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Normal Families of Meromorphic Functions with Sharing Functions

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Abstract. Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let k be a positive integer, and let $h(z)(\not\equiv 0,\infty)$ be a meromorphic function in D such that any $f \in \mathscr{F}$ have neither common zeros nor common poles with $h(z)$. If, for each $f \in \mathscr{F}$, the multiplicity of the zeros f is at least k, and $f = 0 \Leftrightarrow f^{(k)} = 0$, and $f^{(k)}(z) = h(z) \Rightarrow f(z) = h(z)$, then $\mathscr{F}$ is normal in D. This improves the results due to Xia and Xu.

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1. Introduction and main result

Let f and g be two meromorphic functions on a domain D in $\mathbb{C}$, and let a be a complex number. If $g(z) = a$ whenever $f(z) = a$, we denote it by $f = a \Rightarrow g = a$. $f = a \Leftrightarrow g = a$ means $f(z) = a$ if and only if $g(z) = a$, and we say that f and g share a.

Let D be a domain in $\mathbb{C}$ and $\mathscr{F}$ a family of meromorphic functions in D. $\mathscr{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathscr{F}$ has a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D, to a meromorphic function or ∞ (see [5, 10, 16]).

In 1992, Schwick [11] found a connection between normality and shared values, and proved the following well-known result.

Theorem 1.1. *Let $\mathscr{F}$ be a family of meromorphic functions defined in D, and let a_1, a_2, a_3 be distinct complex numbers. If for each $f \in \mathscr{F}$, $f = a_i \Leftrightarrow f' = a_i, i = 1, 2, 3$, then $\mathscr{F}$ is normal in D.*

Since then, many scholars studied the normality criteria concerning shared values, such as Pang and Zalcman [8], Chen and Fang [1], Pang [7], Meng [6], Qi, Ding and Yang [9], etc, and have got several more general normality criteria concerning shared values.

In 2001, Fang [2] improved Theorem 1.1 from sharing values to sharing functions, and proved.

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Theorem 1.2. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, and let $\psi(z)(\neq 0)$ be an analytic function in D. If for each $f \in \mathscr{F}$, $f = 0 \Leftrightarrow f'(z) = 0$, and $f'(z) = \psi(z) \Rightarrow f(z) = \psi(z)$, then $\mathscr{F}$ is normal in D.*

In 2006, Xu [14] proved the following result.

Theorem 1.3. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, and k be a positive integer, and let $\psi(z)(\not\equiv 0)$ be an analytic function in D such that $f \in D$ and $\psi(z)$ has no common zeros and $\psi(z)$ has no simple zeros in D . If for each $f \in \mathscr{F}$, all zeros of f have multiplicity at least k, $f = 0 \Leftrightarrow f^{(k)}(z) = 0$, and $f^{(k)}(z) = \psi(z) \Rightarrow f(z) = \psi(z)$, then $\mathscr{F}$ is normal in D.*

In [14], an example was given to show that the condition any $f \in \mathscr{F}$ and $\psi(z)$ have no common zeros is necessary. In fact, the condition $\psi(z)$ has no simple zeros is not necessary.

In 2010, Xia and Xu [12] proved that Theorem 1.3 still holds for a meromorphic function $\psi(z)(\not\equiv 0)$ as following,

Theorem 1.4. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let k be a positive integer, and let $\psi(z)(\not\equiv 0,\infty)$ be an meromorphic function in D such that $f \in \mathscr{F}$ and $\psi(z)$ have no common zeros and $\psi(z)$ has no simple zeros in D, and all poles of $\psi(z)$ have multiplicity at most k. If for each $f \in \mathscr{F}$, all zeros of f have multiplicity at least k, $f = 0 \Leftrightarrow f^{(k)}(z) = 0$, and $f^{(k)}(z) = \psi(z) \Rightarrow f(z) = \psi(z)$, then $\mathscr{F}$ is normal in D.*

In [12], Xia and Xu also proposed a conjecture that the restriction "all poles of $\psi(z)$ have multiplicity at most k" can be relaxed to that $\psi(z)$ has no common poles with any function $f \in \mathscr{F}$. We study this problem, and prove the following result.

Theorem 1.5. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let k be a positive integer, and let $h(z)(\not\equiv 0,\infty)$ be a meromorphic function in D such that f and $h(z)$ have neither common zeros nor common poles for all $f \in \mathscr{F}$. If, for each $f \in \mathscr{F}$, all zeros of f have multiplicity at least k, $f(z) = 0 \Leftrightarrow f^{(k)}(z) = 0$, and $f^{(k)}(z) = h(z) \Rightarrow f(z) = h(z)$, then $\mathscr{F}$ is normal in D.*

An example was also given in [12] to show that the restriction that $h(z)$ has no common poles with any function in $\mathscr{F}$ is indispensable.

