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New Explicit Solutions of the Fifth-Order KdV Equation with Variable Coefficients

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Abstract. By means of the modified CK's direct method, we give out the relationship between variable coefficients of the fifth-order KdV equation and the corresponding constant coefficient ones. At the same time, we have studied the generalized fifth-order KdV equation with constants coefficients using the Lie symmetry group methods. By applying the nonclassical symmetry method we found that the analyzed model does not admit supplementary, nonclassical type, symmetries. At last, we give some exact analytic solutions by using the power series method.

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1. Introduction

It is well known that the celebrated KdV types of equations have been around for a very long time. Lot of studies have been conducted with these types of equations [2, 8–10, 20, 23–25]. However, in multifarious real physical backgrounds, nonlinear partial differential equations (NPDEs) with variable coefficients often provide more powerful and realistic models than their constant coefficient counterparts when the inhomogeneities of media is considered. So it is of great importance to seek exact solutions of NPDEs with variable coefficients. In this paper, the generalized fifth-order KdV equation

$$u_t + uu_x + \alpha(t)u + \beta(t)u_{xxxxx} = 0, \tag{1.1}$$

of time dependent variable coefficients of the linear damping and dispersion is investigated. Here in (1.1) the first term represents the evolution term while the second term represents the nonlinear term. The third term represents the linear damping while the fourth term is the dispersion term. The time dependent coefficients of damping and dispersion are, respectively, $\alpha(t)$ and $\beta(t)$ are arbitrary smooth functions of the variable t. A number of special cases of the Equation (1.1) have been successfully used to model physically significant nonlinear problems in mathematical physics, nonlinear dynamics and plasma physics. These

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fifth-order KdV types of equations have been derived to model many physical phenomena, such as gravity-capillary waves on a shallow layer and magneto-sound propagation in plasmas, and so on. In [12] similarity solutions for some classes of the Equation (1.1) were considered. The paper [5] is mainly concerned with the local well-posedness of the initial-value problems for the Kawahara and the modified Kawahara equations in Sobolev spaces. Soliton solutions of a generalized fifth-order KdV equation with t-dependent coefficients are obtained in [21].

Lie group analysis [4, 7, 18, 19] is the most powerful tool to find the general solution of partial differential equations (PDEs). It is well known that symmetry methods for differential equations, was originally developed by Lie [11] at the end of the nineteenth century, which was called classical Lie method. Later in 1969, the notion of nonclassical symmetry was introduced by Bluman and Cole [3]. In this case, one generalizes and includes the classical method for obtaining solutions of PDEs. Afterwards, in 1989, the Clarkson and Kruskal put forward the CK direct method [6]. Lou *et al.* modified the CK's direct method and proposed a simple method (called the modified CK's direct method) to find non-Lie point symmetry group [15]. A great of many authors have used the classical, nonclassical symmetry method and the CK direct method to solve PDEs [14, 16, 17, 22, 26, 27].

Our aim in the present work is to perform the variable coefficients version of the fifth-order KdV equation with the help of the improved direct reduction method [15] and classical and nonclassical symmetries method. Then we obtain the corresponding relationship between explicit solutions of fifth-order KdV equation and ones of corresponding reduced equation, and then get group-invariant solutions.

2. Equivalence transformations of fifth-order KdV equation

In this section, we will utilize the improved direct reduction method for finding the relation between variable coefficient fifth-order KdV equation and the corresponding constant coefficient ones.

We firstly use the improved direct reduction method to look for the equivalence transformations between Equation (1.1) and the following equation

$$u_t + uu_x + au + bu_{xxxxx} = 0, \tag{2.1}$$

where a, b are arbitrary constants. Suppose that Equation (1.1) has the following form solution:

$$u(x,t) = A + BU(X,T), \tag{2.2}$$

where $A = A(x,t), B = B(x,t), X = X(x,t), T = T(x,t)$ are functions to be determined by requiring $U(X,T)$ satisfies the same fifth-order KdV equation as $u = u(x,t)$ with the transformation $\{u,x,t\} \to \{U,X,T\}$. That is to say, restricting $\{U,X,T\}$ to satisfy the following equation with constant coefficients

