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Bayesian Parameter and Reliability Estimate of Weibull Failure Time Distribution

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Abstract. Bayes and frequentist estimators are obtained for the two-parameter Weibull failure time distribution with uncensored observations as well as the survival/reliability and hazard function. The Weibull distribution is used extensively in life testing and reliability/survival analysis. The Bayes approach is obtained using Lindleys approximation technique with standard non-informative (vague) prior and a proposed generalisation of the non-informative prior. A simulation study is carried out to compare the performances of the methods. It is observed from the study that the unknown parameters, the reliability and hazard functions are best estimated by Bayes using linear exponential loss with the proposed prior followed by general entropy loss function.

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1. Introduction

As a result of the adaptability in fitting time-to-failure of a very widespread multiplicity to multifaceted mechanisms, the Weibull distribution has assumed the centre stage especially in the field of life-testing and reliability/survival analysis. It has shown to be very useful for modeling and analysing life time data in medical, biological and engineering sciences, Lawless [11]. Much of the popularity of the Weibull distribution is due to the wide variety of shapes it can assume by varying its parameters. We have considered here a situation where all the units under investigation either failed before or at the end of the study. In a situation of this sort, one has at his/her disposal exact values of the observed units. In other words uncensored observations.

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Maximum likelihood estimator (MLE) is quite efficient and very popular both in literature and practise. The Bayesian approach has also recently being employed for estimating parameters. Research have been done to compare MLE and that of the Bayesian approach in estimating the Weibull parameters. Al Omari and Ibrahim [4] conducted a study on Bayesian survival estimator for Weibull distribution with censored data. Pandey *et al.* [13], compared Bayesian and maximum likelihood estimation of the scale parameter of Weibull with known shape parameter under LINEX and Syuan-Rong and Shuo-Jye [10] considered Bayesian estimation and prediction for Weibull model with progressive censoring. Similar work can be seen in [2, 3, 18].

The aim of this paper is two fold. The maximum likelihood estimator of the reliability function and the hazard rate is considered. In other to obtain the estimates of the reliability function and the hazard rate, the MLE of the Weibull two parameters are obtained. It is observed that the MLEs cannot be obtained in closed form, we therefore propose to use the Newton-Raphson numerical approximation method to compute the MLEs via the Taylor series, and the proposed method works quite well.

The second aim of this paper is to consider the Bayesian inference also for the unknown parameters, the reliability function and the hazard rate. It is remarkable that most of the Bayesian inference procedures have been developed with the usual squared error loss function, which is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude Vasile *et al.* [14]. However, such a restriction may be impractical in most situations of practical importance. For example, in the estimation of the hazard rate function, an overestimation is usually much more serious than an underestimation. In this case, the use of a symmetric loss function might be inappropriate as also emphasised by Basu and Ebrahimi [5]. In this paper, the Bayes estimates are obtained under the linear exponential (LINEX) loss, general entropy and squared error loss function using Lindley's approximation technique with standard non-informative prior and a proposed generalised non-informative prior.

The rest of the paper is arranged as follows: Section 2 contains the derivative of the parameters based on which the reliability function and the hazard rate are determined under maximum likelihood estimator, Section 3 is the Bayesian estimator with non-informative and generalised non-informative priors. Section 4 is the linear exponential (LINEX) loss function followed by Lindley approximation in Section 5 and then general entropy loss function in Section 6. Section 7 depicts the symmetric loss function next to it is the simulation study in Section 8. Results and discussion are in Section 9 and conclusion can be found in Section 10.

2. Maximum likelihood estimation

Suppose $T = t_1, \ldots, t_n$, is an uncensored observation from a sample of n units or individuals under examination. Also assume that the uncensored observations (data) follow the Weibull model. Where the two-parameter Weibull failure time distribution of α and β has a probability density function (pdf) and a cumulative distribution function (cdf) given respectively by

$$f(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right], \tag{2.1}$$

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right].$$

The reliability function $R(t)$, and the hazard rate $H(t)$ also known as the instantaneous failure rate are given as

$$R(t) = \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right],$$
$$H(t) = \left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}.$$

The likelihood function of the pdf from Equation (2.1) is given as

$$L(t_i,\alpha,\beta) = \prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t_i}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t_i}{\alpha}\right)^{\beta}\right]\right\}. \tag{2.2}$$

where β represents the shape parameter and α the scale parameter. The log-likelihood function may be written as

$$\ell = n\ln(\beta) - n\beta\ln(\alpha) + (\beta-1)\sum_{i=1}^{n}\ln(t_i) - \sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta}. \tag{2.3}$$

From Equation (2.3), differentiating the log-likelihood equation for the parameters α and β we have

$$\frac{\partial\ell}{\partial\alpha} = -n\left(\frac{\beta}{\alpha}\right) + \left(\frac{\beta}{\alpha}\right)\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right) = 0, \tag{2.4}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln\left(\frac{t_i}{\alpha}\right) - \sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta}\ln\left(\frac{t_i}{\alpha}\right) = 0. \tag{2.5}$$

From Equation (2.4) $\hat{\alpha}$ is obtained in-terms of $\hat{\beta}$ in the form

$$\hat{\alpha} = \left[\frac{1}{n}\sum_{i=1}^{n}(t_i)^{\hat{\beta}}\right]^{1/\hat{\beta}}.$$

From Equation (2.5) $\hat{\beta}$ is obtained by making use of Newton-Raphson method.

The corresponding maximum likelihood estimators of the reliability function and the hazard rate are respectively

$$\hat{S}_{ML}(t) = \exp\left[-\left(\frac{t}{\hat{\alpha}_{ML}}\right)^{\hat{\beta}_{ML}}\right],$$
$$\hat{F}_{ML}(t) = \frac{\hat{\beta}_{ML}}{\hat{\alpha}_{ML}}\left(\frac{t}{\hat{\alpha}_{ML}}\right)^{\hat{\beta}_{ML}-1}.$$

3. Bayesian estimation

Bayesian estimation approach has received a lot of attention for analysing failure time data. It makes use of ones prior knowledge about the parameters and also takes into consideration the data available. If one's prior knowledge about the parameter is available, it is suitable to make use of an informative prior but in a situation where one does not have any prior knowledge about the parameter and cannot obtain vital information from experts to this regard, then a non-informative prior will be a suitable alternative to use, Guure *et al.* [7]. In this study a non-informative prior approach to the parameters is employed. Given a sample $T = (t_1, t_2, \ldots, t_n)$, the likelihood (L) function follows Equation (2.2).

The Bayes estimates of the unknown parameters, the reliability function and the hazard rate are considered with different loss functions given respectively as

$$\text{Linex loss}: L(\hat{\theta} - \theta) \propto \exp(c\hat{\theta}) E_\theta[\exp(-c\theta)] - c(\hat{\theta} - E_\theta(\theta)) - 1, \tag{3.1}$$

$$\text{Generalised entropy loss}: L(\hat{\theta} - \theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^k - k \ln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \tag{3.2}$$

$$\text{Squared error loss}: L(\hat{\theta} - \theta) = (\hat{\theta} - \theta)^2. \tag{3.3}$$

3.1. Non-informative prior

Consider a likelihood function to be of the form $L(\theta)$, with its Fisher Information $I(\theta) = -E\left(\partial^2 \log L(\theta)/\partial\theta^2\right)$. The Fisher Information measures the sensitivity of an estimator.

Jeffreys (1961) suggested that $\pi(\theta) \propto \det(I(\theta))^{1/2}$ be considered as a prior for the likelihood function $L(\theta)$. The Jeffreys prior is justified on the grounds of its invariance under parametrization according to Sinha [16].

Under the two-parameter Weibull distribution the non-informative (vague) prior according to Sinha and Sloan [15] is given as

$$v(\alpha, \beta) \propto \left(\frac{1}{\alpha\beta}\right).$$

Both Jeffreys prior and reference prior are special cases of the vague prior, when $\beta = 1$, we have a standard Jeffreys prior for the Weibull distribution given in Equation (2.1).

Let the likelihood equation which is $L(t_i|\alpha, \beta)$ be the same as (2.2). The joint posterior of (α, β) is given by

$$\pi(\alpha, \beta | t) \propto L(t|\alpha, \beta) v(\alpha, \beta). \tag{3.4}$$

The marginal distribution function is the double integral of Equation (3.4).

Therefore, the posterior probability density function of α and β given the data $(t_1, t_2, \ldots, t_n)$ is obtained by dividing the joint posterior density function over the marginal distribution function as

$$\pi^*(\alpha, \beta | t) = \frac{L(t|\alpha, \beta) v(\alpha, \beta)}{\int_0^\infty \int_0^\infty L(t|\alpha, \beta) v(\alpha, \beta) d\alpha d\beta}. \tag{3.5}$$

3.2. Generalised non-informative prior

We propose a generalised non-informative prior such that,

$$v_1(\theta) \propto [\pi(\theta)]^a, \; a \neq 1, > 0.$$

This is a generalisation of the non-informative prior, when $a = 1$, we have the standard non-informative prior and undefined when $a = 0$. Since our knowledge on the parameters is limited as a result of which a non-informative prior approach is employed on both parameters, it is important that one ensures the prior does not significantly influence the final result. If our limited or lack of knowledge influences the results, one may end-up giving wrong interpretation which could affect whatever it is we seek to address. It is as a result of this that the generalised prior is considered.

