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On the Strong Representability of the Generalized Parallel Sum

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Abstract. We give several regularity conditions, both closedness and interior point type, that ensure the maximal monotonicity of the generalized parallel sum of two maximal monotone operators of Gossez type (D), and we extend some recent results concerning on the same problem.

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1. Introduction and preliminaries

It is well known that in a reflexive Banach space the sum of two (set-valued) maximal monotone operators is still maximal monotone, provided the domain of one of them intersects the interior of the domain of the other (cf. Rockafellar see [35]), but in the nonreflexive case it is still unknown whether this condition is sufficient. However, there are several results, that in particular validate this conjecture. Motivated by a study of parallel connection of electrical multiports, Anderson and Duffin (see [1]) introduced the concept of parallel addition of matrices. Passty (see [27]) approached the parallel sum of arbitrary nonlinear monotone operators starting from the following situation arising from electricity: if two resistors having resistance S and T are connected in parallel, Kirchhoff's law and Ohm's law can be combined to show that their joint resistance is $(S^{-1}+T^{-1})^{-1}$. The same considerations apply to parallel connections of electrical multiports. Instead of resistances which are positive real numbers, however, one must work with impedance operators which map a finite- or infinite dimensional space into itself. There then arises the issue of proper extension of the joint resistance formula given above. Motivated from above, but also inspired from the significant number of results concerning on the problem of maximality of the sum of two maximal monotone operators, Penot and Zălinescu in [31] introduced the concepts of generalized parallel sums.

In what follows X, respectively Y will be real nonzero Banach spaces, and X^*, respectively Y^* will denote their topological dual spaces. Let $S : X \rightrightarrows X^*$, respectively $T : Y \rightrightarrows Y^*$

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be two monotone operators. Moreover, consider the continuous, linear operator $A : X \longrightarrow Y$, and let us denote by A^* its adjoint operator. Recall that the generalized parallel sum $S||_A T$, (see [31]), of the monotone operators S, respectively T is defined as

$$S||_A T : X \rightrightarrows X^*, S||_A T := (S^{-1} + (A^*TA)^{-1})^{-1}.$$

The generalized parallel sum $S||^A T : Y \rightrightarrows Y^*$ (see [31]), is defined by

$$S||^A T := (AS^{-1}A^* + T^{-1})^{-1}.$$

Obviously, when $X = Y$ and $A \equiv id_X$, both sums reduce to the parallel sum introduced by Passty, that is

$$S||T : X \rightrightarrows X^*, S||T := (S^{-1} + T^{-1})^{-1}.$$

Recently, Boţ and László has obtained some conditions, both closedness and interior point type, that ensure the maximal monotonicity of the parallel sum $S||_A T$, (see [13]).

In a recent paper of Simons (see [39]) is given an interior point condition, that ensures the maximal monotonicity of the generalized sum $S||^A T$. Observe, that in reflexive spaces one can easily obtain regularity conditions that ensure the maximal monotonicity of this sum from existing ones, by interchanging the operators with their inverses. However, this is not the case in nonreflexive Banach spaces. Concerning on the generalized parallel sum $S||_A T$, regularity conditions that ensure its maximal monotonicity cannot be obtained from existing ones even in a reflexive Banach space context.

In this paper, we give a closdness type regularity condition that ensures the maximal monotonicity of the generalized parallel sum $S||^A T$, and, we show that our condition is weaker than that given in [39]. Nevertheless, using the same technique we obtain and extend the results from [39] as well. Our results are based on the concepts of representative function and Fenchel conjugate, while the technique used to establish closedness type, respectively interior-point type regularity conditions, that ensure the maximal monotonicity of this generalized parallel sum, is stable strong duality. We deal with the sum problem involving strongly representable operators in nonreflexive Banach spaces, hence, according to a recent result of Marques Alves and Svaiter, our results also hold for operators of negative infimum type (see [36]) and of Gossez type (D) in arbitrary Banach spaces, (see Remark 1.3).

We give an useful application of the stable strong duality for the problem involving the function $f \circ A + g$, where f and g are proper, convex and lower semicontinuous functions, and A is a linear and continuous operator. We also introduce some new generalized infimal convolution formulas, and establish some results concerning on their Fenchel conjugate.

The paper is organized as follows. In the remaining of this section we recall some elements of convex analysis and introduce the necessary apparatus of notions and results referring to monotone operators in general Banach spaces. In Section 2 we introduce some generalized bivariate infimal convolution formulas for which we provide equivalent closedness-type regularity conditions, but also sufficient interiority-type ones. This formula will be used in Section 3 for establishing the maximal monotonicity of Gossez type (D) of a generalized parallel sum of the maximal monotone operators of Gossez type (D) S and T, defined by making use of their extensions to the corresponding biduals. The maximal monotonicity of Gossez type (D) of $S||^A T$ will follow as a particular instance of this general result. A special attention will be also given to the formulation of further sufficient conditions for the interiority-type regularity condition and to the situation when these became equivalent. Finally, in Section 4, as a particular instance of the general result on the maximal monotonicity of $S||^A T$, the maximal monotonicity of the parallel sum of S and T is considered.

1.1. Interiority notions and regularity conditions for stable strong duality

Consider X a separated locally convex space and let X^* be its topological dual space. We denote by w^* the weak* topology on X^* induced by X. We say that the function $f : X \longrightarrow \overline{\mathbb{R}}$ is convex if

$$\forall x,y \in X,\ \forall t \in [0,1]\ : f(tx+(1-t)y) \leq tf(x)+(1-t)f(y),$$

with the conventions $(+\infty)+(-\infty) = +\infty$, $0\cdot(+\infty) = +\infty$ and $0\cdot(-\infty) = 0$ (see [45]). We consider $\text{dom} f = \{x \in X : f(x) < +\infty\}$ the *domain* of f and $\text{epi} f = \{(x,r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\text{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. By $\text{cl} f$ we denote the *lower semicontinuous hull* of f, namely the function whose epigraph is the closure of $\text{epi} f$ in $X \times \mathbb{R}$, that is $\text{epi}(\text{cl} f) = \text{cl}(\text{epi} f)$. We consider also $\text{co} f$, the *convex hull* of f, which is the greatest convex function majorized by f.