$$U_T + UU_X + aU + bU_{XXXXX} = 0. \tag{2.3}$$

Substituting (2.2) into (1.1) and requiring $U(X,T)$ satisfying Equation (2.3). Letting the coefficients of U and their derivatives be zero, after tedious calculation, we have

$$X = (d_2 + d_4 x)\mathrm{e}^{-aT} + d_3 x + d_1, T = T(t), \tag{2.4}$$

$$B = \frac{T_t}{d_3 + d_4\mathrm{e}^{-aT}}, A = \frac{(d_4 x + d_2)aT_t\mathrm{e}^{-aT}}{d_3 + d_4\mathrm{e}^{-aT}}, \tag{2.5}$$

with the conditions

$$\alpha(t) = \frac{T_t^2 a\left(d_3 - d_4 \mathrm{e}^{-aT}\right) - T_{tt}\left(d_3 + d_4 \mathrm{e}^{-aT}\right)}{T_t\left(d_3 + d_4 \mathrm{e}^{-aT}\right)}, \beta(t) = \frac{bT_t}{\left(d_3 + d_4 \mathrm{e}^{-aT}\right)^5}, \tag{2.6}$$

where $T(t)$ is an arbitrary function of t, d_1, d_2, d_3 and d_4 are arbitrary constants. With the help of Equation (2.2), we can obtain new exact solutions for the Equation (1.1) as follows:

$$u = \frac{(d_4 x + d_2) a T_t \mathrm{e}^{-aT}}{d_3 + d_4 \mathrm{e}^{-aT}} + \frac{T_t}{d_3 + d_4 \mathrm{e}^{-aT}} U\left((d_2 + d_4 x)\mathrm{e}^{-aT} + d_3 x + d_1, T(t)\right). \tag{2.7}$$

Obviously we have the following symmetry group theorem:

Theorem 2.1. *If $U = U(X,T)$ is a solution of fifth-order KdV Equation* (2.1)*, then*

$$u(x,t) = A + BU(X,T),$$

is also a solution of Equation (1.1)*, where A, B, X, T are decided by* (2.4)–(2.5).

3. Classical and nonclassical symmetry for the fifth-order KdV equation with constant coefficients

In fact, Equation (2.7) is a non-auto-Bäcklund transformation, which gives out the relationship between variable coefficients of the fifth-order KdV equation and constant coefficient ones. In order to solve Equation (1.1) by using Equation (2.1), we need to get exact solutions of Equation (2.1).

In this section, we will look for exact solutions of Equation (2.1) by using classical and nonclassical symmetry method.

3.1. Classical symmetry analysis of (2.1)

If (2.1) is invariant under a one parameter Lie group of point transformations

$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2),\ x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),\ u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2), \tag{3.1}$$

with infinitesimal generator (the vector field)

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}, \tag{3.2}$$

where the coefficient functions $\tau(x,t,u)$, $\xi(x,t,u)$, $\eta(x,t,u)$ of the vector field are to be determined. If the vector field (3.2) generates a symmetry of the fifth-order KdV equation with constant coefficients Equation (2.1), then V must satisfy Lie's symmetry condition

$$pr^{(5)}V(\Delta)|_{\Delta=0} = 0, \tag{3.3}$$

where $\Delta = u_t + uu_x + au + bu_{xxxxx}$. Applying the fifth prolongation $pr^{(5)}V$ to the Equation (2.1), we find the following system of symmetry equations then the invariant condition reads as

$$\eta^t + u\eta^x + \eta u + a\eta + b\eta^{xxxxx} = 0,$$

where

$$\begin{aligned}
\eta^t &= D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau),\\
\eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),\\
\eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi),\\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi),\\
\eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt} D_x(\tau) - u_{xxxx} D_x(\xi),\\
\eta^{xxxxx} &= D_x(\eta^{xxxx}) - u_{xxxxt} D_x(\tau) - u_{xxxxx} D_x(\xi).
\end{aligned} \tag{3.5}$$

Here, D_i denotes the total derivative operator and is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots \ i = 1,2,$$

and $(x^1, x^2) = (t, x)$.