When

$$v_1(\theta) \propto \left(\frac{1}{\theta}\right)^a, \quad \text{then,}$$

$$v_1(\alpha,\beta) \propto \left(\frac{1}{\alpha\beta}\right)^a.$$

The likelihood function from Equation (2.2) is

$$L(t_i|\alpha,\beta) = \prod_{i=1}^{n}\left[\left(\frac{\beta}{\alpha}\right)\left(\frac{t_i}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t_i}{\alpha}\right)^{\beta}\right]\right\}.$$

With Bayes theorem the joint posterior distribution of the parameters α and β is

$$\pi^*(\alpha,\beta|t) = \left(\frac{k}{\alpha\beta}\right)^a \prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t_i}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t_i}{\alpha}\right)^{\beta}\right]\right\}$$

and the marginal distribution is

$$= \int_0^{\infty}\int_0^{\infty}\left(\frac{k}{\alpha\beta}\right)^a \prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t_i}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t_i}{\alpha}\right)^{\beta}\right]\right\} d\alpha d\beta.$$

where k is the normalizing constant that makes π^* a proper pdf.

The posterior density function is obtained by using Equation (3.5).

4. Linear exponential loss function

This loss function according to Soliman *et al.* [17] rises approximately exponentially on one side of zero and approximately linearly on the other side. The sign and magnitude of the shape parameter c represents the direction and degree of symmetry respectively. There is overestimation if $c > 0$ and underestimation if $c < 0$ but when $c \approxeq 0$, the LINEX loss function is approximately the squared error loss function.

The Bayes Estimator of θ, which is denoted by $\hat{\theta}_{BL}$ under LINEX loss function that minimizes Equation (3.1) is given as

$$\hat{\theta}_{BL} = -\frac{1}{c}\ln\left\{E_\theta\left[\exp(-c\theta)\right]\right\},$$

provided that $E_\theta(.)$ exist and is finite.

The posterior density of the unknown parameters, the reliability function and the hazard rate under this loss function are given respectively as

$$E\left[e^{-c\alpha}, e^{-c\beta}|t\right] = \frac{\int_0^{\infty}\int_0^{\infty}\left(\frac{1}{\alpha\beta}\right)\left[\exp(-c\alpha),\exp(-c\beta)\right]\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty}\left(\frac{1}{\alpha\beta}\right)\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta},$$

$$\begin{aligned} R(t)_{BL} &= E\left\{\exp\left\{-c\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}|t\right\} \\ &= \frac{\int_0^{\infty}\int_0^{\infty}\left(\frac{1}{\alpha\beta}\right)\exp\left\{-c\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty}\left(\frac{1}{\alpha\beta}\right)\prod_{i=1}^{n}\left[\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right]d\alpha d\beta}, \end{aligned}$$

$$H(t)_{BL} = E\left\{\exp\left[\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\right]|t\right\}$$
$$= \frac{\int_0^\infty\int_0^\infty \left(\frac{1}{\alpha\beta}\right)\exp\left[-c\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\right]\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta}{\int_0^\infty\int_0^\infty \left(\frac{1}{\alpha\beta}\right)\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta}.$$

It can be observed that the above equations contain ratio of integrals which cannot be obtained analytically and as a result we make use of Lindley approximation procedure to evaluate the integrals involved.

5. Lindley approximation

According to Abdel-Wahid [1], Lindley proposed a ratio of integral of the form

$$\frac{\int \omega(\theta)\exp\{L(\theta)\}\,d\theta}{\int v(\theta)\exp\{L(\theta)\}\,d\theta}$$

where $L(\theta)$ is the log-likelihood and $\omega(\theta), v(\theta)$ are arbitrary functions of θ. In applying this procedure, it is assumed that $v(\theta)$ is the prior distribution for θ and $\omega(\theta) = u(\theta).v(\theta)$ with $u(\theta)$ being some function of interest. The posterior expectation of the Lindley approximation can be obtained from Sinha [16] and Guure and Ibrahim [8].

Taking the two parameters into consideration the following equation can be applied

$$\hat{\theta} = u + \frac{1}{2}[(u_{11}\sigma_{11}) + (u_{22}\sigma_{22})] + u_1\rho_1\sigma_{11} + u_2\rho_2\sigma_{22} + \frac{1}{2}[(L_{30}u_1\sigma_{11}^2) + (L_{03}u_2\sigma_{22}^2)] \tag{5.1}$$

where L is the log-likelihood equation in (2.3). To estimate the unknown parameters, the following are considered and substituted into Equation (5.1).

$$\rho(\alpha,\beta) = -\ln(\alpha) - \ln(\beta)$$
$$\rho_1 = \frac{\partial\rho}{\partial\alpha} = -\left(\frac{1}{\alpha}\right)$$
$$\rho_2 = \frac{\partial\rho}{\partial\beta} = -\left(\frac{1}{\beta}\right)$$
$$L_{20} = n\left(\frac{\beta}{\alpha^2}\right) - \left(\frac{\beta^2}{\alpha^2}\right)\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta} - \left(\frac{\beta}{\alpha^2}\right)\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta}$$
$$\sigma_{11} = (-L_{20})^{-1}$$
$$L_{30} = -2n\left(\frac{\beta}{\alpha^3}\right) + 2\left(\frac{\beta^2}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta} + \left(\frac{\beta^3}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta} + 2\left(\frac{\beta}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta} + \left(\frac{\beta^2}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta}$$
$$L_{02} = -\left(\frac{n}{\beta^2}\right) - \sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta}\ln^2\left(\frac{t_i}{\alpha}\right)$$
$$\sigma_{22} = (-L_{02})^{-1}$$

$$L_{03} = 2\left(\frac{n}{\beta^3}\right) - \sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta}\ln^3\left(\frac{t_i}{\alpha}\right)$$

$$u(\alpha) = \exp(-c\alpha), \quad u_1 = \frac{\partial u}{\partial \alpha} = -c\exp(-c\alpha)$$

$$u_{11} = \frac{\partial^2 u}{\partial \alpha^2} = c^2\exp(-c\alpha), \quad u_2 = u_{22} = 0$$

$$u(\beta) = \exp(-c\beta), \quad u_2 = \frac{\partial u}{\partial \beta} = -c\exp(-c\beta)$$

$$u_{22} = \frac{\partial^2 u}{\partial \beta^2} = c^2\exp(-c\beta), \quad u_1 = u_{11} = 0.$$

The reliability function is estimated by obtaining the following derivatives and replacing them with that of the unknown parameters

$$u = \exp\left\{-c\exp\left[-\left(\frac{t}{\hat{\alpha}}\right)^{\beta}\right]\right\}, \quad q = \exp\left[-\left(\frac{t}{\hat{\alpha}}\right)^{\beta}\right]$$

$$u_1 = \frac{\partial u}{\partial \alpha} = c\left(\frac{\beta}{\alpha}\right)\left(\frac{-t}{\alpha}\right)^{\beta} qu$$

$$u_{11} = \frac{\partial^2 u}{\partial \alpha^2} = -c\left(\frac{\beta^2}{\alpha^2}\right)\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right] qu - c\left(\frac{\beta}{\alpha^2}\right)\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right] qu$$
$$-c\left(\frac{\beta^2}{\alpha^2}\right)\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]^2 qu + c^2\left(\frac{\beta^2}{\alpha^2}\right)\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]^2 q^2u.$$

In a similar approach $u_2 = \partial u/\partial\beta$ and $u_{22} = \partial^2 u/\partial\beta^2$ can be obtained.

For the Hazard Rate

$$u = \exp\left[-c\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\right], \quad p = \left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}$$

$$u_2 = \frac{\partial u}{\partial \beta} = \left[-cp\ln\left(\frac{t}{\alpha}\right) - \left(\frac{c}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\right] u$$

$$u_{22} = \frac{\partial^2 u}{\partial \beta^2} = \left[-cp\ln^2\left(\frac{t}{\alpha}\right) - 2\left(\frac{c}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\ln\left(\frac{t}{\alpha}\right)\right] u$$
$$+\left[-cp\ln\left(\frac{t}{\alpha}\right) - \left(\frac{c}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\right]^2 u$$

$u_1 = \partial u/\partial\alpha$ and $u_{11} = \partial^2 u/\partial\alpha^2$ follow in like manner.

6. General entropy loss function

This is another useful asymmetric loss function that is used to determine whether there is overestimation or underestimation. It is a generalization of the entropy loss.