2. Some lemmas

For the proof of Theorem 1.5, we require the following lemmas.

Lemma 2.1. [4, 8, 17] *Let k be a positive integer and let $\mathscr{F}$ be a family of meromorphic functions in a domain D, such that each function $f \in \mathscr{F}$ has only zeros of multiplicity at least k, and suppose that there exist $A \geq 1$ such that $|f^{(k)}| \leq A$ whenever $f(z) = 0$, $f \in \mathscr{F}$. If $\mathscr{F}$ is not normal at $z_0 \in D$, then for each $\alpha, 0 \leq \alpha \leq k$, there exist*

a) *points $z_j \in D$, $z_j \to z_0$;*
b) *functions $f_j \in \mathscr{F}$; and*
c) *positive numbers $\rho_j \to 0$*

such that $g_n(\xi) = \rho_j^{-\alpha} f_j(z_j + \rho_j \xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a non-constant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$. Moreover, g has order at most 2.

Lemma 2.2. [2] *Let f be a meromorphic function of finite order in the plane $\mathbb{C}$. If $f = 0 \Leftrightarrow f^{(k)} = 0$, $f' \neq 1$, then f is a constant.*

Lemma 2.3. [3] *Let f be a meromorphic function of finite order in the plane, $k \geq 2$ be a positive integer. If all zeros of f are of order at least $k+1$, and $f^{(k)}(z) = 0 \Leftrightarrow f(z) = 0$, then $f(z)$ is a constant.*

Lemma 2.4. [13] *Let f be a transcendental meromorphic function, let $R(z)(\not\equiv 0)$ be a rational function, and let k be a positive integer. If all zeros of f have multiplicity at least $k+1$, except for finitely many, and $f^{(k)} = 0 \Rightarrow f = 0$, then $f^{(k)} - R(z)$ has infinitely many zeros.*

Lemma 2.5. [15] *Let k, l be two positive integers, and let $Q(z)$ be a rational function all of whose zeros are of order at least k. If $Q^{(k)}(z) \neq z^{-l}$, then $Q(z)$ is a constant.*

3. Proof of Theorem 1.5

Since normality is a local property, we only need to prove that $\mathscr{F}$ is normal at every pole of $h(z)$. Without loss of generality, we may assume that $D = \triangle = \{z : |z| < 1\}, h(z) = b(z)/z^l$, and $b(0) = 1, b(z) \neq 0, \infty, z \in D$, where l is a positive integer. We only need to prove that $\mathscr{F}$ is normal at $z = 0$. Suppose that $\mathscr{F}$ is not normal at $z = 0$. We consider two cases: $l \leq k$ and $l \geq k+1$. In fact, the case $l \leq k$ can be proved as same as Theorem 1.4, so we only need to prove the case $l \geq k+1$.

Consider the family $\mathscr{G} = \{g(z) = f(z)/h(z) : f \in \mathscr{F}, z \in \triangle\}$. Since $z = 0$ is a pole of $h(z)$, and $f(z)$ and $h(z)$ have no common poles, $z = 0$ is a zero of $g(z)$ of order at least $l(\geq k+1)$. Thus all zeros of $g(z)$ have multiplicity at least $k+1$ for the zeros of $f(z)$ are of order at least $k+1$.

We first prove that $\mathscr{G}$ is normal in $\triangle$. Suppose that $\mathscr{G}$ is not normal at $z_0 \in D$. By Lemma 2.1, there exist a sequence of functions g_n, a sequence of points $z_n \to z_0$, and a sequence of positive numbers $\rho_n \to 0^+$, such that

$$G_n(\zeta) = \frac{g_n(z_n + \rho_n \zeta)}{\rho_n^k} \to G(\zeta)$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where $G(\zeta)$ is a nonconstant meromorphic function on $\mathbb{C}$, and $G(\zeta)$ is of order at most two. By Hurwitz's theorem, all zeros of $G(\zeta)$ have multiplicity at least $k+1$.

Two cases are considered in the following.

Case 1 $z_n/\rho_n \to \infty$.

