1 + \sum_2,$$

where

$$\sum_1 = \frac{1}{2n} A_{0,n-1,q}\, b_{n-q-1} - \frac{1}{2(n+1)} A_{q,n+1,q}\, b_{n+q+1} + \sum_{m=1}^{q} \left\{ \frac{1}{2n} A_{m,n-1,q} - \frac{1}{2n(n+1)} B_{m,n,q} - \frac{1}{2(n+1)} A_{m-1,n+1,q} \right\} b_{2m+n-q-1},$$

and

$$\sum_2 = \left\{ \frac{1}{2n} B_{1,n-1,q} - \frac{1}{2n(n+1)} A_{0,n,q} \right\} b_{n-q} - \left\{ \frac{1}{2n(n+1)} A_{q,n,q} + \frac{1}{2(n+1)} B_{q,n+1,q} \right\} b_{n+q} + \sum_{m=2}^{q} \left\{ \frac{1}{2n} B_{m,n-1,q} - \frac{1}{2n(n+1)} A_{m-1,n,q} - \frac{1}{2(n+1)} B_{m-1,n+1,q} \right\} b_{2m+n-q-2}.$$

It is not difficult to see that

$$A_{0,n,q+1} = \frac{1}{2n} A_{0,n-1,q}, \quad A_{q+1,n,q+1} = \frac{-1}{2(n+1)} A_{q,n+1,q},$$

$$A_{m,n,q+1} = \frac{1}{2n} A_{m,n-1,q} - \frac{1}{2n(n+1)} B_{m,n,q} - \frac{1}{2(n+1)} A_{m-1,n+1,q},$$

and

$$B_{1,n,q+1} = \frac{1}{2n} B_{1,n-1,q} - \frac{1}{2n(n+1)} A_{0,n,q},$$

$$B_{q+1,n,q} = -\left\{ \frac{1}{2n(n+1)} A_{q,n,q} + \frac{1}{2(n+1)} B_{q,n+1,q} \right\},$$

$$B_{m,n,q+1} = \frac{1}{2n} B_{m,n-1,q} - \frac{1}{2n(n+1)} A_{m-1,n,q} - \frac{1}{2(n+1)} B_{m-1,n+1,q}.$$

Therefore, we have

$$\sum_1 = \sum_{m=0}^{q+1} A_{m,n,q+1}\, b_{2m+n-q-1}, \quad \sum_2 = \sum_{m=1}^{q+1} B_{m,n,q+1}\, b_{2m+n-q-2},$$

and then

$$b_n^{(q+1)} = \sum_{m=0}^{q+1} A_{m,n,q+1}\, b_{2m+n-q-1} + \sum_{m=1}^{q+1} B_{m,n,q+1}\, b_{2m+n-q-2}.$$

This proves relation (3.7), and hence completes the proof of the Theorem. ∎

Remark 3.1. It is to be noted here that relation (3.6) may be written in the alternative equivalent form

$$b_n^{(q)} = \sum_{j=0}^{2q} G_{j,n,q}\, b_{j+n-q}, \tag{3.9}$$

where

$$G_{j,n,q} = \frac{q!}{2^q} \begin{cases} \dfrac{(-1)^{\frac{j}{2}} \left(n-q+\frac{j}{2}\right)!}{\left(\frac{j}{2}\right)! \left(n+\frac{j}{2}\right)! \left(q-\frac{j}{2}\right)!}, & \text{j even}, \\ \dfrac{(-1)^{\frac{j+1}{2}} \left(n-q+\frac{j-1}{2}\right)!}{\left(\frac{j-1}{2}\right)! \left(n+\frac{j+1}{2}\right)! \left(q-\left(\frac{j+1}{2}\right)\right)!}, & \text{j odd}. \end{cases} \tag{3.10}$$