The Bayes estimator of θ, denoted by $\hat{\theta}_{BG}$ is the value of $\hat{\theta}$ which minimizes (3.2) and given as

$$\hat{\theta}_{BG} = \left[E_{\theta}(\theta)^{-k}\right]^{-1/k}$$

provided $E_{\theta}(.)$ exist and is finite.

The posterior density function of the unknown parameters, the reliability function and the hazard rate under general entropy loss are given respectively as

$$E\left\{u\left[(\alpha)^{-k},(\beta)^{-k}\right]|t\right\}=\frac{\int_0^\infty\int_0^\infty\left(\frac{1}{\alpha\beta}\right)u\left[(\alpha)^{-k},(\beta)^{-k}\right]\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta}{\int_0^\infty\int_0^\infty\left(\frac{1}{\alpha\beta}\right)\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta},$$

$$\begin{aligned}R(t)_{BG}&=E\left\{\left\{\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}^{-k}|t\right\}\\&=\frac{\int_0^\infty\int_0^\infty\left(\frac{1}{\alpha\beta}\right)\left\{\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}^{-k}\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta}{\int_0^\infty\int_0^\infty\left(\frac{1}{\alpha\beta}\right)\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta},\\H(t)_{BG}&=E\left\{\left[\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\right]|t\right\}\\&=\frac{\int_0^\infty\int_0^\infty\left(\frac{1}{\alpha\beta}\right)\left[\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\right]^{-k}\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta}{\int_0^\infty\int_0^\infty\left(\frac{1}{\alpha\beta}\right)\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}d\alpha d\beta}.\end{aligned}$$

By making use of Lindley procedure as in (5.1), where u_1, u_{11}, and u_2, u_{22} represent the first and second derivatives of the unknown parameters, the reliability function and the hazard rate, the following equations are obtained.

$$\begin{aligned}u&=(\alpha)^{-k},\quad u_1=\frac{\partial u}{\partial\alpha}=-k(\alpha)^{-k-1}\\u_{11}&=\frac{\partial^2 u}{\partial(\alpha)^2}=-(-k^2-k)(\alpha)^{-k-2},\quad u_2=u_{22}=0\\u&=(\beta)^{-k},\quad u_2=\frac{\partial u}{\partial\beta}=-k(\beta)^{-k-1}\\u_{22}&=\frac{\partial^2 u}{\partial(\beta)^2}=-(-k^2-k)(\beta)^{-k-2},\quad u_1=u_{11}=0\end{aligned}$$

The following derivatives are to be considered in estimating the reliability function under this loss function.

$$\begin{aligned}u&=\left\{\exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\right\}^{-k},\quad e=\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]\\u_1&=\frac{\partial u}{\partial\alpha}=uk\left(\frac{\beta}{\alpha}\right)e\\u_{11}&=\frac{\partial^2 u}{\partial\alpha^2}=uk^2\left(\frac{\beta^2}{\alpha^2}\right)e^2-k\left(\frac{\beta^2}{\alpha^2}\right)eu-k\left(\frac{\beta}{\alpha^2}\right)eu.\end{aligned}$$

Hence $u_2=\partial u/\partial\beta$ and $u_{22}=\partial^2 u/\partial\beta^2$ follows.

For the hazard rate

$$u=\left[\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\right]^{-k},\quad r=\left(\frac{t}{\alpha}\right)^{\beta-1}$$

$$u_1 = \frac{\partial u}{\partial \alpha} = -\frac{uk\left[-\left(\frac{\beta}{\alpha^2}\right)r - (\beta-1)\left(\frac{\beta}{\alpha^2}\right)r\right]\alpha}{\beta r}$$

$$u_{11} = \frac{\partial^2 u}{\partial \alpha^2} = \frac{uk^2\left[-\left(\frac{\beta}{\alpha^2}\right)r - (\beta-1)\left(\frac{\beta}{\alpha^2}\right)r\right]^2\alpha^2}{\beta^2 r^2}$$
$$- \frac{uk\left[2\left(\frac{\beta}{\alpha^3}\right)r + 3\left(\frac{\beta}{\alpha^3}\right)(\beta-1)r + \left(\frac{\beta}{\alpha^3}\right)r(\beta-1)^2\right]\alpha}{\beta r}$$
$$- \frac{uk\left[-\left(\frac{\beta}{\alpha^2}\right)r - (\beta-1)\left(\frac{\beta}{\alpha^2}\right)r\right]}{\beta r}$$
$$- \frac{uk\left[-\left(\frac{\beta}{\alpha^2}\right)r - (\beta-1)\left(\frac{\beta}{\alpha^2}\right)r\right](\beta-1)}{\beta r}.$$

With the same approach as given above $u_{22} = \partial u/\partial \beta$ and $u_{22} = \partial^2 u/\partial \beta^2$ are obtained.

7. Symmetric loss function

The squared error loss denotes the punishment in using $\hat{\theta}$ to estimate θ and is given by $L(\hat{\theta}-\theta) = (\hat{\theta}-\theta)^2$. This loss function is symmetric in nature i.e. it gives equal weightage to both over and under estimation. In real life, we encounter many situations where over-estimation may be more serious than under-estimation or vice versa.

The posterior density function of the unknown parameters, the Reliability function and the Hazard rate under the symmetric loss function are given as

$$E\left[u(\alpha,\beta)|t\right] = \frac{\int_0^\infty\int_0^\infty \left(\frac{1}{\alpha\beta}\right) u(\alpha,\beta) \prod_{i=1}^n \left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]\right\} d\alpha d\beta}{\int_0^\infty\int_0^\infty \left(\frac{1}{\alpha\beta}\right) \prod_{i=1}^n \left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]\right\} d\alpha d\beta},$$

$$R(t)_{BS} = E\left\{\exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]|t\right\}$$
$$= \frac{\int_0^\infty\int_0^\infty \left(\frac{1}{\alpha\beta}\right)\left\{\exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]\right\} \prod_{i=1}^n \left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]\right\} d\alpha d\beta}{\int_0^\infty\int_0^\infty \left(\frac{1}{\alpha\beta}\right) \prod_{i=1}^n \left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]\right\} d\alpha d\beta},$$

$$H(t)_{BS} = E\left[\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}|t\right]$$
$$= \frac{\int_0^\infty\int_0^\infty \left(\frac{1}{\alpha\beta}\right)\left[\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\right] \prod_{i=1}^n \left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]\right\} d\alpha d\beta}{\int_0^\infty\int_0^\infty \left(\frac{1}{\alpha\beta}\right) \prod_{i=1}^n \left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[\left(\frac{t}{\alpha}\right)^\beta\right]\right\} d\alpha d\beta}.$$

Applying the same Lindley approach here as in (5.1) with u_1, u_{11} and u_2, u_{22} being the first and second derivatives for the unknown parameters, the $R(t)_{BS}$ and $H(t)_{BS}$. For the unknown parameters

$$u = \alpha, \quad u_1 = \frac{\partial u}{\partial \alpha} = 1$$
$$u_{11} = u_2 = u_{22} = 0$$
$$u = \beta, \quad u_2 = 1$$
$$u_{22} = u_1 = u_{11} = 0.$$

For the reliability function and the hazard rate we have respectively

$$u = \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]$$
$$u_1 = \frac{\partial u}{\partial \alpha} = -\left(\frac{\beta}{\alpha}\right) eu$$
$$u_{11} = \frac{\partial^2 u}{\partial \alpha^2} = \left(\frac{\beta}{\alpha^2}\right) eu + \left(\frac{\beta^2}{\alpha^2}\right) eu + \left(\frac{\beta^2}{\alpha^2}\right) e^2 u.$$

In a similar approach $u_2 = \partial u/\partial \beta$ and $u_{22} = \partial^2 u/\partial \beta^2$ can be obtained.

And

$$u = \left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}, \quad d = \ln\left(\frac{t}{\alpha}\right)$$
$$u_2 = \frac{\partial u}{\partial \beta} = \left(\frac{1}{\alpha}\right) r + ud$$
$$u_{22} = \frac{\partial^2 u}{\partial \beta^2} = \left(\frac{2}{\alpha}\right) rd^2 + ud.$$

$u_1 = \partial u/\partial \alpha$ and $u_{11} = \partial^2 u/\partial \alpha^2$ follow in like manner.

With respect to the generalised non-informative prior, the same procedures as above are also employed but $\rho = \log[v(\alpha,\beta)]$ is substituted by $\rho = \log[v_1(\alpha,\beta)]$.