We denote by $\langle x^*, x\rangle$ the value of the continuous linear functional $x^* \in X^*$ at $x \in X$. Consider the *identity function* on X, $\text{id}_X : X \longrightarrow X, \text{id}_X(x) = x$ for all $x \in X$. For a function $f : U \times V \longrightarrow \overline{\mathbb{R}}$ we denote by $f^\top$ the *transpose* of f, namely the function $f^\top : V \times U \longrightarrow \overline{\mathbb{R}}, f^\top(v,u) = f(u,v)$ for all $(v,u) \in V \times U$. For E and F two nonempty sets we consider the *projection operator* $\text{pr}_E : E \times F \to E$, $\text{pr}_E(e,f) = e$ for all $(e,f) \in E \times F$. For G and H two further nonempty sets and $k : E \to G$ and $l : F \to H$ two given functions we denote by $k \times l : E \times F \to G \times H$ the function defined as $k \times l(e,f) = (k(e),l(f))$ for all $(e,f) \in E \times F$.

The *indicator function* of U, denoted by δ_U, is defined as $\delta_U : X \longrightarrow \overline{\mathbb{R}}$,

$$\delta_U(x) = \begin{cases} 0, & \text{if } x \in U, \\ +\infty, & \text{otherwise.} \end{cases}$$

The *Fenchel-Moreau conjugate* of the function $f : X \to \overline{\mathbb{R}}$ is the function $f^* : X^* \longrightarrow \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\}\ \forall x^* \in X^*.$$

We mention here some important properties of conjugate functions. We have the so-called *Young-Fenchel inequality*

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle\ \forall x \in X\ \forall x^* \in X^*.$$

The *Fenchel-Moreau Theorem* states that if f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$ (see [45]). Moreover, if f is convex and $(\text{cl} f)(x) > -\infty$ for all $x \in X$, then $f^{**} = \text{cl} f$ (see [45, Theorem 2.3.4]).

For a non-empty set $D \subseteq X$, we denote by $\text{co}(D), \text{cone}(D), \text{aff}(D), \text{lin}(D), \text{inte}(D), \text{cl}(D)$, its *convex hull, conic hull, affine hull, linear hull, interior*, and *closure*, respectively. We say that a set $C \subseteq X$ is closed regarding D, if $C \cap D = \text{cl}(C) \cap D$. We have $\text{cone}(D) = \cup_{t\geq 0} tD$ and if $0 \in D$ then obviously $\text{cone}(D) = \cup_{t>0} tD$. The set $\text{rint}(D)$ is the interior of D relative to $\text{aff}(D)$. Then, the relative interior of D is

$$ri(D) = \begin{cases} \text{rint}(D), & \text{if aff}(D) \text{ is a closed set,} \\ \emptyset, & \text{otherwise.} \end{cases}.$$

The *algebraic interior* (the *core*) of D is the set (see [19, 34, 45])

$$\text{core}(D) = \{u \in X \mid \forall x \in X,\ \exists \delta > 0 \text{ such that } \forall \lambda \in [0,\delta] : u + \lambda x \in D\},$$

while its *relative algebraic interior* (sometimes called also *intrinsic core*) is the set (see [19, 45])

$$\text{icr}(D) = \{u \in X \mid \forall x \in \text{aff}(D - D), \exists \delta > 0 \text{ such that } \forall \lambda \in [0,\delta] : u + \lambda x \in D\}.$$

We consider also the *strong quasi-relative interior* (sometimes called *intrinsic relative algebraic interior*) of D (see [7, 20, 45, 46]), denoted by ${}^{\text{ic}}(D)$,

$${}^{\text{ic}}(D) = \begin{cases} \text{icr}(D), & \text{if aff}(D) \text{ is a closed set,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Obviously, we have $\text{rint}(D) \subseteq \text{icr}(D)$, hence, if $\text{aff}(D)$ is closed, we have $\text{ri}(D) = \text{rint}(D) \subseteq \text{icr}(D) = {}^{\text{ic}}(D)$.

In the case when D is a convex set, the above generalized interiority notions can be characterized as follows:

- $\text{core}(D) = \{x \in D : \text{cone}(D - x) = X\}$ (see [34, 45]);
- $\text{icr}(D) = \{x \in D : \text{cone}(D - x) \text{ is a linear subspace of } X\}$ (see [8, 19, 45]);
- ${}^{\text{ic}}(D) = \{x \in D : \text{cone}(D - x) \text{ is a closed linear subspace of } X\}$ (see [7, 20, 45, 46]);
- $x \in {}^{\text{ic}}(D)$ if and only if $x \in \text{icr}(D)$ and $\text{aff}(D - x)$ is a closed linear subspace of X (see [45, 46]).

We have the following inclusions for a set $D \subseteq X$:

$$\text{inte}(D) \subseteq \text{core}(D) \subseteq {}^{\text{ic}}(D) \subseteq \text{icr}(D) \subseteq D,$$

in general the inclusions being strict. Let us suppose in the following that D is a convex set. In case $\text{inte}(D) \neq \emptyset$, all the generalized interiority notions mentioned above coincide with $\text{inte}(D)$ (see [6, Corollary 2.14]). Let us mention that if X is a Banach space and D is a closed set then $\text{core}(D) = \text{inte}(D)$ (see [34]). For other useful properties of generalized interiority notions see [16].

Consider Y another separated locally convex space and let Y^* be its topological dual space. For a given continuous linear mapping $A : X \longrightarrow Y$, its *adjoint operator*, $A^* : Y^* \longrightarrow X^*$ is defined by $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$ for all $y^* \in Y^*$ and $x \in X$. When X and Y are normed spaces, the *biadjoint operator* of A, $A^{**} : X^{**} \longrightarrow Y^{**}$, is defined as being the adjoint operator of A^*.