Using (3.4) with the help of (3.5) we obtain the determining equations

$$\begin{aligned}
&\tau = \tau(t),\ \xi_u = 0,\ \eta_{uu} = 0,\\
&b\tau_t - 5b\xi_x = 0,\\
&\eta_{xu} - 2\xi_{xx} = 0,\\
&\tau_t u - \xi_x u + \eta - \xi_t + b(5\eta_{xxxxu} - \xi_{xxxxx}) = 0,\\
&\eta_t + a\eta + au\tau_t - au\eta_u + u\eta_x + b\eta_{xxxxx} = 0.
\end{aligned} \tag{3.6}$$

Solving Equations (3.6), it follows that

$$\tau = c_3,\ \xi_= c_2 + c_1 \mathrm{e}^{-at},\ \eta = -c_1 a \mathrm{e}^{-at}, \tag{3.7}$$

where c_1, c_2 and c_3 are arbitrary constants.

3.2. Nonclassical symmetry analysis of (2.1)

In this section, we will apply the so called nonclassical symmetry method [3]. Therefore, we expect that nonclassical symmetries are much more numerous than classical ones, since any classical symmetry is clearly a nonclassical one. Beside the classical symmetry, we must add the invariance surface condition to the given equation:

$$\Delta_1 = \eta - \xi u_x - \tau u_t. \tag{3.8}$$

The vector field (3.2) is a nonclassical symmetry of (2.1) if

$$pr^{(5)} V(\Delta)|_{\Delta=0,\Delta_1=0} = 0. \tag{3.9}$$

From the nature of the invariant surface condition (3.8), there are two cases to arise: (i)$\tau = 0$, $\xi \neq 0$; (ii)$\tau \neq 0$. In case(i), there are not any nontrivial solutions. Without loss of generality, in case (ii), we set $\tau = 1$. Then from the invariance surface condition (3.8) one can get

$$u_t = \eta - \xi u_x, \tag{3.10}$$

and from (2.1) we get

$$u_{xxxxx} = -(u_t + uu_x + au)/b. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9) leads to

$$\begin{aligned}&\tau=1,\ \xi_u=0,\ \eta_{uu}=0,\\&\eta_{xu}-2\xi_{xx}=0,\\&\eta-\xi_t-5\xi_x\xi+4u\xi_x=0,\\&\eta_t+a\eta+5\xi_x\eta-au\eta_u+u\eta_x+5a\xi_x u=0.\end{aligned}\tag{3.12}$$

The analysis of the associated overdetermined systems Equations (3.12), we obtain

$$\tau=1,\ \xi_{=}c_2+\frac{c_1}{a}\mathrm{e}^{-at},\ \eta=-c_1\mathrm{e}^{-at}.\tag{3.13}$$

where c_1, c_2 are arbitrary constants.

As is easy to see from (3.13), we can find out that the only solutions we found were exactly the solution obtained through the classical symmetry approach. This means that no supplementary symmetries, of non-classical type, are specific for our model. Therefore no new explicit solutions can be obtained by using the nonclassical symmetry method.

Now we shall deal with (3.7).

From (3.7), one can get the symmetry algebra of the fifth-order KdV equation is spanned by the three vector fields

$$V_1=\frac{\partial}{\partial x},\ V_2=\frac{\partial}{\partial t},\ V_3=\mathrm{e}^{-at}\frac{\partial}{\partial x}-a\mathrm{e}^{-at}\frac{\partial}{\partial u}.\tag{3.14}$$

Their commutator table is

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	$-aV_3$
V_3	0	aV_3	0

To obtain the group transformation which is generated by the infinitesimal generators V_i for $i=1,2,3$ we need to solve the following initial problems

$$\begin{aligned}&\frac{d(\bar{x}(\varepsilon))}{d\varepsilon}=\xi\left(\bar{x}(\varepsilon),\bar{t}(\varepsilon),\bar{u}(\varepsilon)\right),\bar{x}(0)=x,\\&\frac{d(\bar{t}(\varepsilon))}{d\varepsilon}=\tau\left(\bar{x}(\varepsilon),\bar{t}(\varepsilon),\bar{u}(\varepsilon)\right),\bar{t}(0)=t,\\&\frac{d(\bar{u}(\varepsilon))}{d\varepsilon}=\xi\left(\bar{x}(\varepsilon),\bar{t}(\varepsilon),\bar{u}(\varepsilon)\right),\bar{u}(0)=u,\end{aligned}\tag{3.15}$$