8. Simulation study

In this section, we perform a numerical study to compare the proposed estimates of α, β, $R(t)$ and $H(t)$. Comparison among the different estimators is made with their mean squared errors and absolute error values. The Bayesian estimates of the unknown parameters, the reliability function and the hazard rate are derived with respect to three loss functions, which are LINEX, general entropy and squared error loss functions. A sample of size n = 25, 50 and 100 were considered to represent relatively small, medium and large data set. The reliability function and the hazard rate and that of the parameters were determined with Bayes using non-informative prior approach. The values of the parameters chosen were α = 0.5 and 1.5, and β = 0.8 and 1.2. The values for the loss parameters were $c = k = \pm$ 0.7 and $\pm$ 1.6. The loss parameters are chosen by taken into consideration the facts that they must either be below or above zero in order to determine whether the estimates obtained are under or above the actual parameters values. Also, the loss parameters of both LINEX and general entropy are chosen such that they do not respectively equal to zero (0) and negative one (-1), if they do, both loss functions turn to be approximately the squared error loss, Guure *et al.* [9]. We have considered the generalised non-informative prior to be a = 0.9, and 1.8 without loss of generality. The choice of the constant for the generalised non-informative prior is subjective since it is used to consider the proportion in which one will prefer the prior to influence the posterior density function. These were iterated 1000 times. See Calabria and Pulcini [6] for further details on how to chose the loss parameter values. Simulation results with respect to the mean squared errors and absolute errors are presented in the form of tables only for the reliability and hazard functions and that of the absolute errors for the parameters but we have summarised the observations made about the MSEs on the unknown parameters in the results and discussion section. The average mean squared errors and average absolute error values are determined and presented for the purpose of comparison.

9. Results and discussion

It has been observed from the results that Bayes estimator with linear exponential loss function under the proposed generalised non-informative prior provides the smallest values in most cases for the scale parameter especially when the loss parameter is less than zero (0) that is, $c = k = -1.6$ indicating that underestimation is more serious than overestimation but as the sample size increases both the maximum likelihood estimator and Bayes estimator under all the loss functions have a corresponding decrease in average MSEs. The average abolute errors are presented in Table 9.

For the shape parameter, the Bayesian estimator with LINEX and GELF again with the generalised non-informative prior give a better or smaller average mean squared error and the average absolute errors presented in Table 10, as compared to the others but this happens when the loss parameter $c = k = 1.6$, implying overestimation since the loss parameter is greater than zero (0). It is observed again that as the sample size increases the average mean squared errors of the general entropy loss function decreases to smaller values than any of the others but it must be stated that the other loss functions also have their average MSEs decreasing with increasing sample size.

Similarly, it has also been observed that the estimator that gives the smallest average absolute error over all the other estimators in majority of the cases is Bayes estimator under the generalised prior with LINEX loss function. This is followed by the general entropy loss function.

We present in Table 1, the average mean squared errors of the reliability function with the vague prior while Table 3, are average mean squared errors of the generalised non-informative prior. The average absolute errors of the reliability function are presented in Tables 5 and 7, for both the vague and generalised non-informative priors. It has been observed that Bayes estimator with LINEX loss function has the smallest average mean squared errors and that of the average absolute errors as compared to maximum likelihood estimator and Bayes with squared error and general entropy loss function. The smallest average MSEs and average absolute errors for the LINEX loss function occur at all cases with the proposed generalised non-informative prior. This is followed by general entropy loss function also with the generalised non-informative prior. As the sample size increases, it is observed that all the estimators average MSEs almost converged to one another.

With regards to the average mean squared errors for the hazard rate which are illustrated in Table 2, with the vague prior and Table 4, for the generalised non-informative prior, Bayes estimator under LINEX loss function is a better estimate of the hazard rate also known as the instantaneous failure rate. However, LINEX loss function overestimates the hazard rate, this is because most of the smallest average MSEs occur at where the lost parameter is greater than zero. This largely happened with the generalised non-informative prior. It has been observed that as the sample size increases maximum likelihood estimation and Bayes estimation under all the loss functions have their average mean squared errors decreasing unswervingly. The proposed generalised non-informative prior (data-dependent prior) outperform the vague prior. The same observation is made taking into consideration Tables 6 and 8 which contain the average absolute errors for the hazard rate.

According to Vasile *et al.* [14] the Bayes estimators under squared error loss and general entropy loss functions performed better than their corresponding maximum likelihood estimator and that of the Bayes estimators under linear exponential loss function. Their study was considered with progressive censored data which tern to contradict our findings with uncensored data whereby Bayes with linear loss function performed better than squared error and general entropy loss functions. Our findings support the study of Al-Aboud [2] that the Bayes estimators relative to asymmetric loss functions LINEX and general entropy are sensitive to the values of the loss parameters and perform better than that of maximum likelihood and Bayes via squared error loss function. Though his premise was based on extreme value distribution.

As may be expected since all the priors are non-informative, it is observed that the performances of all estimators become better when the sample size increases. It is also observed in terms of MSEs that, for large sample sizes the Bayes estimates and the MLEs become closer.

The loss parameters for both the LINEX and general entropy loss functions give one the opportunity to estimate the unknown parameters, the reliability and hazard functions with more flexibility. The estimations of α, β, $R(t)$ and $H(t)$ relative to asymmetric loss functions are sensitive to the values of the loss parameters. The problem of choosing values of the loss parameters for both the LINEX and general entropy loss functions was discussed in detail in Calabria and Pulcini [6].

The analytical ease with which results can be obtained using asymmetric loss functions makes them attractive for use in applied problems and in assessing the effects of departures from that of assumed symmetric loss functions which do not consider the practicability of over-and-under estimation.

Finally we should mention that, although we have in this paper considered only uncensored observations in comparing the maximum likelihood, standard non-informative and our proposed prior, it is possible to consider our proposed prior on general class of censored observations.

Table 1. Average MSEs for $\hat{R}(t)$ with vague prior

		$\alpha = 0.5$		$\alpha = 1.5$	
n	Estimators	$\beta = 0.8$	$\beta = 1.2$	$\beta = 0.8$	$\beta = 1.2$
25	ML	0.004539	0.004439	0.004650	0.004541
	BS	0.004649	0.004547	0.004586	0.004677
	$BL(c = 0.7)$	0.004741	0.004606	0.004482	0.004844
	$BL(c = -0.7)$	0.004917	0.004279	0.004374	0.004595
	$BL(c = 1.6)$	0.004551	0.004370	0.004666	0.004544
	$BL(c = -1.6)$	0.004949	0.004457	0.004337	0.004585
	$BG(k = 0.7)$	0.004906	0.004927	0.004727	0.004517
	$BG(k = -0.7)$	0.004547	0.004673	0.004369	0.004523
	$BG(k = 1.6)$	0.004847	0.004762	0.004391	0.004554
	$BG(k = -1.6)$	0.004527	0.004590	0.004572	0.004549
50	ML	0.002311	0.002298	0.002329	0.002319
	BS	0.002459	0.002319	0.002293	0.002319
	$BL(c = 0.7)$	0.002187	0.002153	0.002110	0.002141
	$BL(c = -0.7)$	0.002257	0.002349	0.002211	0.002186
	$BL(c = 1.6)$	0.002219	0.002309	0.002220	0.002325
	$BL(c = -1.6)$	0.002226	0.002129	0.002279	0.002286
	$BG(k = 0.7)$	0.002120	0.002197	0.002112	0.002126
	$BG(k = -0.7)$	0.002122	0.002240	0.002181	0.002230
	$BG(k = 1.6)$	0.002222	0.002234	0.002217	0.002213
	$BG(k = -1.6)$	0.002173	0.002127	0.002160	0.002386
100	ML	0.001119	0.001153	0.001114	0.001084
	BS	0.001132	0.001155	0.001100	0.001090
	$BL(c = 0.7)$	0.001136	0.001086	0.001079	0.001119
	$BL(c = -0.7)$	0.001138	0.001078	0.001067	0.001034
	$BL(c = 1.6)$	0.001086	0.001067	0.001157	0.001099
	$BL(c = -1.6)$	0.001138	0.001136	0.001057	0.001133
	$BG(k = 0.7)$	0.001069	0.001081	0.001090	0.001022
	$BG(k = -0.7)$	0.001105	0.001116	0.001039	0.001020
	$BG(k = 1.6)$	0.001116	0.001058	0.001131	0.001066
	$BG(k = -1.6)$	0.001127	0.001155	0.001065	0.001069

ML = Maximum Likelihood, BG = General Entropy Loss Function,
BL = LINEX Loss Function, BS = Squared Error Loss Function