In what follows consider the proper, convex and lower semicontinuous functions $f : X \longrightarrow \overline{\mathbb{R}}$ and $g : Y \longrightarrow \overline{\mathbb{R}}$. Moreover, let $A : Y \longrightarrow X$ be a linear and continuous operator and let $A^* : X^* \longrightarrow Y^*$ be its adjoint operator.

The next result ensures stable strong duality between the problems $(P^A) :\ \inf_{y \in Y}\{(f \circ A + g)(y)\}$ and $(D^A) :\ \sup_{x^* \in X^*}\{-(f^*(x^*) + g^*(-A^* x^*))\}$, that is

$$\sup_{y \in Y}\{\langle y^*, y\rangle - (f \circ A + g)(y)\} = \min_{x^* \in X^*}\{f^*(x^*) + g^*(y^* - A^* x^*)\} \text{ for all } y^* \in Y^*.$$

Theorem 1.1. *Assume that X and Y are Fréchet spaces. Suppose that the feasibility condition $A^{-1}(dom(f)) \cap dom(g) \neq \emptyset$ is fulfilled and $0 \in {}^{ic}(dom(f) - A(dom(g)))$. Then,*

$$\sup_{y \in Y}\{\langle y^*, y\rangle - (f \circ A + g)(y)\} = \min_{x^* \in X^*}\{f^*(x^*) + g^*(y^* - A^* x^*)\} \textit{ for all } y^* \in Y^*.$$

Proof. Let us introduce the perturbation function

$$\Phi_A : Y \times X \longrightarrow \overline{\mathbb{R}}, \Phi_A(y,x) = f(x+Ay) + g(y).$$

Obviously, Φ_A is proper, convex and lower semicontinuous. It can be easily verified, that for all $(y^*,x^*) \in Y^* \times X^*$, $\Phi_A^*(y^*,x^*) = f^*(x^*) + g^*(y^* - A^*x^*)$. It is an easy verification that $\mathrm{pr}_X(\mathrm{dom}\Phi_A) = \{x \in X | (\exists)\, y \in \mathrm{dom}(g), \text{such that } x + Ay \in \mathrm{dom}(f)\} = \{u - Av | u \in \mathrm{dom}(f), v \in \mathrm{dom}(g)\} = \mathrm{dom}(f) - A(\mathrm{dom}(g))$. Since $A^{-1}(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$, we have $0 \in \mathrm{pr}_X(\mathrm{dom}\Phi_A)$.

According to Theorem 5.5 from [9], if $0 \in {}^{\mathrm{ic}}(\mathrm{pr}_X(\mathrm{dom}\Phi_A))$, then

$$\sup_{y\in Y}\{\langle y^*,y\rangle - \Phi_A(y,0)\} = \min_{x^*\in X^*} \Phi_A^*(y^*,x^*)\, \forall y^* \in Y^*.$$

In other words, if $0 \in {}^{\mathrm{ic}}(\mathrm{dom}(f) - A(\mathrm{dom}(g)))$, then

$$\sup_{y\in Y}\{\langle y^*,y\rangle - (f\circ A + g)(y)\} = \min_{x^*\in X^*}\{f^*(x^*) + g^*(y^* - A^*x^*)\} \text{ for all } y^* \in Y^*.$$ ∎

Remark 1.1. According to Proposition 4 from [44] if X and Y are Fréchet spaces and $0 \in {}^{\mathrm{ic}}(\mathrm{pr}_X(\mathrm{dom}\Phi_A))$ then ${}^{\mathrm{ic}}(\mathrm{pr}_X(\mathrm{dom}\Phi_A)) = \mathrm{ri}(\mathrm{pr}_X(\mathrm{dom}\Phi_A))$, hence

$${}^{\mathrm{ic}}(\mathrm{dom}(f) - A(\mathrm{dom}(g))) = \mathrm{ri}(\mathrm{dom}(f) - A(\mathrm{dom}(g)))$$

In what follows we give an equivalent condition to stable strong duality for the problems (P^A) and (D^A) considered above.

Theorem 1.2. *Let U be a nonempty subset of Y^* and assume that the feasibility condition $A^{-1}(dom(f)) \cap dom(g) \neq \emptyset$ is fulfilled. The following statements are equivalent:*

(i) $\sup_{y\in Y}\{\langle y^*,y\rangle - (f\circ A+g)(y)\} = \min_{x^*\in X^*}\{f^*(x^*)+g^*(y^*-A^*x^*)\}$ *for all* $y^* \in U$.
(ii) *The set* $\{(A^*x^*+y^*,r) : f^*(x^*)+g^*(y^*) \le r\}$ *is closed regarding* $U\times\mathbb{R}$ *in* $(Y^*,w^*)\times\mathbb{R}$ *topology.*

Proof. Let us introduce the perturbation function Φ_A as in the proof of Theorem 1.1. It is an easy verification that $\mathrm{pr}_{Y^*\times\mathbb{R}}(\mathrm{epi}\Phi_A^*) = \{(A^*x^* + y^*, r) : f^*(x^*) + g^*(y^*) \le r, x^* \in X^*, y^* \in Y^*\}$. According to Theorem 2 from [10], the following conditions are equivalent:

(a) $\sup_{y\in Y}\{\langle y^*,y\rangle - \Phi_A(y,0)\} = \min_{x^*\in X^*}\Phi_A^*(y^*,x^*)$, for all $y^* \in U$.
(b) The set $\mathrm{pr}_{Y^*\times\mathbb{R}}(\mathrm{epi}(\Phi_A^*))$ is closed regarding $U\times\mathbb{R}$ in $(Y^*,w^*)\times\mathbb{R}$ topology.