3.2. Computation of q times repeated integration of $V_n(x)$

Theorem 3.2. *If we define q times repeated integration of $V_n(x)$ by*

$$I_n^{(q)}(x) = \overbrace{\int\int\cdots\int}^{q \text{ times}} V_n(x) \overbrace{dx\,dx\cdots dx}^{q \text{ times}},$$

then

$$I_n^{(q)}(x) = \sum_{k=q}^{n+q} E_{k,n,q} V_k(x) + \pi_{q-1}(x), \quad n \geq q \geq 1, \tag{3.11}$$

where

$$E_{k,n,q} = \frac{q!}{2^q} \begin{cases} \dfrac{(-1)^{\frac{n-k+q}{2}} \left(\frac{n+k-q}{2}\right)!}{\left(\frac{k-n+q}{2}\right)! \left(\frac{n-k+q}{2}\right)! \left(\frac{k+n+q}{2}\right)!}, & (n+k+q) \textit{ even}, \\ \dfrac{(-1)^{\frac{n-k+q+1}{2}} \left(\frac{n+k-q-1}{2}\right)!}{\left(\frac{k-n+q-1}{2}\right)! \left(\frac{n-k+q-1}{2}\right)! \left(\frac{k+n+q+1}{2}\right)!}, & (n+k+q) \textit{ odd}, \end{cases} \tag{3.12}$$

and $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

Proof. If we integrate Equation (3.3) q times with respect to x, we get

$$f(x) = \sum_{n=0}^{\infty} b_n I_n^{(q)}(x) + \bar{\pi}_{q-1}(x), \tag{3.13}$$

where $\bar{\pi}_{q-1}(x)$ is a polynomial of degree at most $(q-1)$. Making use of formula (3.9) and substitution into (3.1), gives

$$f(x) = \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{2q} G_{j,n,q} b_{n+j-q} \right\} V_n(x). \tag{3.14}$$

Expanding (3.14) and collecting similar terms, enables one to put Equation (3.14) in the form

$$f(x) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n+q} G_{n-k+q,k,q} V_k \right\} b_n,$$

then comparison with Equation (3.13) yields

$$I_n^{(q)}(x) = \sum_{k=q}^{n+q} G_{k,n-k+q,q} V_k(x) + \pi_{q-1}(x), \tag{3.15}$$

where $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$. Equation (3.15) may be written as

$$I_n^{(q)}(x) = \sum_{k=q}^{n+q} E_{k,n,q} V_k(x) + \pi_{q-1}(x),$$

where $E_{k,n,q}$ is given by (3.12). This completes the proof of Theorem 3.2. ∎

As an immediate consequence of Theorem 3.2, the q times repeated integration of the shifted Chebyshev third kind $V_n^*(x)$ can be easily obtained. This result is given in the following corollary.

Corollary 3.1. *If we define the q times repeated integration of $V_n^*(x)$ by*

$$\bar{I}_n^{(q)}(x) = \overbrace{\int\int\cdots\int}^{q \text{ times}} V_n^*(x) \overbrace{dx\,dx\cdots dx}^{q \text{ times}},$$

then

$$\bar{I}_n^{(q)}(x) = \left(\frac{b-a}{2}\right)^q \sum_{k=q}^{n+q} E_{k,n,q} V_k^*(x) + \sigma_{q-1}(x),$$

where $E_{k,n,q}$ is as given in (3.12)*, and $\sigma_{q-1}(x)$ a polynomial of degree at most $(q-1)$.*

4. The coefficients of integrated expansions and integrals of $W_n(x)$

Let $c_n^{(q)}$, $q \geq 1$, denote the fourth kind Chebyshev expansion coefficients of $f(x), x \in [-1,1]$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} c_n^{(q)} W_n(x),$$

and let $f(x)$ be an infinitely differentiable function, then we may express the qth derivative of $f(x)$ in the form

$$f^{(q)}(x) = \sum_{n=0}^{\infty} c_n W_n(x), \quad c_n = c_n^{(0)}.$$

Following a similar procedure to that followed in Section 3.1, we get the following difference equation

$$c_n^{(q)} = \frac{1}{2n} c_{n-1}^{(q-1)} + \frac{1}{2n(n+1)} c_n^{(q-1)} - \frac{1}{2(n+1)} c_n^{(q-1)}, \quad \forall n \geq 1,\ q \geq 1.$$

Now, we give without proof the corresponding results to those given in Section 3.

Theorem 4.1. *Let $f(x)$ be an infinitely differentiable function defined on $[-1,1]$. The fourth kind Chebyshev coefficients $c_n^{(q)}$ of $f(x)$ are related to fourth kind Chebyshev coefficients c_n of the qth derivative of $f(x)$ by*

$$c_n^{(q)} = \sum_{j=0}^{2q} M_{j,n,q}\, c_{j+n-q}, \quad \text{where} \quad M_{n,j,q} = (-1)^j G_{j,n,q},$$

and $G_{j,n,q}$ is as defined in (3.10).