Table 2. Average MSEs for $\hat{H}(t)$ with vague prior

		$\alpha = 0.5$		$\alpha = 1.5$	
n	Estimators	$\beta = 0.8$	$\beta = 1.2$	$\beta = 0.8$	$\beta = 1.2$
25	ML	3.783420	0.560094	0.178732	0.064335
	BS	4.913962	0.543178	0.453892	0.063931
	$BL(c = 0.7)$	2.723836	0.430512	0.101904	0.055452
	$BL(c = -0.7)$	4.011972	0.643023	0.362907	0.065747
	$BL(c = 1.6)$	2.301949	0.392819	0.156331	0.048229
	$BL(c = -1.6)$	4.013245	0.648014	0.621963	0.072735
	$BG(k = 0.7)$	1.813694	0.483790	0.236298	0.055523
	$BG(k = -0.7)$	4.709160	0.528657	0.257529	0.056453
	$BG(k = 1.6)$	1.548734	0.455916	0.125755	0.053361
	$BG(k = -1.6)$	3.791071	0.506485	0.517603	0.059592
50	ML	2.265619	0.224225	0.206696	0.026346
	BS	2.198552	0.209081	0.168144	0.025864
	$BL(c = 0.7)$	1.142458	0.197646	0.186539	0.024681
	$BL(c = -0.7)$	1.358746	0.240983	0.183011	0.026024
	$BL(c = 1.6)$	1.780576	0.191138	0.121118	0.023976
	$BL(c = -1.6)$	3.344647	0.251547	0.136765	0.029309
	$BG(k = 0.7)$	1.718245	0.217423	0.182593	0.022839
	$BG(k = -0.7)$	2.086132	0.200302	0.143292	0.024236
	$BG(k = 1.6)$	1.367952	0.207346	0.059657	0.023501
	$BG(k = -1.6)$	1.647876	0.234730	0.242375	0.025636
100	ML	0.301270	0.104144	0.033570	0.011559
	BS	0.501418	0.096940	0.073336	0.011562
	$BL(c = 0.7)$	0.214610	0.097575	0.023751	0.011150
	$BL(c = -0.7)$	0.485988	0.115443	0.050981	0.012031
	$BL(c = 1.6)$	0.349674	0.093267	0.025393	0.010395
	$BL(c = -1.6)$	0.346782	0.113941	0.069610	0.011549
	$BG(k = 0.7)$	0.251469	0.095601	0.043867	0.010644
	$BG(k = -0.7)$	0.370011	0.097999	0.040944	0.011346
	$BG(k = 1.6)$	0.216079	0.098692	0.025571	0.011078
	$BG(k = -1.6)$	0.275517	0.105664	0.078485	0.010508

ML = Maximum Likelihood, BG = General Entropy Loss Function,
BL = LINEX Loss Function, BS = Squared Error Loss Function

Table 3. Average MSEs for $R(t)$ with gen. non-informative prior

n	Estimators	$\alpha = 0.5$, $\beta = 0.8$, $a = 0.9$, [1.8]	$\alpha = 0.5$, $\beta = 1.2$	$\alpha = 1.5$, $\beta = 0.8$, $a = 0.9$, [1.8]	$\alpha = 1.5$, $\beta = 1.2$
25	BS	0.004514 [0.004351]	0.004861 [0.004541]	0.004438 [0.004239]	0.004678 [0.004858]
	$BL(c=0.7)$	0.004761 [0.004411]	0.004880 [0.004256]	0.004482 [0.004053]	0.004393 [0.004260]
	$BL(c=-0.7)$	0.004567 [0.003992]	0.004862 [0.004253]	0.004197 [0.004421]	0.004808 [0.004483]
	$BL(c=1.6)$	0.004517 [0.004338]	0.004375 [0.004343]	0.004302 [0.004527]	0.004756 [0.004339]
	$BL(c=-1.6)$	0.004584 [0.004316]	0.004571 [0.004035]	0.004527 [0.004261]	0.004339 [0.004479]
	$BG(k=0.7)$	0.004669 [0.004652]	0.004853 [0.004167]	0.004418 [0.004332]	0.004689 [0.004374]
	$BG(k=-0.7)$	0.004834 [0.004233]	0.004458 [0.004370]	0.004150 [0.004321]	0.004682 [0.004414]
	$BG(k=1.6)$	0.005029 [0.004743]	0.004961 [0.004344]	0.004461 [0.004544]	0.004859 [0.004270]
	$BG(k=-1.6)$	0.004972 [0.004338]	0.004432 [0.004304]	0.004109 [0.004521]	0.004591 [0.004597]
50	BS	0.002317 [0.002263]	0.002353 [0.002169]	0.002213 [0.002113]	0.002259 [0.002339]
	$BL(c=0.7)$	0.002403 [0.002222]	0.002218 [0.002201]	0.002155 [0.002287]	0.002258 [0.002151]
	$BL(c=-0.7)$	0.002366 [0.002291]	0.002199 [0.002160]	0.002198 [0.002130]	0.002247 [0.002195]
	$BL(c=1.6)$	0.002358 [0.002011]	0.002112 [0.002102]	0.002079 [0.002052]	0.002353 [0.002173]
	$BL(c=-1.6)$	0.002245 [0.002064]	0.002243 [0.002037]	0.002125 [0.002148]	0.002120 [0.002284]
	$BG(k=0.7)$	0.002284 [0.002294]	0.002173 [0.001998]	0.002288 [0.002033]	0.002300 [0.002039]
	$BG(k=-0.7)$	0.002177 [0.002139]	0.002229 [0.002077]	0.002215 [0.002259]	0.002172 [0.002165]
	$BG(k=1.6)$	0.002169 [0.002231]	0.002169 [0.002261]	0.002234 [0.002017]	0.002286 [0.002236]
	$BG(k=-1.6)$	0.002229 [0.001908]	0.002329 [0.002090]	0.002190 [0.002125]	0.002286 [0.002286]
100	BS	0.001127 [0.001034]	0.001033 [0.001035]	0.001146 [0.001121]	0.001087 [0.001113]
	$BL(c=0.7)$	0.001168 [0.001137]	0.001098 [0.001124]	0.001059 [0.001065]	0.001080 [0.001039]
	$BL(c=-0.7)$	0.001083 [0.001060]	0.001073 [0.001061]	0.001083 [0.001084]	0.001094 [0.001054]
	$BL(c=1.6)$	0.001152 [0.001099]	0.001109 [0.001012]	0.001179 [0.001104]	0.001082 [0.001054]
	$BL(c=-1.6)$	0.001112 [0.001062]	0.001184 [0.001025]	0.000981 [0.001119]	0.001102 [0.001120]
	$BG(k=0.7)$	0.001127 [0.001138]	0.001062 [0.001045]	0.001152 [0.001011]	0.001122 [0.001111]
	$BG(k=-0.7)$	0.001105 [0.001034]	0.001054 [0.001066]	0.001109 [0.001034]	0.001079 [0.001070]
	$BG(k=1.6)$	0.001096 [0.001084]	0.001092 [0.001102]	0.001130 [0.001096]	0.001082 [0.001104]
	$BG(k=-1.6)$	0.001097 [0.001109]	0.001158 [0.001079]	0.001044 [0.001045]	0.001043 [0.001064]

ML = Maximum Likelihood, BG = General Entropy Loss Function,

BL = LINEX Loss Function, BS = Squared Error Loss Function

Table 4. Average MSEs for $\hat{H}(t)$ with gen. non-informative prior.