In other words, the following statements are equivalent:

(i) $\sup_{y\in Y}\{\langle y^*,y\rangle - (f\circ A+g)(y)\} = \min_{x^*\in X^*}\{f^*(x^*)+g^*(y^*-A^*x^*)\}$ for all $y^* \in U$.
(ii) The set $\{(A^*x^*+y^*,r) : f^*(x^*)+g^*(y^*) \le r\}$ is closed regarding $U\times\mathbb{R}$ in $(Y^*,w^*)\times\mathbb{R}$ topology. ∎

Remark 1.2. Observe that if X and Y are Fréchet spaces, then the condition $\{(A^*x^* + y^*,r) : f^*(x^*)+g^*(y^*) \le r\}$ is closed in $(Y^*,w^*)\times\mathbb{R}$ topology is weaker than the condition $0 \in^{ic} (dom(f) - A(dom(g)))$

1.2. Maximal monotone operators and representative functions

Consider further X a nontrivial Banach space, let X^* be its topological dual space and let X^{**} be its bidual space. A set-valued operator $S : X \rightrightarrows X^*$ is said to be *monotone* if

$$\langle y^* - x^*, y - x\rangle \ge 0, \text{ whenever } y^* \in S(y) \text{ and } x^* \in S(x).$$

The monotone operator S is called *maximal monotone* if its graph

$$G(S) = \{(x,x^*) : x^* \in S(x)\} \subseteq X \times X^*$$

is not properly contained in the graph of any other monotone operator $S' : X \rightrightarrows X^*$. For S we consider also its *domain* $D(S) = \{x \in X : S(x) \neq \emptyset\} = \mathrm{pr}_X(G(S))$ and its *range* $R(S) = \cup_{x \in X} S(x) = \mathrm{pr}_{X^*}(G(S))$.

The classical example of a maximal monotone operator is the subdifferential of a proper, convex and lower semicontinuous function (this result is due to Rockafellar, see [35]). However, there exist maximal monotone operators which are not subdifferentials (see [36, 37]).

To an arbitrary monotone operator $S : X \rightrightarrows X^*$ we associate the *Fitzpatrick function* $\varphi_S : X \times X^* \longrightarrow \overline{\mathbb{R}}$, defined by

$$\varphi_S(x,x^*) = \sup\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in S(y)\},$$

which is obviously convex and strong-weak* lower semicontinuous (it is even weak-weak* lower semicontinuous) in the corresponding topology on $X \times X^*$. Introduced by Fitzpatrick in 1988 (see [17]) and rediscovered after some years in [15, 21], it proved to be very important in the theory of maximal monotone operators, revealing important connections between convex analysis and monotone operators (see [3–5], [11, 12], [15, 22], [30, 31, 36, 40], [28, 29, 41, 47] and the references therein).

Considering the function $c : X \times X^* \to \mathbb{R}$, $c(x,x^*) = \langle x^*, x\rangle$ for all $(x,x^*) \in X \times X^*$, we get the equality $\varphi_S(x,x^*) = c_S^*(x^*,x)$ for all $(x,x^*) \in X \times X^*$, where $c_S = c + \delta_{G(S)}$ and we are considering the natural injection $X \subseteq X^{**}$. Let us recall the most important properties of the Fitzpatrick function.

Lemma 1.1. *(*see [17]*) Let $S : X \rightrightarrows X^*$ be a maximal monotone operator. Then*

(i) $\varphi_S(x,x^*) \geq \langle x^*, x\rangle$ *for all* $(x,x^*) \in X \times X^*$,
(ii) $G(S) = \{(x,x^*) \in X \times X^* : \varphi_S(x,x^*) = \langle x^*, x\rangle\}$.

Motivated by these properties of the Fitzpatrick function, the notion of *representative function* of a monotone operator was introduced and studied in the literature.

Definition 1.1. *For $S : X \rightrightarrows X^*$ a monotone operator, we call* representative function *of S a convex and lower semicontinuous function $h_S : X \times X^* \longrightarrow \overline{\mathbb{R}}$ (in the strong topology of $X \times X^*$) fulfilling*

$$h_S \geq c \text{ and } G(S) \subseteq \{(x,x^*) \in X \times X^* : h_S(x,x^*) = \langle x^*, x\rangle\}.$$

We observe that if $G(S) \neq \emptyset$ (in particular if S is maximal monotone), then every representative function of S is proper. It follows immediately that the Fitzpatrick function associated to a maximal monotone operator is a representative function of the operator. The following proposition is a direct consequence of some results given in [15].

Proposition 1.1. *Let $S : X \rightrightarrows X^*$ be a maximal monotone operator and h_S be a representative function of S. Then*

(i) $\varphi_S \leq h_S$,
(ii) *the canonical restriction of $h_S^{*\top}$ to $X \times X^*$ is also a representative function of S,*
(iii) $\{(x,x^*) \in X \times X^* : h_S(x,x^*) = \langle x^*, x\rangle\} = \{(x,x^*) \in X \times X^* : h_S^{*\top}(x,x^*) = \langle x^*, x\rangle\} = G(S)$.

Let us give the following maximality criteria valid in reflexive Banach spaces (cf. [14, Theorem 3.1] and [31, Proposition 2.1]; see also [37] for other maximality criteria in reflexive spaces). We refer to [23, Theorem 4.2] for a generalization of the next result to arbitrary Banach spaces.

Theorem 1.3. (cf. [14, 31]) *Let X be a reflexive Banach space and $f : X \times X^* \longrightarrow \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function such that $f \geq c$. Then the operator whose graph is the set $\{(x,x^*) \in X \times X^* : f(x,x^*) = \langle x^*,x\rangle\}$ is maximal monotone if and only if $f^{*\top}|_{X\times X^*} \geq c$.*

The following particular class of maximal monotone operators has been recently introduced in [23], being also studied in [42].

Definition 1.2. *An operator $S : X \rightrightarrows X^*$ is said to be* strongly-representable *whenever there exists a proper, convex and strong lower semicontinuous function $h : X \times X^* \longrightarrow \overline{\mathbb{R}}$ such that*

$$h \geq c, h^*(x^*,x^{**}) \geq \langle x^{**},x^*\rangle \forall (x^*,x^{**}) \in X^* \times X^{**}$$

and

$$G(S) = \{(x,x^*) \in X \times X^* : h(x,x^*) = \langle x^*,x\rangle\}.$$

In this case h is called a strong-representative *of S.*

The following result is a generalization of Theorem 1.3 (see [23, Theorem 4.2]).