Theorem 4.2. *If we define q times repeated integration of $W_n(x)$ by*

$$J_n^{(q)}(x) = \overbrace{\int\int\cdots\int}^{q\ times} W_n(x)\overbrace{dx\,dx\cdots dx}^{q\ times},$$

then

$$J_n^{(q)}(x) = \sum_{k=q}^{n+q} S_{k,n,q}\,W_k(x) + \rho_{q-1}(x), \tag{4.1}$$

where

$$S_{k,n,q} = (-1)^{n+k+q}E_{k,n,q}. \tag{4.2}$$

and $\rho_{q-1}(x)$ is a polynomial of degree at most $(q-1)$ and $E_{k,n,q}$ is as defined in (3.12).

Corollary 4.1. *If we define the q times repeated integration of $W_n^*(x)$ by*

$$\bar{J}_n^{(q)}(x) = \overbrace{\int\int\cdots\int}^{q\ times} W_k^*(x)\overbrace{dx\,dx\cdots dx}^{q\ times},$$

then

$$\bar{J}_n^{(q)}(x) = \left(\frac{b-a}{2}\right)^q \sum_{k=q}^{n+q} S_{k,n,q}\,W_k^*(x) + \delta_{q-1}(x),$$

where $S_{k,n,q}$ is as given in (4.2)*, and $\delta_{q-1}(x)$ a polynomial of degree at most $(q-1)$.*

5. Reduction formulae for terminating hypergeometric functions of the type ${}_3F_2(1)$

In [7], a formula expressing explicitly the integrals of Jacobi polynomials of any degree and for any order in terms of the Jacobi polynomials themselves is given. This result is stated in the following theorem.

Theorem 5.1. *If we define the q times repeated integration of the classical Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ by*

$$I_n^{(q,\alpha,\beta)}(x) = \overbrace{\int\int\cdots\int}^{q\ times} P_n^{(\alpha,\beta)}(x)\overbrace{dx\,dx\cdots dx}^{q\ times},$$

then

$$I_n^{(q,\alpha,\beta)}(x) = \frac{2^q}{(n-q+\alpha+\beta+1)_q}\sum_{k=q}^{n+q} C_{n+q,k}(\alpha-q,\beta-q,\alpha,\beta)P_k^{(\alpha,\beta)}(x) + \pi_{q-1}(x),$$

$q \geq 0,\ n \geq q+1$ *for* $\alpha = \beta = -\frac{1}{2}$; $q \geq 0,\ n \geq q$ *for* $\alpha \neq -\frac{1}{2}$ *or* $\beta \neq -\frac{1}{2}$, *and*

(5.1)

$$C_{n+q,k}(\alpha-q,\beta-q,\alpha,\beta) = \frac{(n-q+\alpha+\beta+1)_k\ (k-q+\alpha+1)_{n-k+q}\ \Gamma(k+\alpha+\beta+1)}{(n-k+q)!\ \Gamma(2k+\alpha+\beta+1)}$$
$$\times {}_3F_2\left(\begin{matrix} k-n-q,\ k+n-q+\alpha+\beta+1,\ k+\alpha+1 \\ k-q+\alpha+1,\ 2k+\alpha+\beta+2 \end{matrix}\ \middle|\ 1\right).$$

Remark 5.1. It is to be noted here that although the ${}_3F_2(1)$ in (5.1) is terminated, it can not be summed in closed form except for certain special values of its parameters. For the special case correspond to $\beta = \alpha$, this ${}_3F_2(1)$ can be summed in a closed form with the aid of Watson's identity (see [8]). The two formulae (3.11) and (4.1) enable one to deduce two new closed forms for the ${}_3F_2(1)$ in (5.1) for the two nonsymmetric cases correspond to $\beta = -\alpha = \pm 1/2$. These two reduction formulae are given in the following corollary.