n	Estimators	$\alpha = 0.5$, $\beta = 0.8$, $a = 0.9$, [1.8]	$\alpha = 0.5$, $\beta = 1.2$	$\alpha = 1.5$, $\beta = 0.8$, $a = 0.9$, [1.8]	$\alpha = 1.5$, $\beta = 1.2$
25	BS	6.162885 [7.070932]	0.541834 [0.512479]	0.893700 [0.895048]	0.063392 [0.061908]
	$BL(c=0.7)$	2.378734 [1.333984]	0.476059 [0.406154]	0.353531 [0.377706]	0.053889 [0.053417]
	$BL(c=-0.7)$	3.493634 [2.903458]	0.726778 [0.630562]	0.689086 [0.410405]	0.067490 [0.067991]
	$BL(c=1.6)$	4.447621 [4.795282]	0.426760 [0.439095]	0.665762 [0.755662]	0.049852 [0.051064]
	$BL(c=-1.6)$	3.546875 [3.721305]	0.765891 [0.541278]	0.547275 [0.611891]	0.075004 [0.070765]
	$BG(k=0.7)$	1.314662 [3.055011]	0.506001 [0.633310]	0.372011 [0.223407]	0.055012 [0.058685]
	$BG(k=-0.7)$	6.470203 [4.397155]	0.561801 [0.622212]	0.849058 [0.524798]	0.061184 [0.067194]
	$BG(k=1.6)$	0.854643 [0.971416]	0.507121 [0.522118]	0.127378 [0.119673]	0.050048 [0.054119]
	$BG(k=-1.6)$	6.113259 [7.941836]	0.578219 [0.610981]	0.715303 [0.435711]	0.065594 [0.068357]
50	BS	1.099980 [3.152638]	0.222548 [0.220448]	0.137102 [0.505778]	0.024875 [0.026084]
	$BL(c=0.7)$	1.927603 [1.627315]	0.201077 [0.193187]	0.121378 [0.192231]	0.023597 [0.025428]
	$BL(c=-0.7)$	1.595877 [0.862981]	0.239591 [0.228354]	0.369632 [0.176571]	0.024378 [0.027583]
	$BL(c=1.6)$	1.163671 [0.625549]	0.191433 [0.183036]	0.127998 [0.119826]	0.023899 [0.023312]
	$BL(c=-1.6)$	1.041237 [1.773712]	0.263088 [0.278687]	0.218235 [0.138254]	0.026042 [0.028918]
	$BG(k=0.7)$	1.219203 [1.051208]	0.214923 [0.224566]	0.164129 [0.148459]	0.024371 [0.023003]
	$BG(k=-0.7)$	1.430543 [2.177439]	0.233314 [0.225804]	0.388349 [0.209519]	0.024107 [0.024473]
	$BG(k=1.6)$	0.576417 [0.616461]	0.209880 [0.227224]	0.062305 [0.062459]	0.024426 [0.023518]
	$BG(k=-1.6)$	3.408256 [3.061849]	0.246163 [0.250584]	0.249856 [0.167228]	0.024099 [0.026635]
100	BS	0.828949 [0.468291]	0.092059 [0.096484]	0.105431 [0.050118]	0.011208 [0.011076]
	$BL(c=0.7)$	0.219983 [0.679051]	0.092645 [0.098504]	0.028705 [0.027313]	0.010711 [0.010105]
	$BL(c=-0.7)$	0.412189 [0.970795]	0.106671 [0.105212]	0.085741 [0.051188]	0.010672 [0.011897]
	$BL(c=1.6)$	0.394538 [0.324953]	0.093859 [0.087551]	0.035253 [0.045600]	0.010474 [0.010510]
	$BL(c=-1.6)$	0.543434 [0.554014]	0.113636 [0.106709]	0.045853 [0.061020]	0.011788 [0.012620]
	$BG(k=0.7)$	0.300170 [0.326592]	0.096168 [0.103724]	0.043880 [0.030176]	0.011490 [0.011364]
	$BG(k=-0.7)$	0.459765 [0.660640]	0.098755 [0.106063]	0.045283 [0.064522]	0.011162 [0.011162]
	$BG(k=1.6)$	0.190678 [0.194472]	0.094005 [0.104792]	0.033504 [0.031192]	0.011374 [0.010566]
	$BG(k=-1.6)$	0.805029 [0.637823]	0.109503 [0.103108]	0.068016 [0.074382]	0.011129 [0.011701]

ML = Maximum Likelihood, BG = General Entropy Loss Function,
BL = LINEX Loss Function, BS = Squared Error Loss Function

Table 5. Average Absolute Errors for $\hat{R}(t)$ with vague prior

		$\alpha = 0.5$		$\alpha = 1.5$	
n	Estimators	$\beta = 0.8$	$\beta = 1.2$	$\beta = 0.8$	$\beta = 1.2$
25	ML	0.0526	0.0526	0.0540	0.0539
	BS	0.0534	0.0530	0.0531	0.0542
	$BL(c = 0.7)$	0.0528	0.0531	0.0515	0.0541
	$BL(c = -0.7)$	0.0527	0.0538	0.0515	0.0524
	$BL(c = 1.6)$	0.0542	0.0523	0.0503	0.0526
	$BL(c = -1.6)$	0.0536	0.0522	0.0518	0.0539
	$BG(k = 0.7)$	0.0547	0.0536	0.0505	0.0522
	$BG(k = -0.7)$	0.0506	0.0527	0.0518	0.0529
	$BG(k = 1.6)$	0.0558	0.0551	0.0532	0.0531
	$BG(k = -1.6)$	0.0523	0.0524	0.0528	0.0505
50	ML	0.0360	0.0354	0.0364	0.0366
	BS	0.0366	0.0355	0.0362	0.0368
	$BL(c = 0.7)$	0.0365	0.0359	0.0362	0.0375
	$BL(c = -0.7)$	0.0366	0.0366	0.0351	0.0366
	$BL(c = 1.6)$	0.0357	0.0364	0.0359	0.0360
	$BL(c = -1.6)$	0.0359	0.0363	0.0357	0.0354
	$BG(k = 0.7)$	0.0372	0.0372	0.0361	0.0361
	$BG(k = -0.7)$	0.0366	0.0357	0.0364	0.0358
	$BG(k = 1.6)$	0.0363	0.0372	0.0372	0.0354
	$BG(k = -1.6)$	0.0358	0.0355	0.0362	0.0358
100	ML	0.0245	0.0257	0.0265	0.0249
	BS	0.0245	0.0258	0.0265	0.0251
	$BL(c = 0.7)$	0.0251	0.0259	0.0252	0.0247
	$BL(c = -0.7)$	0.0255	0.0254	0.0253	0.0258
	$BL(c = 1.6)$	0.0258	0.0262	0.0251	0.0260
	$BL(c = -1.6)$	0.0252	0.0256	0.0251	0.0253
	$BG(k = 0.7)$	0.0255	0.0249	0.0259	0.0249
	$BG(k = -0.7)$	0.0256	0.0253	0.0249	0.0255
	$BG(k = 1.6)$	0.0249	0.0259	0.0263	0.0249
	$BG(k = -1.6)$	0.0252	0.0247	0.0258	0.0258

ML = Maximum Likelihood, BG = General Entropy Loss Function,
BL = LINEX Loss Function, BS = Squared Error Loss Function

Table 6. Average Absolute Errors for $\hat{H}(t)$ with vague prior

		$\alpha = 0.5$		$\alpha = 1.5$	
n	Estimators	$\beta = 0.8$	$\beta = 1.2$	$\beta = 0.8$	$\beta = 1.2$
25	ML	0.5822	0.5267	0.1868	0.1830
	BS	0.7870	0.5232	0.2278	0.1827
	$BL(c = 0.7)$	0.5376	0.5209	0.1702	0.1761
	$BL(c = -0.7)$	0.5803	0.5524	0.1927	0.1802
	$BL(c = 1.6)$	0.5471	0.4913	0.1596	0.1698
	$BL(c = -1.6)$	0.5823	0.5604	0.1891	0.1850
	$BG(k = 0.7)$	0.5278	0.5379	0.1747	0.1737
	$BG(k = -0.7)$	0.6160	0.5273	0.1896	0.1792
	$BG(k = 1.6)$	0.4749	0.5320	0.1755	0.1733
	$BG(k = -1.6)$	0.6741	0.5376	0.2259	0.1747
50	ML	0.3664	0.3464	0.1195	0.1154
	BS	0.4182	0.3465	0.1357	0.1153
	$BL(c = 0.7)$	0.3376	0.3437	0.1148	0.1193
	$BL(c = -0.7)$	0.3784	0.3596	0.1289	0.1190
	$BL(c = 1.6)$	0.3412	0.3368	0.1112	0.1104
	$BL(c = -1.6)$	0.3756	0.3677	0.1254	0.1168
	$BG(k = 0.7)$	0.3429	0.3507	0.1184	0.1122
	$BG(k = -0.7)$	0.3749	0.3411	0.1252	0.1153
	$BG(k = 1.6)$	0.3360	0.3568	0.1150	0.1126
	$BG(k = -1.6)$	0.3709	0.3476	0.1290	0.1165
100	ML	0.2355	0.2395	0.0852	0.0788
	BS	0.2516	0.2392	0.0910	0.0786
	$BL(c = 0.7)$	0.2277	0.2450	0.0792	0.0783
	$BL(c = -0.7)$	0.2518	0.2430	0.0830	0.0806
	$BL(c = 1.6)$	0.2439	0.2408	0.0760	0.0810
	$BL(c = -1.6)$	0.2483	0.2423	0.0852	0.0817
	$BG(k = 0.7)$	0.2469	0.2330	0.0805	0.0794
	$BG(k = -0.7)$	0.2457	0.2414	0.0811	0.0786
	$BG(k = 1.6)$	0.2328	0.2451	0.0816	0.0792
	$BG(k = -1.6)$	0.2447	0.2358	0.0842	0.0826

ML = Maximum Likelihood, BG = General Entropy Loss Function,
BL = LINEX Loss Function, BS = Squared Error Loss Function

Table 7. Average Absolute Errors for $\hat{R}(t)$ under gen. non-informative prior. Absolute Errors for a = 1.8 are in the parenthesis.