Theorem 1.4. *Let X be a nonzero Banach space and $h : X \times X^* \longrightarrow \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function such that $h \geq c$ and $h^*(x^*,x^{**}) \geq \langle x^{**},x^*\rangle$ for all $(x^*,x^{**}) \in X^* \times X^{**}$. Then the operator whose graph is the set $\{(x,x^*) \in X \times X^* : h(x,x^*) = \langle x^*,x\rangle\}$ is maximal monotone and it holds $\{(x,x^*) \in X \times X^* : h(x,x^*) = \langle x^*,x\rangle\} = \{(x,x^*) \in X \times X^* : h^*(x^*,x) = \langle x^*,x\rangle\}$.*

Hence, if $S : X \rightrightarrows X^*$ is strongly-representable, then S is maximal monotone (see also [42, Theorem 8]), and φ_S is a strong-representative of S.

Definition 1.3. (see [18]) *Gossez's monotone closure of a maximal monotone operator $S : X \rightrightarrows X^*$ is $\overline{S} : X^{**} \rightrightarrows X^*$,*

$$G(\overline{S}) = \{(x^{**},x^*) \in X^{**} \times X^* : \langle x^* - y^*, x^{**} - y\rangle \geq 0, (\forall)(y,y^*) \in G(S)\}.$$

A maximal monotone operator $S : X \rightrightarrows X^*$ is of Gossez type (D) if for any $(x^{**},x^*) \in G(\overline{S})$, there exists a bounded net $\{(x_\alpha,x^*_\alpha)\}_{\alpha\in\mathfrak{I}} \subseteq G(S)$ which converges to (x^{**},x^*) in the $w^* \times \|\cdot\|$ topology of $X^{**} \times X^*$.

In [38] Simons introduced a new class of maximal monotone operators, called operators of negative infimum type (NI).

Definition 1.4. (see [38]) *A maximal monotone operator $S : X \rightrightarrows X^*$ is of Simons type (NI) if*

$$\inf_{(y,y^*)\in G(S)} \langle y^* - x^*, y - x^{**}\rangle \geq 0, (\forall)(x^*,x^{**}) \in X^* \times X^{**}.$$

Remark 1.3. Marques Alves and Svaiter recently proved that the class of strongly-representable operators, the class of maximal monotone operators of type (NI) and the class of maximal monotone operators of Gossez type (D) coincide (cf. [24, Theorem 1.2] and [25, Theorem 4.4]).

We will need further the following result of Simons, adapted to our purposes.

Proposition 1.2. *(*see [36] Lemma 20.4(b).*) Let* $S : X^{**} \rightrightarrows X^*$ *be maximal monotone operator and* $(x^*, x^{**}) \in X^* \times X^{**}$*, with* $\langle x^{**}, x^* \rangle = 0$*. Suppose that there exists* $u \in \mathbb{R}$ *such that* $\langle x^{**}, s^* \rangle + \langle s^{**}, x^* \rangle = \langle (x^*, x^{**}), (s^{**}, s^*) \rangle = u$ *for all* $(s^{**}, s^*) \in G(S)$*. Then* $\langle (x^*, x^{**}), (s^{**}, s^*) \rangle = u$ *for all* $(s^{**}, s^*) \in dom\varphi_S$*, where* φ_S *denotes the Fitzpatrick function of* S*.*

2. About a generalized bivariate infimal convolution formula

Let X and Y be two normed spaces, let X^* and Y^*, respectively X^{**} and Y^{**} be their topological duals, respectively their topological biduals, and consider the proper, convex and lower semicontinuous functions $f : X \times X^* \longrightarrow \overline{\mathbb{R}}$ and $g : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$. Moreover, let $A : X \longrightarrow Y$ be a linear and continuous operator and $A^* : Y^* \longrightarrow X^*$, respectively $A^{**} : X^{**} \longrightarrow Y^{**}$ be its adjoint, respectively its biadjoint operator.

Consider the following generalized infimal convolution formulas, $f \square_1^A g : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$

$$(f \square_1^A g)(y, y^*) = \inf\{f(x, A^* y^*) + g(y - Ax, y^*) : x \in X\},$$

respectively, $f^* \square_2^A g^* : Y^* \times Y^{**} \longrightarrow \overline{\mathbb{R}}$,

$$(f^* \square_2^A g^*)(y^*, y^{**}) = \inf\{f^*(A^* y^*, x^{**}) + g^*(y^*, y^{**} - A^{**} x^{**}) : x^{**} \in X^{**}\}.$$

Due to our best knowledge $\square_1^A$, respectively $\square_2^A$ were not considered till now in the literature. Obviously, when $A \equiv id_X$, $X = Y$ we obtain the classical bivariate infimal convolutions $f \square_1 g$ and $f^* \square_2 g^*$, respectively (see, for instance, [10, 36, 40, 42]), that is

$$(f \square_1 g)(x, x^*) = \inf\{f(u, x^*) + g(x - u, x^*) : u \in X\},$$

respectively,

$$(f^* \square_2 g^*)(x^*, x^{**}) = \inf\{f^*(x^*, u^{**}) + g^*(x^*, x^{**} - u^{**}) : u^{**} \in X^{**}\}.$$

The following result provides a closedness type regularity condition that not only ensures that $(f \square_1^A g)^*(y^*, y^{**}) = (f^* \square_2^A g^*)(y^*, y^{**})$ and $f^* \square_2^A g^*$ is exact for every $(y^*, y^{**}) \in Y^* \times Y^{**}$, that is, the infimum in its definition is attained, but is also equivalent to it.