Corollary 5.1. *For all $k,n,q \in \mathbb{Z}^{\geq 0}$ and $k \leq n+q$, we have*

$$
(5.2)\quad {}_3F_2\left(\begin{matrix} k-n-q, k+n-q+1, k+\frac{1}{2} \\ k-q+\frac{1}{2}, 2k+2 \end{matrix} \,\middle|\, 1 \right) = \frac{(2k+1)!\,q!\,\Gamma(k-q+\frac{1}{2})}{2^{2k+1}\Gamma(k+\frac{3}{2})} \times
\begin{cases} \dfrac{(-1)^{\frac{n-k+q}{2}}\Gamma\left(\frac{n-k+q+1}{2}\right)}{\Gamma\left(\frac{n+k-q+1}{2}\right)\Gamma\left(\frac{k-n+q+2}{2}\right)\Gamma\left(\frac{k+n+q+2}{2}\right)}, & (k+n+q)\ \textit{even}, \\[2ex] \dfrac{(-1)^{\frac{n-k+q+1}{2}}\Gamma\left(\frac{n-k+q+2}{2}\right)}{\Gamma\left(\frac{n+k-q+2}{2}\right)\Gamma\left(\frac{k-n+q+1}{2}\right)\Gamma\left(\frac{k+n+q+3}{2}\right)}, & (k+n+q)\ \textit{odd}, \end{cases}
$$

and

$$
(5.3)\quad {}_3F_2\left(\begin{matrix} k-n-q, k+n-q+1, k+\frac{3}{2} \\ k-q+\frac{3}{2}, 2k+2 \end{matrix} \,\middle|\, 1 \right) = \frac{(-1)^{n+k+q}(2k+1)!\,q!\,\Gamma(k-q+\frac{3}{2})}{2^{2k}(2n+1)\Gamma(k+\frac{3}{2})} \times
\begin{cases} \dfrac{(-1)^{\frac{n-k+q}{2}}\Gamma\left(\frac{n-k+q+1}{2}\right)}{\Gamma\left(\frac{n+k-q+1}{2}\right)\Gamma\left(\frac{k-n+q+2}{2}\right)\Gamma\left(\frac{k+n+q+2}{2}\right)}, & (k+n+q)\ \textit{even}, \\[2ex] \dfrac{(-1)^{\frac{n-k+q+1}{2}}\Gamma\left(\frac{n-k+q+2}{2}\right)}{\Gamma\left(\frac{n+k-q+2}{2}\right)\Gamma\left(\frac{k-n+q+1}{2}\right)\Gamma\left(\frac{k+n+q+3}{2}\right)}, & (k+n+q)\ \textit{odd}. \end{cases}
$$

Proof. Substituting by the two identities (2.1) and (2.2) in the two formulae (3.11) and (4.1), and comparing the results with those obtained from Theorem 5.1 for the two special cases correspond to $\alpha = -\beta = -1/2$ and $\alpha = -\beta = 1/2$ respectively, the two reduction formulae (5.2) and (5.3) can be immediately deduced. ∎

Remark 5.2. From the two identities (5.2) and (5.3), the following transformation formula holds for all $k,n,q \in \mathbb{Z}^{\geq 0}$ and $k \leq n+q$,

$$
{}_3F_2\left(\begin{matrix} k-n-q, k+n-q+1, k+\frac{3}{2} \\ k-q+\frac{3}{2}, 2k+2 \end{matrix} \,\middle|\, 1 \right)
= \frac{(-1)^{n+k+q}(2k-2q+1)}{2n+1}\, {}_3F_2\left(\begin{matrix} k-n-q, k+n-q+1, k+\frac{1}{2} \\ k-q+\frac{1}{2}, 2k+2 \end{matrix} \,\middle|\, 1 \right).
$$

6. Solution of the integrated forms of sixth-order two point boundary value problem

Even order boundary value problems of higher order have been investigated by a large number of authors because of both their mathematical importance and their potential for applications in hydrodynamic and hydromagnetic stability. Sixth-order boundary-value problems (BVPs) are known to arise in astrophysics; the narrow convecting layers bounded by stable layers, which are believed to surround A-type stars, may be modeled by sixth-order BVPs (see, for instance, [1,23]).