n	Estimators	$\alpha = 0.5$		$\alpha = 1.5$	
		$\beta = 0.8$	$\beta = 1.2$	$\beta = 0.8$	$\beta = 1.2$
		$a = 0.9$ and (1.8)		$a = 0.9$ and (1.8)	
25	BS	0.0532 (0.0525)	0.0531 (0.0496)	0.0509 (0.0486)	0.0533 (0.0518)
	$BL(c = 0.7)$	0.0536 (0.0522)	0.0523 (0.0466)	0.0528 (0.0490)	0.0529 (0.0495)
	$BL(c = -0.7)$	0.0529 (0.0501)	0.0523 (0.0505)	0.0490 (0.0475)	0.0521 (0.0522)
	$BL(c = 1.6)$	0.0528 (0.0550)	0.0520 (0.0489)	0.0516 (0.0493)	0.0522 (0.0507)
	$BL(c = -1.6)$	0.0531 (0.0499)	0.0515 (0.0498)	0.0526 (0.0497)	0.0515 (0.0503)
	$BG(k = 0.7)$	0.0526 (0.0541)	0.0522 (0.0510)	0.0516 (0.0504)	0.0515 (0.0509)
	$BG(k = -0.7)$	0.0542 (0.0509)	0.0525 (0.0485)	0.0535 (0.0484)	0.0532 (0.0501)
	$BG(k = 1.6)$	0.0537 (0.0538)	0.0545 (0.0555)	0.0513 (0.0471)	0.0543 (0.0509)
	$BG(k = -1.6)$	0.0557 (0.0499)	0.0534 (0.0479)	0.0527 (0.0506)	0.0517 (0.0509)
50	BS	0.0363 (0.0349)	0.0347 (0.0355)	0.0370 (0.0351)	0.0355 (0.0363)
	$BL(c = 0.7)$	0.0353 (0.0346)	0.0351 (0.0328)	0.0365 (0.0347)	0.0365 (0.0359)
	$BL(c = -0.7)$	0.0363 (0.0356)	0.0353 (0.0350)	0.0359 (0.0338)	0.0358 (0.0346)
	$BL(c = 1.6)$	0.0355 (0.0346)	0.0364 (0.0346)	0.0344 (0.0349)	0.0359 (0.0360)
	$BL(c = -1.6)$	0.0369 (0.0341)	0.0369 (0.0343)	0.0352 (0.0337)	0.0365 (0.0366)
	$BG(k = 0.7)$	0.0366 (0.0350)	0.0347 (0.0349)	0.0374 (0.0352)	0.0361 (0.0363)
	$BG(k = -0.7)$	0.0372 (0.0343)	0.0359 (0.0345)	0.0356 (0.0346)	0.0372 (0.0371)
	$BG(k = 1.6)$	0.0375 (0.0377)	0.0362 (0.0347)	0.0365 (0.0353)	0.0368 (0.0370)
	$BG(k = -1.6)$	0.0359 (0.0341)	0.0359 (0.0344)	0.0363 (0.0354)	0.0359 (0.0370)
100	BS	0.0252 (0.0254)	0.0250 (0.0250)	0.0251 (0.0252)	0.0244 (0.0257)
	$BL(c = 0.7)$	0.0260 (0.0241)	0.0252 (0.0247)	0.0249 (0.0246)	0.0261 (0.0249)
	$BL(c = -0.7)$	0.0259 (0.0245)	0.0262 (0.0245)	0.0265 (0.0249)	0.0251 (0.0248)
	$BL(c = 1.6)$	0.0258 (0.0244)	0.0250 (0.0241)	0.0255 (0.0247)	0.0256 (0.0249)
	$BL(c = -1.6)$	0.0241 (0.0242)	0.0257 (0.0247)	0.0254 (0.0248)	0.0249 (0.0244)
	$BG(k = 0.7)$	0.0251 (0.0256)	0.0250 (0.0252)	0.0256 (0.0257)	0.0253 (0.0248)
	$BG(k = -0.7)$	0.0256 (0.0241)	0.0256 (0.0241)	0.0261 (0.0240)	0.0257 (0.0248)
	$BG(k = 1.6)$	0.0260 (0.0257)	0.0254 (0.0248)	0.0250 (0.0251)	0.0261 (0.0255)
	$BG(k = -1.6)$	0.0252 (0.0251)	0.0253 (0.0244)	0.0246 (0.0252)	0.0256 (0.0246)

ML = Maximum Likelihood, BG = General Entropy Loss Function,
BL = LINEX Loss Function, BS = Squared Error Loss Function

Table 8. Average Absolute Errors for $\hat{H}(t)$ under gen. non-informative prior. Absolute Errors for a = 1.8 are in the parenthesis.

n	Estimators	$\alpha = 0.5$		$\alpha = 1.5$	
		$\beta = 0.8$	$\beta = 1.2$	$\beta = 0.8$	$\beta = 1.2$
		$a = 0.9$ and (1.8)		$a = k = 0.9$ and (1.8)	
25	BS	0.8340 (0.7004)	0.5304 (0.5098)	0.2414 (0.3184)	0.1751 (0.1750)
	$BL(c = 0.7)$	0.5032 (0.4896)	0.4872 (0.4699)	0.1738 (0.1719)	0.1699 (0.1715)
	$BL(c = -0.7)$	0.6023 (0.6207)	0.5489 (0.5862)	0.1900 (0.1917)	0.1726 (0.1784)
	$BL(c = 1.6)$	0.6113 (0.4839)	0.4895 (0.4837)	0.1816 (0.1773)	0.1661 (0.1610)
	$BL(c = -1.6)$	0.6031 (0.6042)	0.5541 (0.5600)	0.1937 (0.1994)	0.1816 (0.1796)
	$BG(k = 0.7)$	0.5226 (0.9114)	0.5135 (0.6495)	0.1736 (0.1964)	0.1648 (0.1809)
	$BG(k = -0.7)$	0.5837 (0.7545)	0.5206 (0.6600)	0.1917 (0.2212)	0.1733 (0.1793)
	$BG(k = 1.6)$	0.4748 (0.7829)	0.5274 (0.6558)	0.1726 (0.1666)	0.1737 (0.1769)
	$BG(k = -1.6)$	0.6068 (0.7264)	0.5435 (0.6499)	0.2831 (0.2144)	0.1839 (0.1828)
50	BS	0.4347 (0.3772)	0.3377 (0.3374)	0.1399 (0.1373)	0.1154 (0.1188)
	$BL(c = 0.7)$	0.3391 (0.3195)	0.3380 (0.3124)	0.1168 (0.1117)	0.1107 (0.1148)
	$BL(c = -0.7)$	0.3627 (0.3515)	0.3591 (0.3493)	0.1317 (0.1260)	0.1138 (0.1141)
	$BL(c = 1.6)$	0.3476 (0.3306)	0.3341 (0.3252)	0.1107 (0.1133)	0.1136 (0.1127)
	$BL(c = -1.6)$	0.3719 (0.3789)	0.3667 (0.3712)	0.1201 (0.1228)	0.1185 (0.1203)
	$BG(k = 0.7)$	0.3497 (0.4445)	0.3339 (0.3886)	0.1179 (0.1162)	0.1168 (0.1223)
	$BG(k = -0.7)$	0.3708 (0.4431)	0.3489 (0.3784)	0.1194 (0.1234)	0.1199 (0.1210)
	$BG(k = 1.6)$	0.3338 (0.4312)	0.3491 (0.3786)	0.1143 (0.1173)	0.1140 (0.1204)
	$BG(k = -1.6)$	0.3720 (0.4580)	0.3506 (0.3917)	0.1277 (0.1522)	0.1162 (0.1221)
100	BS	0.2590 (0.2479)	0.2352 (0.2403)	0.0877 (0.0931)	0.0753 (0.0814)
	$BL(c = 0.7)$	0.2458 (0.2229)	0.2313 (0.2350)	0.0785 (0.0802)	0.0800 (0.0798)
	$BL(c = -0.7)$	0.2606 (0.2409)	0.2455 (0.2388)	0.0890 (0.0840)	0.0785 (0.0798)
	$BL(c = 1.6)$	0.2324 (0.2229)	0.2311 (0.2243)	0.0785 (0.0781)	0.0778 (0.0774)
	$BL(c = -1.6)$	0.2400 (0.2424)	0.2506 (0.2413)	0.0843 (0.0819)	0.0813 (0.0783)
	$BG(k = 0.7)$	0.2418 (0.2804)	0.2340 (0.2550)	0.0800 (0.0823)	0.0795 (0.0792)
	$BG(k = -0.7)$	0.2431 (0.2665)	0.2465 (0.2498)	0.0866 (0.0784)	0.0815 (0.0784)
	$BG(k = 1.6)$	0.2398 (0.2588)	0.2376 (0.2488)	0.0785 (0.0797)	0.0799 (0.0798)
	$BG(k = -1.6)$	0.2438 (0.2844)	0.2417 (0.2480)	0.0823 (0.0833)	0.0804 (0.0779)

ML = Maximum Likelihood, BG = General Entropy Loss Function,
BL = LINEX Loss Function, BS = Squared Error Loss Function