Since $f_n(z) = g_n(z)h(z)$ and $h(z) = b(z)/z^l$, by simple calculation, we have

$$g_n^{(k)}(z) = \frac{f_n^{(k)}(z)}{h(z)} - \sum_{j=1}^{k} C_k^j g_n^{(k-j)}(z) \frac{h^{(j)}(z)}{h(z)}.$$

By mathematical induction, we have

$$\frac{h^{(j)}(z)}{h(z)} = \frac{(-1)^j l(l+1)\cdots(l+j-1)}{z^j} + \frac{C_j^1 (-1)^{j-1} l(l+1)\cdots(l+j-2)}{z^{j-1}} \frac{b'(z)}{b(z)} + \cdots + \frac{C_j^{j-1}(-1)l}{z} \frac{b^{(j-1)}(z)}{b(z)} + \frac{b^{(j)}(z)}{b(z)}.$$

and

$$G_n^{(k)}(\zeta) = g_n^{(k)}(z_n + \rho_n \zeta) \to G^{(k)}(\zeta)$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of G.

Thus

$$\begin{aligned} G_n^{(k)}(\zeta) = &\frac{f_n^{(k)}(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)} - C_k^1 g_n^{(k-1)}(z_n+\rho_n\zeta)\left[\frac{-l}{z_n+\rho_n\zeta} + \frac{b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)}\right] \\ &- \cdots - g_n(z_n+\rho_n\zeta)\left[\frac{(-1)^k l(l+1)\cdots(l+k-1)}{(z_n+\rho_n\zeta)^k} + \right. \\ &\left. \frac{C_k^1(-1)^{k-1}l(l+1)\cdots(l+k-2)}{(z_n+\rho_n\zeta)^{k-1}}\frac{b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)} + \cdots + \frac{b^{(k)}(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)}\right] \\ = &\frac{f_n^{(k)}(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)} - C_k^1 \frac{g_n^{(k-1)}(z_n+\rho_n\zeta)}{\rho_n}\left[\frac{-l}{z_n/\rho_n+\zeta} + \frac{\rho_n b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)}\right] \\ &- \cdots - \frac{g_n(z_n+\rho_n\zeta)}{\rho_n^k}\left[\frac{(-1)^k l(l+1)\cdots(l+k-1)}{(z_n/\rho_n+\zeta)^k} + \right. \\ &\left. \frac{k(-1)^{k-1}(l+1)\cdots(l+k-2)}{(z_n/\rho_n+\zeta)^{k-1}}\frac{\rho_n b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)} + \cdots + \frac{\rho_n^k b^{(k)}(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)}\right]. \end{aligned}$$

On the other hand,

$$\lim_{n\to\infty} \frac{1}{z_n/\rho_n+\zeta} = 0,$$

and

$$\lim_{n\to\infty} \frac{\rho_n^j b^{(j)}(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)} = 0 (j = 1, 2, \ldots, k),$$

uniformly on compact subsets of $\mathbb{C}$. Noting that $g^{(k-j)}(z_n+\rho_n\zeta)/\rho_n^j$ is locally bounded on $\mathbb{C}$ disjoint from the poles of $G(\zeta)$ since $g_n(z_n+\rho_n\zeta)/\rho_n^k \to G(\zeta)$.

Therefore,

$$\frac{f_n^{(k)}(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)} \to G^{(k)}(\zeta)$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of $G(\zeta)$.

We claim that

(i) $G(\zeta) = 0 \Leftrightarrow G^{(k)}(\zeta) = 0$, and
(ii) $G^{(k)}(\zeta) \neq 1$.

Since all zeros of $G(\zeta)$ are of multiplicity at least $k+1$, $G^{(k)}(\zeta) = 0$ whenever $G(\zeta) = 0$. Suppose that $G^{(k)}(\zeta_0) = 0$. Clearly, $G^{(k)}(\zeta) \not\equiv 0$. Otherwise, $G(\zeta)$ would be a polynomial of degree less than k, which contradicts the condition that the multiplicity of the zeros of $G(\zeta)$ is at least $k+1$. Then by Hurwitz's theorem, there exist $\zeta_n, \zeta_n \to \zeta_0$, such that, for n sufficiently large,

$$\frac{f_n^{(k)}(z_n+\rho_n\zeta_n)}{h(z_n+\rho_n\zeta_n)} = 0.$$

Thus $f_n(z_n+\rho_n\zeta_n)=0$ or $h(z_n+\rho_n\zeta_n)=\infty$. If $h(z_n+\rho_n\zeta_n)=\infty$, then $z_n+\rho_n\zeta_n=0$, and $\zeta_n=-z_n/\rho_n\to\infty$, which contradicts the fact that $\zeta_n\to\zeta_0$, ζ_0 is a finite number. So we have $f_n(z_n+\rho_n\zeta_n)=0$. Since f_n and $f_n^{(k)}$ share 0. It follows that

$$G(\zeta_0)=\lim_{n\to\infty}G_n(\zeta_n)=\lim_{n\to\infty}\frac{f_n(z_n+\rho_n\zeta_n)}{\rho_n^k h(z_n+\rho_n\zeta)}=0.$$

This shows that $G(\zeta)=0$ whenever $G^{(k)}(\zeta)=0$.