In this section, we are interested in using SC3GM and SC4GM to solve the following sixth-order two point boundary value problem:

$$-u^{(6)}(x)+\sum_{i=0}^{5} c_i\, u^{(i)}(x)=f(x), \quad a<x<b, \tag{6.1}$$

subject to the nonhomogeneous boundary conditions

$$u^{(j)}(a)=\alpha_j, \quad u^{(j)}(b)=\beta_j, \quad j=0,1,2. \tag{6.2}$$

In such case and with the aid of a suitable transformation, namely

$$U(x)=u(x)+\sum_{i=0}^{5}\gamma_i x^i,$$

where $\gamma_i,\ i=0,1,\dots,5$, are coefficients should be determined such that $U(x)$ satisfies the homogeneous boundary conditions, namely

$$U^{(j)}(a)=U^{(j)}(b)=0, \quad j=0,1,2. \tag{6.3}$$

It can be easily shown that problem (6.1) subject to the nonhomogeneous boundary conditions (6.2) is equivalent to a modified problem of the form

$$-U^{(6)}(x)+\sum_{i=0}^{5} c_i\, U^{(i)}(x)=g(x), \quad a<x<b, \tag{6.4}$$

subject to the homogeneous boundary conditions (6.3),where

$$g(x)=f(x)+\sum_{i=0}^{5} d_i x^i,$$

and $d_i,\ 0\le i\le 5$ are some constants that should be determined as part of the solution.
Now, we consider the integrated form for Equation (6.4) subject to the homogeneous boundary conditions (6.3), namely,

$$\left.\begin{aligned} &-U(x)+\sum_{i=0}^{5} c_i \int^{(6-i)} U(x)(dx)^{6-i}=h(x)+\sum_{i=0}^{5} e_i x^i, \quad x\in(a,b),\\ &U^{(j)}(a)=U^{(j)}(b)=0, \quad j=0,1,2, \quad h(x)=\int^{(6)} g(x)(dx)^6, \end{aligned}\right\} \tag{6.5}$$

where

$$\int^{(q)} U(x)(dx)^q = \overbrace{\int\int\cdots\int}^{q\ times} U(x)\overbrace{dx\,dx\cdots dx}^{q\ times}.$$

Now, define the following spaces

$$\begin{aligned}
S_N &= \operatorname{span}\{V_0^*(x), V_1^*(x), V_2^*(x), \dots, V_N^*(x)\},\\
\bar{S}_N &= \operatorname{span}\{W_0^*(x), W_1^*(x), W_2^*(x), \dots, W_N^*(x)\},\\
Y_N &= \{y(x) \in S_N \ : D^j y(\pm 1) = 0,\ j = 0,1,2\},\\
\bar{Y}_N &= \{\bar{y}(x) \in \bar{S}_N \ : D^j \bar{y}(\pm 1) = 0,\ j = 0,1,2\},
\end{aligned}$$

then, the shifted Chebyshev third kind-Galerkin and shifted Chebyshev fourth kind-Galerkin procedures for solving (6.5) are to find $U_N(x) \in Y_N$ and $\bar{U}_N(x) \in \bar{Y}_N$ such that

$$\begin{aligned}
&-\big(U_N(x), y(x)\big)_{w_1} + \sum_{i=0}^{5} c_i \left(\int^{6-i} U_N(x)(dx)^{6-i}, y(x)\right)_{w_1}\\
&\qquad = \left(h(x) + \sum_{i=0}^{5} r_i V_i^*(x), y(x)\right)_{w_1}, \quad \forall\, y(x) \in Y_N,
\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}
&-\big(\bar{U}_N(x), \bar{y}(x)\big)_{w_2} + \sum_{i=0}^{5} c_i \left(\int^{(6-i)} \bar{U}_N(x)(dx)^{6-i}, \bar{y}(x)\right)_{w_2}\\
&\qquad = \left(h(x) + \sum_{i=0}^{5} \bar{r}_i W_i^*(x), \bar{y}(x)\right)_{w_2}, \quad \forall\, \bar{y}(x) \in \bar{Y}_N,
\end{aligned} \tag{6.7}$$

where $(u,v)_w = \int_a^b w\,u\,v\,dx$, is the scalar inner product in the weighted space $L_w^2(a,b)$, $w_1 = \sqrt{(x-a)/(b-x)}$ and $w_2 = \sqrt{(b-x)/(x-a)}$.
Now, we choose the following two bases functions $\phi_k(x) \in Y_N$ and $\psi_k(x) \in \bar{Y}_N$ to be of the forms

$$\phi_k(x) = \sum_{m=0}^{6} d_{m,k} V_{k+m}^*(x), \quad \psi_k(x) = \sum_{m=0}^{6} \bar{d}_{m,k} W_{k+m}^*(x), \quad k = 0,1,\dots,N-6,$$