where ε is a parameter. So we can obtain the Lie symmetry group

$$g:(x,t,u)\rightarrow(\bar{x},\bar{t},\bar{u}).\tag{3.16}$$

Exponentiating the infinitesimal symmetries of Equation (2.1), we get the one-parameter groups $g_i(\varepsilon)$ generated by V_i for $i=1,2,3$

$$\begin{aligned}&g_1:(x,t,u)\mapsto(x+\varepsilon,t,u),\\&g_2:(x,t,u)\mapsto(x,t+\varepsilon,u),\\&g_3:(x,t,u)\mapsto(x+\mathrm{e}^{-at}\varepsilon,t,u-a\mathrm{e}^{-at}\varepsilon).\end{aligned}\tag{3.17}$$

The symmetry groups g_1 and g_2 demonstrate the time- and space-invariance of the equation. Consequently, we can obtain the corresponding Theorem:

Theorem 3.1. *If $u = f(x,t)$ is a solution of fifth-order KdV Equation* (2.1)*, so are the functions*

$$g_1(\varepsilon)\cdot f(x,t) = f(x-\varepsilon,t),$$
$$g_2(\varepsilon)\cdot f(x,t) = f(x,t-\varepsilon), \tag{3.18}$$
$$g_3(\varepsilon)\cdot f(x,t) = f(x+\mathrm{e}^{-at}\varepsilon,t) - a\mathrm{e}^{-at}\varepsilon.$$

4. Symmetry reductions and exact group-invariant solutions of Equation (1.1)

In this section we apply Theorem 2.1 to look for some exact solutions of Equation (1.1).

4.1.1. $c_1 = c_3 = 0$ (V_1).

The group-invariant solution corresponding to V_1 is $u = f(\xi)$, where $\xi = t$ is the group-invariant, the substitution of this solution into the Equation (2.1) gives the trivial solution

$$u(x,t) = C, \tag{4.1}$$

C is a constant.

From Equation (2.7), one can get the solution of Equation (1.1)

$$u = \frac{(d_4x+d_2)aT_t\mathrm{e}^{-aT}}{d_3+d_4\mathrm{e}^{-aT}} + \frac{T_t}{d_3+d_4\mathrm{e}^{-aT}}C. \tag{4.2}$$

4.1.2. $V_2 + cV_1$.

For the linear combination $V_2 + cV_1$, we have

$$u = f(\xi), \tag{4.3}$$

where $\xi = x - ct$ is the group-invariant. Substitution of (4.3) into the Equation (2.1), we reduce it to the following ODE

$$-cf' + ff' + af + bf^{(5)} = 0. \tag{4.4}$$

Now, we seek a solution of Equation (4.4) in a power series of the form

$$f(\xi) = \sum_{n=0}^{\infty} c_n\xi^n. \tag{4.5}$$

Substituting (4.5) into (4.4), we get

$$\begin{aligned} &120bc_5 + b\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}\xi^n + ac_0 + a\sum_{n=1}^{\infty}c_n\xi^n \\ &+c_0c_1 + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}(n-k+1)c_kc_{n-k+1}\right)\xi^n - cc_1 - c\sum_{n=1}^{\infty}(n+1)c_{n+1}\xi^n = 0. \end{aligned} \tag{4.6}$$

From (4.6), comparing coefficients, for $n = 0$, one can get

$$c_5 = \frac{cc_1 - c_0c_1 - ac_0}{120b}. \tag{4.7}$$

Generally, for $n \geq 1$, we have

$$c_{n+5} = \frac{1}{b(n+1)(n+2)(n+3)(n+4)(n+5)}\left(c(n+1)c_{n+1} - \sum_{k=0}^{n}(n-k+1)c_kc_{n-k+1} - ac_n\right). \tag{4.8}$$

From (4.7) and (4.8), we can get all the coefficients $c_n (n \geq 5)$ of the power series (4.5). For arbitrary chosen constant numbers c_0, c_1, c_2, c_3 and c_4, the other terms can be determined successively from (4.7) and (4.8) in a unique way. In addition, it is easy to prove that the convergence of the power series (4.5) with the coefficients give by (4.7) and (4.8) [13]. The details are omitted here. For this reason, this power series solution is an exact analytic solution.