Next we prove (ii). Suppose that $G^{(k)}(\zeta_0)=1$. We claim that $G^{(k)}(\zeta)\not\equiv 1$. Otherwise, $G(\zeta)$ would be a polynomial of degree k, which is a contradiction with the fact that the zeros of $G(\zeta)$ have multiplicity at least $k+1$. Then by Hurwitz's theorem, there exist $\zeta_n,\zeta_n\to\zeta_0$, such that, for n sufficiently large,

$$\frac{f_n^{(k)}(z_n+\rho_n\zeta_n)-h(z_n+\rho_n\zeta_n)}{h(z_n+\rho_n\zeta_n)}=0.$$

Thus $f_n^{(k)}(z_n+\rho_n\zeta_n)=h(z_n+\rho_n\zeta_n)$ or $h(z_n+\rho_n\zeta_n)=\infty$. As the same as the previous proof, we know that $h(z_n+\rho_n\zeta_n)\neq\infty$. Thus $f_n(z_n+\rho_n\zeta_n)=h(z_n+\rho_n\zeta_n)$ for $f_n^{(k)}(z)=h(z)\Rightarrow f_n(z)=h(z)$. It follows that

$$G(\zeta_0)=\lim_{n\to\infty}G_n(\zeta_n)=\lim_{n\to\infty}\frac{f_n(z_n+\rho_n\zeta_n)}{\rho_n^k h(z_n+\rho_n\zeta_n)}=\infty,$$

which contradicts that $G^{(k)}(\zeta_0)=1$. This prove (ii).

However, it follows by Lemma 2.2 and Lemma 2.3 that $G(\zeta)$ is a constant, a contradiction.

Case 2 $z_n/\rho_n\to\alpha$, where α is a finite complex number.

We have

$$\frac{g_n(\rho_n\zeta)}{\rho_n^k}=\frac{g_n[z_n+\rho_n(\zeta-z_n/\rho_n)]}{\rho_n^k}=G_n(\zeta-\frac{z_n}{\rho_n})\to G(\zeta-\alpha)=\tilde{G}(\zeta)$$

spherically uniformly on compact subsets of $\mathbb{C}$. Since $f(z)$ and $h(z)$ have neither common zeros nor common poles, and the zeros of $f(z)$ have multiplicity at least $k+1$, then $\zeta=0$ is a zero of $\tilde{G}(\zeta)$ with multiplicity $l(\geq k+1)$, or $l+m(m\geq k+1)$. So all zeros of $\tilde{G}(\zeta)$ have multiplicity at least $k+1$.

Set

$$H_n(\zeta)=\frac{f_n(\rho_n\zeta)}{\rho_n^{k-l}}.$$

Then

$$H_n(\zeta)=\frac{f_n(\rho_n\zeta)}{h(\rho_n\zeta)\rho_n^k}\cdot\frac{h(\rho_n\zeta)}{\rho_n^{-l}}=\frac{g_n(\rho_n\zeta)}{\rho_n^k}\cdot\frac{h(\rho_n\zeta)}{\rho_n^{-l}}.$$

Noting that

$$\frac{h(\rho_n\zeta)}{\rho_n^{-l}}\to 1/\zeta^l$$

uniformly on every compact subset of $\mathbb{C}/\{0\}$, thus

$$H_n(\zeta)\to 1/\zeta^l\cdot\tilde{G}(\zeta)=H(\zeta)$$

uniformly on compact subsets of $\mathbb{C}/\{0\}$. Since $z=0$ is the zero of $\tilde{G}(\zeta)$ with multiplicity at least l, and the multiplicity of the other zeros of $\tilde{G}(\zeta)$ is at least $k+1$, the zeros of $H(\zeta)$ have multiplicity at least $k+1$, and $H(0)\neq\infty$.

We claim that

(iii) $H(\zeta)=0 \Leftrightarrow H^{(k)}(\zeta)=0$, and
(iv) $H^{(k)}(\zeta)\neq 1/\zeta^l$.