Theorem 2.1. *Consider the proper, convex and lower semicontinuous functions* $f : X \times X^* \longrightarrow \overline{\mathbb{R}}$ *and* $g : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$*, such that* $pr_{X^*}(dom(f)) \cap A^*(pr_{Y^*}(dom(g))) \neq \emptyset$*.*

(a) *The following statements are equivalent:*
  (i) $(CQ^{\square_1^A})$ *: The set* $\{(x^*, y^*, A^{**} x^{**} + y^{**}, r) : f^*(x^*, x^{**}) + g^*(y^*, y^{**}) \le r\}$ *is closed regarding the set* $\Delta_{A^*}^{Y^*} \times Y^{**} \times \mathbb{R}$ *in the* $(X^*, w^*) \times (Y^*, w^*) \times (Y^{**}, w^*) \times \mathbb{R}$ *topology, where* $\Delta_{A^*}^{Y^*} = \{(A^* y^*, y^*) : y^* \in Y^*\}$*.*
  (ii) $(f \square_1^A g)^*(y^*, y^{**}) = (f^* \square_2^A g^*)(y^*, y^{**})$ *and* $f^* \square_2^A g^*$ *is exact for every* $(y^*, y^{**}) \in Y^* \times Y^{**}$*.*

(b) *If* $(RC^{\square_1^A}) : 0 \in^{ic} (pr_{X^*}(dom(f)) - A^* pr_{Y^*}(dom(g)))$ *holds, then the statements (i) and (ii) are true.*

Proof. Consider the functions $F : X \times Y \times X^* \longrightarrow \overline{\mathbb{R}}$, $F(u, v, u^*) = f(u, u^*)$ and $G : X \times Y \times Y^* \longrightarrow \overline{\mathbb{R}}$, $G(u, v, v^*) = g(v, v^*)$, and the linear continuous operator $M : X \times Y \times Y^* \longrightarrow X \times Y \times X^*$, $M = id_X \times id_Y \times A^*$. Since $\mathrm{pr}_{X^*}(\mathrm{dom}(f)) \cap A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(g))) \neq \emptyset$ we obtain that $M^{-1}(\mathrm{dom}(F)) \cap \mathrm{dom}(G) \neq \emptyset$.

(a) We have

$(f\Box_1^A g)^*(y^*,y^{**}) = \sup_{(s,v^*)\in Y\times Y^*}\{\langle (y^*,y^{**}),(s,v^*)\rangle - \inf_{u\in X}\{f(u,A^*v^*)+g(s-Au,v^*)\}\}$
$= \sup_{(u,s,v^*)\in X\times Y\times Y^*}\{\langle (y^*,y^{**}),(s,v^*)\rangle - f(u,A^*v^*)-g(s-Au,v^*)\}$
$= \sup_{(u,v,v^*)\in X\times Y\times Y^*}\{\langle (y^*,y^{**}),(v+Au,v^*)\rangle - f(u,A^*v^*)-g(v,v^*)\}$
$= \sup_{(u,v,v^*)\in X\times Y\times Y^*}\{\langle (A^*y^*,y^*,y^{**}),(u,v,v^*)\rangle - (F\circ M)(u,v,v^*)-G(u,v,v^*)\}.$

It is an easy computation that $F^*(u^*,v^*,u^{**}) = \delta_{\{0\}}(v^*)+f^*(u^*,u^{**})$ and $G^*(u^*,v^*,v^{**}) = \delta_{\{0\}}(u^*)+g^*(v^*,v^{**})$.

Let $U = \Delta_{A^*}^{Y^*}\times Y^{**}\subseteq X^*\times Y^*\times Y^{**}$. According to Theorem 1.2 the following statements are equivalent:

(1) $\sup_{(u,v,v^*)\in X\times Y\times Y^*}\{\langle (A^*y^*,y^*,y^{**}),(u,v,v^*)\rangle - (F\circ M)(u,v,v^*)-G(u,v,v^*)\} = \min_{(u^*,v^*,u^{**})\in X^*\times Y^*\times X^{**}}\{F^*(u^*,v^*,u^{**})+G^*((A^*y^*,y^*,y^{**})-M^*(u^*,v^*,u^{**}))\}$, for all $(A^*y^*,y^*,y^{**})\in U$.

(2) $\{(M^*(u_1^*,v_1^*,u^{**})+(u_2^*,v_2^*,v^{**}),r) : F^*(u_1^*,v_1^*,u^{**})+G^*(u_2^*,v_2^*,v^{**})\le r\}$ is closed regarding to $U\times\mathbb{R}$ in $(X^*,w^*)\times(Y^*,w^*)\times(Y^{**},w^*)\times\mathbb{R}$ topology. Thus, we obtain that $\sup_{(u,v,v^*)\in X\times Y\times Y^*}\{\langle (A^*y^*,y^*,y^{**}),(u,v,v^*)\rangle - (F\circ M)(u,v,v^*)-G(u,v,v^*)\} = \min_{u^{**}\in X^{**}}\{f^*(A^*y^*,u^{**})+g^*(y^*,y-A^{**}u^{**})\}$ for all $(A^*y^*,y^*,y^{**})\in U$ is equivalent to

$\{(u^*,v^*,A^{**}u^{**}+v^{**},r) : f^*(u^*,u^{**})+g^*(v^*,v^{**})\le r\}$ is closed regarding the set $U\times\mathbb{R} = \Delta_{A^*}^{Y^*}\times Y^{**}\times\mathbb{R}$ in $(X^*,w^*)\times(Y^*,w^*)\times(Y^{**},w^*)\times\mathbb{R}$ topology.

In other words, $(f\Box_1^A g)^*(y^*,y^{**}) = (f^*\Box_2^A g^*)(y^*,y^{**})$ and $f^*\Box_2^A g^*$ is exact for every $(y^*,y^{**})\in Y^*\times Y^{**}$ if, and only if, $\{(u^*,v^*,A^{**}u^{**}+v^{**},r) : f^*(u^*,u^{**})+g^*(v^*,v^{**})\le r\}$ is closed regarding the set $\Delta_{A^*}^{Y^*}\times Y^{**}\times\mathbb{R}$ in $(X^*,w^*)\times(Y^*,w^*)\times(Y^{**},w^*)\times\mathbb{R}$ topology.