Hence, the power series solution of Equation (4.4) can be written as following

$$\begin{aligned} f(\xi) &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + \sum_{n=1}^{\infty} c_{n+5}\xi^{n+5} \\ &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \frac{cc_1 - c_0c_1 - ac_0}{120b}\xi^5 \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{b(n+1)(n+2)(n+3)(n+4)(n+5)} \\ &\quad \left(c(n+1)c_{n+1} - \sum_{k=0}^{n}(n-k+1)c_kc_{n-k+1} - ac_n\right)\xi^{n+5}. \end{aligned} \tag{4.9}$$

Thus, the exact power series solution of Equation (1.1) is

$$\begin{aligned} u(x,t) =& \frac{(d_4x+d_2)aT_t\mathrm{e}^{-aT}}{d_3+d_4\mathrm{e}^{-aT}} + \frac{T_t}{d_3+d_4\mathrm{e}^{-aT}}\Bigg[c_0 + c_1(X-cT) + c_2(X-cT)^2 + c_3(X-cT)^3 \\ & + c_4(X-cT)^4 + \frac{cc_1 - c_0c_1 - ac_0}{120b}(X-cT)^5 + \sum_{n=1}^{\infty} c_{n+5}(X-cT)^{n+5}\Bigg] \\ =& \frac{(d_4x+d_2)aT_t\mathrm{e}^{-aT}}{d_3+d_4\mathrm{e}^{-aT}} + \frac{T_t}{d_3+d_4\mathrm{e}^{-aT}}\Bigg[c_0 + c_1(X-cT) + c_2(X-cT)^2 + c_3(X-cT)^3 \\ & + c_4(X-cT)^4 + \frac{cc_1 - c_0c_1 - ac_0}{120b}(X-cT)^5 + \sum_{n=1}^{\infty} \frac{1}{b(n+1)(n+2)(n+3)(n+4)(n+5)} \\ & \left(c(n+1)c_{n+1} - \sum_{k=0}^{n}(n-k+1)c_kc_{n-k+1} - ac_n\right)(X-cT)^{n+5}\Bigg]. \end{aligned}$$

where $c_i (i = 0,1,2,3,4)$ are arbitrary constants, the other coefficients $c_n (n \geq 5)$ can be determined successively from (4.7) and (4.8).

Of course, in physical applications, it will be convenient to write the solution of Equation (1.1) in the approximate form

$$\begin{aligned} u(x,t) =& \frac{(d_4x+d_2)aT_t\mathrm{e}^{-aT}}{d_3+d_4\mathrm{e}^{-aT}} + \frac{T_t}{d_3+d_4\mathrm{e}^{-aT}}\Bigg[c_0 + c_1(X-cT) + c_2(X-cT)^2 + c_3(X-cT)^3 \\ & + c_4(X-cT)^4 + \frac{cc_1 - c_0c_1 - ac_0}{120b}(X-cT)^5 + \cdots\Bigg] \end{aligned}$$

in terms of the above computation. Here X and T are given by (2.4).

4.1.3. $V_2 + V_3$.

For this case, we have

$$u = f(\xi) - ax, \tag{4.10}$$

where $\xi = x - \mathrm{e}^{-at}/a$ is the group-invariant. Substituting Equation (4.10) into Equation (2.1), we reduce it to the following ODE

$$bf^{(5)} + ff' - a\xi f' = 0. \tag{4.11}$$

By the same method, we can obtain

$$\begin{aligned} &120bc_5 + b\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}\xi^n - a\sum_{n=1}^{\infty} nc_n\xi^n \\ &+ c_0c_1 + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}(n-k+1)c_kc_{n-k+1}\right)\xi^n = 0. \end{aligned} \tag{4.12}$$

From (4.12), comparing coefficients, we have

$$c_5 = \frac{-c_0c_1}{120b}, \tag{4.13}$$

$$c_{n+5} = \frac{1}{b(n+1)(n+2)(n+3)(n+4)(n+5)}\left(anc_n - \sum_{k=0}^{n}(n-k+1)c_kc_{n-k+1}\right). \tag{4.14}$$