Since all zeros of $H(\zeta)$ are of multiplicity at least $k+1$, $H^{(k)}(\zeta)=0$ whenever $H(\zeta)=0$. Suppose that $H^{(k)}(\zeta_0)=0$. Clearly, $H^{(k)}(\zeta)\not\equiv 0$. Otherwise $H(\zeta)$ would be a polynomial of degree less than k, which contradicts the fact that the multiplicity of the zeros of $H(\zeta)$ is at least $k+1$. Then by Hurwitz's theorem there exist $\zeta_n, \zeta_n\to\zeta_0$, such that, for n sufficiently large,

$$H_n^{(k)}(\zeta_n)=f_n^{(k)}(\rho_n\zeta_n)\rho_n^l=0,$$

so we have $f_n^{(k)}(\rho_n\zeta_n)=0$. Thus $f_n(\rho_n\zeta_n)=0$ since f_n and $f_n^{(k)}$ share 0. It follows that

$$H(\zeta_0)=\lim_{n\to\infty}H_n(\zeta_n)=\lim_{n\to\infty}\frac{f_n(\rho_n\zeta_n)}{\rho_n^{k-l}}=0.$$

This shows that $H(\zeta)=0$ whenever $H^{(k)}(\zeta)=0$. This completes the proof of (iii).

Next we prove (iv). Suppose $H^{(k)}(\zeta_0)=1/\zeta_0^l$. Obviously, $H^{(k)}(\zeta)\not\equiv 1/\zeta^l, \zeta_0\neq 0$ for $H(0)\neq\infty$. Then $H(\zeta)$ is holomorphic at ζ_0 and noting that

$$\rho_n^l[f_n^{(k)}(\rho_n\zeta)-h(\rho_n\zeta)]\to H^{(k)}(\zeta)-1/\zeta^l$$

uniformly on compact subset of $\mathbb{C}/\{0\}$ disjoint from the poles of $H(\zeta)$, then by Hurwitz's theorem there exist $\zeta_n, \zeta_n\to\zeta_0$, such that, for n sufficiently large, $f_n^{(k)}(\rho_n\zeta_n)=h(\rho_n\zeta_n)$. Thus $f_n(\rho_n\zeta_n)=h(\rho_n\zeta_n)$ for $f^{(k)}(z)=h(z)\Rightarrow f(z)=h(z)$. It follows that

$$H(\zeta_0)=\lim_{n\to\infty}H_n(\zeta_n)=\lim_{n\to\infty}\frac{f_n(\rho_n\zeta_n)b(\rho_n\zeta_n)}{\rho_n^k h(\rho_n\zeta_n)}=\infty,$$

which contradicts that $H^{(k)}(\zeta_0)=1/\zeta_0^l$. This proves (iv).

It follows from Lemma 2.4 that $H(\zeta)$ is a rational function. Noting that $H(0)\neq\infty$, by Lemma 2.5, $H(\zeta)$ is a constant. Since $H(\zeta)=0\Leftrightarrow H^{(k)}(\zeta)=0$, $H\equiv 0$. For $H(\zeta)=1/\zeta^l\cdot\tilde{G}(\zeta)$, thus we have $\tilde{G}(\zeta)=G(\zeta-\alpha)=0$, which is a contradiction. We thus prove $\mathscr{G}$ is normal on $\triangle$.

It remains to prove that $\mathscr{F}$ is normal at 0. Since $\mathscr{G}$ is normal on $\triangle$, then the family $\mathscr{G}$ is equicontinuous on $\triangle$ with respect to the spherical distance. On the other hand, $g(0)=0$ for each $g\in\mathscr{G}$, so there exists $\delta>0$ such that $|g(z)|\leq 1$ for all $g\in\mathscr{G}$ and each $z\in\Delta_\delta=\{z:|z|<\delta\}$. It follow that $f(z)$ is holomorphic on $\triangle_\delta$ for all $f\in\mathscr{F}$. Since $\mathscr{F}$ is normal on $\triangle'$, but it is not normal at $z=0$, there exists a sequence $\{f_n\}\subset\mathscr{F}$ which converges locally uniformly on $\triangle'_\delta$, but not on $\triangle_\delta$. By the maximum modulus principle, we have $f_n\to\infty$ on $\triangle'_\delta$, and hence so does $\{g_n\}\subset\mathscr{G}$, where $g_n=f_n/h$. But $|g_n(z)|\leq 1$ for $z\in\triangle_\delta$, a contradiction. Thus $\mathscr{F}$ is normal in D. Thus this completes the proof of Theorem 1.5.

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