(b) The assertion is a direct consequence of Theorem 1.1, as obviously,

$(0,0,0)\in^{\mathrm{ic}}(\mathrm{dom}(F)-M\,\mathrm{dom}(G)) \Leftrightarrow 0\in{}^{\mathrm{ic}}(\mathrm{pr}_{X^*}(\mathrm{dom}(f))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(g)))$. ∎

Remark 2.1. According to Remark 1.1 ${}^{\mathrm{ic}}(\mathrm{pr}_{X^*}(\mathrm{dom}(f))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(g))) = \mathrm{ri}(\mathrm{pr}_{X^*}(\mathrm{dom}(f))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(g)))$.

By taking $X=Y$ and $A\equiv id_X$ in Theorem 2.1 we obtain the following result.

Corollary 2.1. Assume that $\mathrm{pr}_{X^*}(\mathrm{dom}(f))\cap\mathrm{pr}_{X^*}(\mathrm{dom}(g))\neq\emptyset$.

(a) The following statements are equivalent:
   (i) $(CQ^{\Box_1}) : \{(u^*,v^*,u^{**}+v^{**},r) : f^*(u^*,u^{**})+g^*(v^*,v^{**})\le r\}$ is closed regarding the set $\Delta_{X^*}\times X^{**}\times\mathbb{R}$ in the $(X^*,w^*)\times(X^*,w^*)\times(X^{**},w^*)\times\mathbb{R}$ topology, where $\Delta_{X^*} = \{(x^*,x^*) : x^*\in X^*\}$.
   (ii) $(f\Box_1 g)^*(x^*,x^{**}) = (f^*\Box_2 g^*)(x^*,x^{**})$ and $f^*\Box_2 g^*$ is exact for every $(x^*,x^{**})\in X^*\times X^{**}$.

(b) If $(RC^{\Box_1}) : 0\in{}^{\mathrm{ic}}(\mathrm{pr}_{X^*}(\mathrm{dom}(f))-\mathrm{pr}_{X^*}(\mathrm{dom}(g)))$ holds, then the statements (i) and (ii) are true.

Remark 2.2. Observe that the condition $(CQ^{\Box_1^A}))$, i.e. $\{(x^*,y^*,A^{**}x^{**}+y^{**},r) : f^*(x^*,x^{**})+g^*(y^*,y^{**})\le r\}$ is closed regarding the set $\Delta_{A^*}^{Y^*}\times Y^{**}\times\mathbb{R}$ in the $(X^*,w^*)\times(Y^*,w^*)\times(Y^{**},w^*)\times\mathbb{R}$ topology is weaker than $(RC^{\Box_1^A})$, i.e. $0\in{}^{\mathrm{ic}}((\mathrm{pr}_{X^*}(\mathrm{dom}(f))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(g))))$.

3. The maximal monotonicity of the generalized parallel sum $S||^A T$

In the sequel, unless is otherwise specified, X and Y are nonzero Banach spaces, and X^* and Y^*, respectively X^{**} and Y^{**} denote their duals, respectively their biduals.

Consider the monotone operators $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ and let $A : X \longrightarrow Y$ be a linear and continuous operator, and let A^*, respectively A^{**} be its adjoint, respectively its biadjoint operator. Let us denote by $\overline{S}$, respectively $\overline{T}$ the Gossez monotone closure of S, respectively T. Then their generalized parallel sum may be introduced as $\overline{S}||^A\overline{T} : Y \rightrightarrows Y^*$,

$$\overline{S}||^A\overline{T} := (A^{**}\overline{S}^{-1}A^* + \overline{T}^{-1})^{-1}.$$

When $X = Y, A \equiv \mathrm{id}_X$ we obtain the parallel sum for the Gossez monotone closure of S, respectively T, that is $\overline{S}||\overline{T} : X \rightrightarrows X^*$,

$$\overline{S}||\overline{T} := (\overline{S}^{-1} + \overline{T}^{-1})^{-1}.$$

Next we will provide some conditions that ensures the maximal monotonicity of the generalized parallel sum $\overline{S}||^A\overline{T}$. Due to our best knowledge, in the literature does not exists so far any closedness type regularity condition that provides this result. However, in a recent paper of Simons, (see [39]), some of the interior point type regularity conditions that will be presented in the sequel are also obtained. In what follows, based on the results presented in Section 2, we will give both an interior point type and a closedness type regularity condition, that ensure the maximal monotonicity of the generalized parallel sum $\overline{S}||^A\overline{T}$.

Theorem 3.1. *Let $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ be two maximal monotone operators of Gossez type* (D)*, with strong representative functions h_S and h_T respectively, and let $A : X \longrightarrow Y$ be a linear and continuous operator, with its adjoint denoted by A^*, and its biadjoint denoted by A^{**}. Assume, that $pr_{X^*}(dom(h_S)) \cap A^*pr_{Y^*}(dom(h_T)) \neq \emptyset$. Assume that one of the following conditions is fulfilled.*

(a) $0 \in {}^{ic}(pr_{X^*}(dom(h_S)) - A^*pr_{Y^*}(dom(h_T)))$.

(b) *The set $\{(x^*, y^*, A^{**}x^{**} + y^{**}, r) : h_S^*(x^*, x^{**}) + h_T^*(y^*, y^{**}) \leq r\}$ is closed regarding the set $\Delta_{A^*}^{Y^*} \times Y^{**} \times \mathbb{R}$ in the $(X^*, w^*) \times (Y^*, w^*) \times (Y^{**}, w^*) \times \mathbb{R}$ topology.*

Then the function $h : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$, $h(y, y^*) = cl_{\|\cdot\| \times \|\cdot\|_*}(h_S \Box_1^A h_T)(y, y^*)$ is a strong representative function of $\overline{S}||^A\overline{T}$ and the generalized parallel sum $\overline{S}||^A\overline{T}$ is maximal monotone operator of Gossez type (D).