Table 9. Average Absolute Errors for $(\hat{\alpha})$

n	α	a	β	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{BG}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{BG}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{BG}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{BG}$
						$c=k=0.7$		$c=k=-0.7$		$c=k=1.6$		$c=k=-1.6$	
25	0.5	0.9	0.8	0.1052	0.1052	0.1074	0.1088	0.1048	0.1067	0.1094	0.1088	0.1004	0.1064
	0.5	0.9	1.2	0.0684	0.0685	0.0723	0.0713	0.0714	0.0722	0.0714	0.0721	0.0682	0.0733
	0.5	1.8	0.8	0.1056	0.1056	0.0981	0.0988	0.0934	0.0899	0.1062	0.1021	0.0889	0.0831
	0.5	1.8	1.2	0.0699	0.0697	0.0687	0.0686	0.0662	0.0655	0.0693	0.0658	0.0662	0.0637
	1.5	0.9	0.8	0.3206	0.3200	0.3252	0.3246	0.3105	0.3171	0.3690	0.3368	0.2913	0.3174
	1.5	0.9	1.2	0.2087	0.2087	0.2136	0.2120	0.2021	0.2192	0.2271	0.2186	0.2146	0.2190
	1.5	1.8	0.8	0.3165	0.3133	0.3050	0.2967	0.2925	0.3010	0.3254	0.3038	0.3014	0.3028
	1.5	1.8	1.2	0.2104	0.2017	0.2095	0.2111	0.2019	0.2080	0.2155	0.2098	0.2105	0.2097
50	0.5	0.9	0.8	0.0736	0.0736	0.0738	0.0763	0.0736	0.0741	0.0754	0.0762	0.0723	0.0738
	0.5	0.9	1.2	0.0495	0.0495	0.0491	0.0502	0.0501	0.0498	0.0507	0.0509	0.0499	0.0499
	0.5	1.8	0.8	0.0756	0.0754	0.0698	0.0695	0.0687	0.0677	0.0728	0.0738	0.0662	0.0663
	0.5	1.8	1.2	0.0493	0.0471	0.0490	0.0485	0.0490	0.0481	0.0492	0.0495	0.0472	0.0476
	1.5	0.9	0.8	0.2202	0.2200	0.2242	0.2274	0.2174	0.2234	0.2371	0.2262	0.2138	0.2242
	1.5	0.9	1.2	0.1454	0.1454	0.1510	0.1522	0.1521	0.1503	0.1560	0.1484	0.1503	0.1484
	1.5	1.8	0.8	0.2247	0.2247	0.2181	0.2135	0.2163	0.2130	0.2251	0.2229	0.2108	0.2187
	1.5	1.8	1.2	0.1463	0.1477	0.1516	0.1506	0.1487	0.1503	0.1531	0.1505	0.1480	0.1505
100	0.5	0.9	0.8	0.0522	0.0522	0.0529	0.0514	0.0522	0.0534	0.0530	0.0529	0.0522	0.0519
	0.5	0.9	1.2	0.0354	0.0354	0.0353	0.0349	0.0352	0.0355	0.0361	0.0354	0.0356	0.0351
	0.5	1.8	0.8	0.0522	0.0520	0.0515	0.0508	0.0505	0.0502	0.0513	0.0520	0.0502	0.0496
	0.5	1.8	1.2	0.0349	0.0349	0.0347	0.0349	0.0352	0.0338	0.0349	0.0348	0.0346	0.0343
	1.5	0.9	0.8	0.1592	0.1588	0.1601	0.1580	0.1560	0.1578	0.1635	0.1585	0.1562	0.1580
	1.5	0.9	1.2	0.1049	0.1047	0.1070	0.1067	0.1073	0.1051	0.1086	0.1063	0.1052	0.1056
	1.5	1.8	0.8	0.1571	0.1571	0.1585	0.1543	0.1550	0.1549	0.1585	0.1577	0.1552	0.1566
	1.5	1.8	1.2	0.1040	0.1040	0.1044	0.1045	0.1055	0.1060	0.1067	0.1047	0.1059	0.1053

ML = Maximum Likelihood, BG = General Entropy Loss Function,
BL = LINEX Loss Function, BS = Squared Error Loss Function

Table 10. Average Absolute Errors for $(\hat{\beta})$

n	α	a	β	$\hat{\beta}_{ML}$	$\hat{\beta}_{BS}$	$\hat{\beta}_{BL}$	$\hat{\beta}_{BG}$	$\hat{\beta}_{BL}$	$\hat{\beta}_{BG}$	$\hat{\beta}_{BL}$	$\hat{\beta}_{BG}$	$\hat{\beta}_{BL}$	$\hat{\beta}_{BG}$
						$c=k=0.7$		$c=k=-0.7$		$c=k=1.6$		$c=k=-1.6$	
25	0.5	0.9	0.8	0.1135	0.1104	0.1084	0.1091	0.1133	0.1153	0.1048	0.1082	0.1134	0.1149
	0.5	0.9	1.2	0.1708	0.1652	0.1568	0.1636	0.1652	0.1663	0.1564	0.1593	0.1761	0.1678
	0.5	1.8	0.8	0.1154	0.1170	0.1148	0.1131	0.1135	0.1159	0.1104	0.1118	0.1186	0.1205
	0.5	1.8	1.2	0.1711	0.1736	0.1654	0.1719	0.1767	0.1771	0.1603	0.1639	0.1845	0.1779
	1.5	0.9	0.8	0.1107	0.1076	0.1061	0.1092	0.1120	0.1108	0.1052	0.1047	0.1141	0.1158
	1.5	0.9	1.2	0.1710	0.1656	0.1589	0.1641	0.1676	0.1677	0.1561	0.1583	0.1724	0.1708
	1.5	1.8	0.8	0.1126	0.1142	0.1150	0.1100	0.1176	0.1166	0.1083	0.1113	0.1187	0.1164
	1.5	1.8	1.2	0.1688	0.1716	0.1671	0.1726	0.1773	0.1749	0.1619	0.1673	0.1826	0.1775
50	0.5	0.9	0.8	0.0748	0.0737	0.0739	0.0732	0.0743	0.0747	0.0712	0.0725	0.0746	0.0753
	0.5	0.9	1.2	0.1136	0.1118	0.1018	0.1097	0.1134	0.1140	0.1070	0.1087	0.1160	0.1115
	0.5	1.8	0.8	0.0745	0.0747	0.0755	0.0766	0.0766	0.0752	0.0733	0.0755	0.0781	0.0777
	0.5	1.8	1.2	0.1112	0.1119	0.1116	0.1103	0.1179	0.1110	0.1094	0.1108	0.1185	0.1137
	1.5	0.9	0.8	0.0743	0.0731	0.0785	0.0721	0.0747	0.0749	0.0729	0.0724	0.0748	0.0764
	1.5	0.9	1.2	0.1126	0.1105	0.1094	0.1093	0.1109	0.1098	0.1076	0.1071	0.1156	0.1105
	1.5	1.8	0.8	0.0753	0.0758	0.0737	0.0758	0.0763	0.0751	0.0739	0.0739	0.0764	0.0778
	1.5	1.8	1.2	0.1106	0.1114	0.1124	0.1136	0.1141	0.1153	0.1115	0.1128	0.1178	0.1180
100	0.5	0.9	0.8	0.0512	0.0508	0.0513	0.0507	0.0508	0.0510	0.0502	0.0496	0.0517	0.0520
	0.5	0.9	1.2	0.0772	0.0765	0.0766	0.0743	0.0759	0.0781	0.0748	0.0757	0.0782	0.0771
	0.5	1.8	0.8	0.0503	0.0504	0.0519	0.0512	0.0512	0.0523	0.0504	0.0507	0.0518	0.0528
	0.5	1.8	1.2	0.0773	0.0776	0.0777	0.0765	0.0781	0.0780	0.0770	0.0774	0.0796	0.0773
	1.5	0.9	0.8	0.0504	0.0499	0.0505	0.0507	0.0512	0.0511	0.0507	0.0507	0.0508	0.0519
	1.5	0.9	1.2	0.0771	0.0764	0.0768	0.0764	0.0778	0.0751	0.0758	0.0766	0.0766	0.0771
	1.5	1.8	0.8	0.0524	0.0526	0.0509	0.0512	0.0517	0.0508	0.0515	0.0518	0.0524	0.0522
	1.5	1.8	1.2	0.0769	0.0773	0.0758	0.0773	0.0778	0.0777	0.0759	0.0767	0.0772	0.0789

ML = Maximum Likelihood, BG = General Entropy Loss Function,
BL = LINEX Loss Function, BS = Squared Error Loss Function

10. Conclusion

Bayes estimators of the unknown parameters, the reliability function and the hazard rate are obtained using vague prior and generalised non-informative prior under both asymmetric and symmetric loss functions via Lindley approximation. Maximum likelihood estimators of the reliability function and the hazard rate are obtained by using Newton-Raphson numerical approach via Taylor series on the parameters.

Comparisons are made between the estimators based on simulation study. The effects of maximum likelihood estimator and Bayes under asymmetric and symmetric loss functions are examined. We observed that the Weibull parameters, reliability function and the hazard rate are best estimated by Bayes using LINEX loss function followed by Bayes using general entropy loss function with uncensored observations. Both occurred with the proposed generalised non-informative prior. We must emphasise that all the estimators seem to converge to the same values with respect to the average mean squared errors as the sample size increases especially with the reliability function and the parameters.

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