Therefore, the power series solution of Equation (4.11) can be written as following

$$\begin{aligned} f(\xi) &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + \sum_{n=1}^{\infty} c_{n+5}\xi^{n+5} \\ &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \frac{-c_0c_1}{120b}\xi^5 \\ &+ \sum_{n=1}^{\infty}\frac{1}{b(n+1)(n+2)(n+3)(n+4)(n+5)}\left(anc_n - \sum_{k=0}^{n}(n-k+1)c_kc_{n-k+1}\right)\xi^{n+5}. \end{aligned}$$

Thus, the exact power series solution of Equation (1.1) is

$$\begin{aligned} u(x,t) =& \frac{(d_4x+d_2)aT_t\mathrm{e}^{-aT}}{d_3+d_4\mathrm{e}^{-aT}} + \frac{T_t}{d_3+d_4\mathrm{e}^{-aT}}\left[c_0 + c_1\left(X - \frac{\mathrm{e}^{-aT}}{a}\right) + c_2\left(X - \frac{\mathrm{e}^{-aT}}{a}\right)^2 + c_3\left(X - \frac{\mathrm{e}^{-aT}}{a}\right)^3\right. \\ &\left. + c_4\left(X - \frac{\mathrm{e}^{-aT}}{a}\right)^4 + \frac{cc_1 - c_0c_1 - ac_0}{120b}\left(X - \frac{\mathrm{e}^{-aT}}{a}\right)^5 + \sum_{n=1}^{\infty} c_{n+5}(X - \frac{\mathrm{e}^{-aT}}{a})^{n+5} - aX\right] \\ =& \frac{(d_4x+d_2)aT_t\mathrm{e}^{-aT}}{d_3+d_4\mathrm{e}^{-aT}} + \frac{T_t}{d_3+d_4\mathrm{e}^{-aT}}\left[c_0 + c_1\left(X - \frac{\mathrm{e}^{-aT}}{a}\right) + c_2\left(X - \frac{\mathrm{e}^{-aT}}{a}\right)^2\right. \\ &+ c_3\left(X - \frac{\mathrm{e}^{-aT}}{a}\right)^3 + c_4\left(X - \frac{\mathrm{e}^{-aT}}{a}\right)^4 + \frac{cc_1 - c_0c_1 - ac_0}{120b}\left(X - \frac{\mathrm{e}^{-aT}}{a}\right)^5 \\ &+ \sum_{n=1}^{\infty}\frac{1}{b(n+1)(n+2)(n+3)(n+4)(n+5)}\left(anc_n - \sum_{k=0}^{n}(n-k+1)c_kc_{n-k+1}\right)\left(X - \frac{\mathrm{e}^{-aT}}{a}\right)^{n+5} - aX \end{aligned}$$

where $c_i (i = 0,1,2,3,4)$ are arbitrary constants, X and T are given by (2.4), the other coefficients $c_n (n \geq 5)$ can be determined successively from (4.13) and (4.14).

Remark 4.1. It is easy to see that the reduced Equations (4.4) and (4.11) are all higher-order nonlinear or nonautonomous ODEs. If we obtain a one-parameter symmetry group of an ODE, then we could reduce the order of the equation by one. But we find out that such reduced ODEs are more complicated than the original equation. In view of this, we can find that the power series method [1,3–7,11–19,21,22,26,27] is an effective tool of solving such higher-order nonlinear or nonautonomous ODEs. Moreover, from our model, we could found that these power series solutions are important for computations in numerical analysis and physical applications. And above all, these power series play an important role in the investigation of physical phenomena and other natural phenomenon.

5. Conclusion

In this paper, we have studied the generalized fifth-order KdV equation with variable coefficients using the modified CK's direct method. At the same time we get the corresponding Lie algebra and the similarity reductions for equations of constants coefficients. By applying the nonclassical symmetry method we concluded that the analyzed model does not admit supplementary, nonclassical type symmetries. At last, we also get some new exact analytic solutions. These conclusions may be useful for the explanation of some practical physical problems.

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