where

$$d_{m,k} = (k+1)(k+2)(k+3)\begin{cases} \dfrac{(-1)^{\frac{m}{2}} \binom{3}{\frac{m}{2}} (k+\frac{m}{2})!}{(k+\frac{m}{2}+3)!}, & \text{m even},\\[2ex] \dfrac{(-1)^{\frac{m+1}{2}} \left(\frac{m+1}{2}\right) \binom{3}{\frac{m+1}{2}} (k+\frac{m-1}{2})!}{(k+\frac{m+7}{2})!}, & \text{m odd}, \end{cases}$$

and

$$\bar{d}_{m,k} = (-1)^m d_{m,k}.$$

Now, the two variational formulations (6.6) and (6.7), are respectively equivalent to

$$
\begin{aligned}
&-\big(U_N(x),\phi_k(x)\big)_{w_1}+\sum_{i=0}^{5}c_i\left(\int^{(6-i)}U_N(x)(dx)^{6-i},\phi_k(x)\right)_{w_1}\\
&\qquad=\left(h(x)+\sum_{i=0}^{5}r_iV_i^*(x),\phi_k(x)\right)_{w_1},\quad k=0,1,\dots,N-6,
\end{aligned}
\tag{6.8}
$$

and

$$
\begin{aligned}
&-\big(\bar{U}_N(x),\psi_k(x)\big)_{w_2}+\sum_{i=0}^{5}c_i\left(\int^{(6-i)}\bar{U}_N(x)(dx)^{6-i},\psi_k(x)\right)_{w_2}\\
&\qquad=\left(h(x)+\sum_{i=0}^{5}\bar{r}_iW_i^*(x),\psi_k(x)\right)_{w_2},\quad k=0,1,\dots,N-6.
\end{aligned}
\tag{6.9}
$$

It is worthy noting that the constants r_i and $\bar{r}_i$, $0\le i\le 5$, would not appear if we take $k\ge 6$ in (6.8) and (6.9), and accordingly we have

$$
\big(-U_N(x),\phi_k(x)\big)_{w_1}+\sum_{i=0}^{5}c_i\left(\int^{(6-i)}U_N(x)(dx)^{6-i},\phi_k(x)\right)_{w_1}=(h(x),\phi_k(x))_{w_1},\quad 6\le k\le N,
\tag{6.10}
$$

and

$$
\big(-\bar{U}_N(x),\psi_k(x)\big)_{w_2}+\sum_{i=0}^{5}c_i\left(\int^{(6-i)}\bar{U}_N(x)(dx)^{6-i},\psi_k(x)\right)_{w_2}=(h(x),\psi_k(x))_{w_2},\quad 6\le k\le N.
\tag{6.11}
$$

Let us denote

$$
\begin{aligned}
&h_k=(h(x),\phi_k(x))_{w_1},\quad \mathbf{h}=(h_6,h_7,\dots,h_N)^T,\quad \bar{h}_k=(h(x),\psi_k(x))_{w_2},\quad \bar{\mathbf{h}}=(\bar{h}_6,\bar{h}_7,\dots,\bar{h}_N)^T,\\
&U_N(x)=\sum_{k=0}^{N-6}p_k\ \phi_k(x),\quad \mathbf{p}=(p_0,p_1,\dots,p_{N-6})^T,\\
&\bar{U}_N(x)=\sum_{k=0}^{N-6}\bar{p}_k\ \psi_k(x),\quad \bar{\mathbf{p}}=(\bar{p}_0,\bar{p}_1,\dots,\bar{p}_{N-6})^T,\\
&A=(a_{kj})_{6\le k,j\le N}=-\big(\phi_{j-6}(x),\phi_k(x)\big)_{w_1},\quad Z=(z_{kj})_{6\le k,j\le N}=-\big(\psi_{j-6}(x),\psi_k(x)\big)_{w_2},\\
&B_{6-i}=(b_{kj}^{6-i})_{6\le k,j\le N}=\left(\int^{(6-i)}\phi_{j-6}(x)(dx)^{6-i},\phi_k(x)\right)_{w_1},\quad 0\le i\le 5,\\
&E_{6-i}=(e_{kj}^{6-i})_{6\le k,j\le N}=\left(\int^{(6-i)}\psi_{j-6}(x)(dx)^{6-i},\psi_k(x)\right)_{w_2},\quad 0\le i\le 5,
\end{aligned}
$$

then the two relations (6.10) and (6.11) are equivalent to the following two matrix systems

$$
\left(A+\sum_{i=0}^{5}c_iB_{6-i}\right)\mathbf{p}=\mathbf{h},
$$

and

$$\left(Z+\sum_{i=0}^{5} c_i E_{6-i}\right)\bar{\mathbf{p}}=\bar{\mathbf{h}},$$

where the nonzero elements of the matrices $A, Z, B_{6-i}, E_{6-i}, 0 \le i \le 5$ can be obtained explicitly with the aid of the two Corollaries 3.1 and 4.1.

7. Numerical results

For the sake of comparison of our two methods with some other techniques discussed by some other authors, we consider the following two examples.

Example 7.1. Consider the following BVP (see [1,23]):

$$\begin{aligned}
&y^{(6)}(x)+y(x)=6(2x\cos(x)+5\sin(x)), \quad x\in[-1,1],\\
&y(-1)=y(1)=0, \quad y^{(1)}(-1)=y^{(1)}(1)=2\sin(1),\\
&y^{(2)}(-1)=-y^{(2)}(1)=-4\cos(1)-2\sin(1).
\end{aligned}$$

The analytic solution of this problem is

$$y(x)=(x^2-1)\sin(x).$$

Table 1 lists the maximum pointwise error E of $u-u_N$ using SC3GM and SC4GM for various values of N. In Table 2, we introduce a comparison between the best errors obtained by our two methods (SC3GM and SC4GM), the spline based technique developed in [1] and Spline collocation method (SCM) developed in [23]. This table shows that our two methods are more accurate if compared with the method developed in [1] and [23].

Table 1. Maximum pointwise errors for $N = 12, 16, 20, 24$.

N	C3GM	C4GM
12	1.866×10^{-8}	1.883×10^{-8}
16	2.384×10^{-13}	2.394×10^{-13}
20	2.797×10^{-16}	2.797×10^{-16}
24	2.71×10^{-16}	3.674×10^{-16}

Table 2. Comparison between different methods for Example 7.1

Error	SC3GM	SC4GM	method in [1]	(SCM) developed in [23]
E	2.71×10^{-16}	3.674×10^{-16}	3.81×10^{-8}	3.1311×10^{-8}

Example 7.2. Consider the following BVP (see [23]):

$$\begin{aligned}
&y^{(6)}(x)+y^{(3)}(x)+y^{(2)}(x)-y(x)=(-15x^2+78x-114)e^{-x}, \quad x\in[0,1],\\
&y(0)=0, \quad y'(0)=0, \quad y''(0)=0, \quad y(1)=\frac{1}{e}, \quad y'(1)=\frac{2}{e}, \quad y''(1)=\frac{1}{e},
\end{aligned}$$

which has the exact solution $y(x)=x^3e^{-x}$.

Table 3 lists the maximum pointwise error E of $u - u_N$ using SC3GM and SC4GM for various values of N. In Table 4, we give a comparison between the best errors obtained by our two methods (SC3GM and SC4GM) and Spline collocation method (SCM) developed in [23]. This table shows that our two methods are more accurate if compared with the method developed in [23].

Table 3. Maximum pointwise errors for $N = 12, 16, 20, 24$

N	SC3GM	SC4GM
12	1.4125×10^{-11}	1.30372×10^{-11}
16	1.54634×10^{-16}	1.01725×10^{-16}
20	1.38778×10^{-16}	9.06122×10^{-17}
24	1.2078×10^{-16}	1.52266×10^{-16}

Table 4. Comparison between different methods for Example 7.2 using different methods

Error	SC3GM	SC4GM	SCM in [23]
E	1.2078×10^{-16}	1.52266×10^{-16}	8.8478×10^{-9}

8. Concluding remarks

This paper deals with formulae relating the coefficients in the integrated expansions of Chebyshev polynomials of third and fourth kinds to those of the original expansion that has been integrated any number of times. It also develops formulae associated with the q times integration of Chebyshev polynomials of third and fourth kinds. In this article, and as an important application, we describe how to use these formulae to solve sixth-order two point boundary value problems. To the best of our knowledge, all the presented theoretical formulae are completely new and we do believe that these formulae may be used to solve some other kinds of high even-order and high-odd order boundary value problems.

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