Proof. Obviously h is convex and lower semicontinuous in $(Y, \|\cdot\|) \times (Y^*, \|\cdot\|_*)$ topology and due to the feasibility condition $\mathrm{pr}_{X^*}(\mathrm{dom}(h_S)) \cap A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(h_T))) \neq \emptyset$ h is not identically $+\infty$. According to Theorem 2.1, if either (a) or (b) holds, then $h^*(y^*, y^{**}) = (h_S \Box_1^A h_T)^*(y^*, y^{**}) = (h_S^* \Box_2^A h_T^*)(y^*, y^{**})$ and the infimal convolution of the right side is exact.

Next we prove that $h(y, y^*) \geq \langle y^*, y\rangle$ for all $(y, y^*) \in Y \times Y^*$, moreover $G(\overline{S}||^A\overline{T}) = \{(y, y^*) : h^*(y^*, y) = \langle y^*, y\rangle\}$ and $h^*(y^*, y^{**}) \geq \langle y^{**}, y^*\rangle$ for all $(y^*, y^{**}) \in Y^* \times Y^{**}$. Then, according to Theorem 1.4, the operator whose graph is $\{(y, y^*) \in Y \times Y^* : h(y, y^*) = \langle y^*, y\rangle\}$ is maximal monotone of Gossez type (D), and $\{(y, y^*) \in Y \times Y^* : h(y, y^*) = \langle y^*, y\rangle\} = \{(y, y^*) \in Y \times Y^* : h^*(y^*, y) = \langle y^*, y\rangle\}$. Hence, the generalized parallel sum $\overline{S}||^A\overline{T}$ is a maximal monotone operator of Gossez type (D).

We have $(h_S \Box_1^A h_T)(y, y^*) = \inf\{h_S(x, A^*y^*) + h_T(y - Ax, y^*) : x \in X\} \geq \inf\{\langle A^*y^*, x\rangle + \langle y^*, y - Ax\rangle : x \in X\} = \langle y^*, y\rangle$. Hence, $h(y, y^*) = \mathrm{cl}_{\|\cdot\| \times \|\cdot\|_*}(h_S \Box_1^A h_T)(y, y^*) \geq \langle y^*, y\rangle$, which implies that $h \geq c$, concomitantly ensuring that h is proper.

We have $h^*(y^*, y^{**}) = (h_S^* \Box_2^A h_T^*)(y^*, y^{**}) = \inf_{x^{**} \in X^{**}}\{h_S^*(A^*y^*, x^{**}) + h_T^*(y^*, y^{**} - A^{**}x^{**})\}$ and the infimum is attained.

Hence, $h^*(y^*,y^{**}) = \min_{x^{**}\in X^{**}}\{h_S^*(A^*y^*,x^{**}) + h_T^*(y^*,y^{**} - A^{**}x^{**})\} = h_S^*(A^*y^*,\overline{u^{**}}) + h_T^*(y^*,y^{**} - A^{**}\overline{u^{**}})$. But $h_S^{*\top}$ and $h_T^{*\top}$ are strong representative functions of $\overline{S}$ and $\overline{T}$, hence $h^*(y^*,y^{**}) \geq \langle \overline{u^{**}},A^*y^*\rangle + \langle y^{**} - A^{**}\overline{u^{**}},y^*\rangle = \langle y^{**},y^*\rangle$.

Let $(y,y^*) \in G(\overline{S}||^A\overline{T})$. Then $y \in A^{**}\overline{S}^{-1}A^*y^* + \overline{T}^{-1}y^*$, hence there exists $v_1^{**} \in A^{**}\overline{S}^{-1}A^*y^*$ and $v_2^{**} \in \overline{T}^{-1}y^*$ such that $y = v_1^{**} + v_2^{**}$. So we have $v_1^{**} \in A^{**}\overline{S}^{-1}A^*y^*$, hence there exists $u^{**} \in \overline{S}^{-1}A^*y^*$ such that $v_1^{**} = A^{**}u^{**}$. But then $(u^{**},A^*y^*) \in G(\overline{S})$. Since $y = v_1^{**} + v_2^{**}$ and $v_2^{**} \in \overline{T}^{-1}y^*$ we obtain that $(y - A^{**}u^{**},y^*) \in G(\overline{T})$. We have $h^*(y^*,y) = \min_{x^{**}\in X^{**}}\{h_S^*(A^*y^*, x^{**}) + h_T^*(y^*,y - A^{**}x^{**})\} \leq h_S^*(A^*y^*,u^{**}) + h_T^*(y^*,y - A^{**}u)$.

Taking into account that $h_S^{*\top}$ and $h_T^{*\top}$ are strong representative functions of $\overline{S}$ and $\overline{T}$ and $(u^{**},A^*y^*) \in G(\overline{S})$, $(y - A^{**}u^{**},y^*) \in G(\overline{T})$ we obtain $h^*(y^*,y) \leq \langle u^{**},A^*y^*\rangle + \langle y - A^{**}u^{**},y^*\rangle = \langle y^*,y\rangle$. On the other hand $h^*(y^*,y) \geq \langle y^*,y\rangle$ for all $(y,y^*) \in Y \times Y^*$, hence equality must be fulfilled. Therefore we have $G(\overline{S}||^A\overline{T}) \subseteq \{(y,y^*) \in Y \times Y^* : h^*(y^*,y) = \langle y^*,y\rangle\}$, which shows that $h^{*\top}\big|_{Y\times Y^*}$ is a representative function of $\overline{S}||^A\overline{T}$.

Conversely, let $h^*(y^*,y) = \langle y^*,y\rangle$. We show that $(y,y^*) \in G(\overline{S}||^A\overline{T})$. We have $\langle y^*,y\rangle = h^*(y^*,y) = h_S^*(A^*y^*,\overline{u^{**}}) + h_T^*(y^*,y - A^{**}\overline{u^{**}})$, for some $\overline{u^{**}} \in X^{**}$. But $h_S^{*\top}$ and $h_T^{*\top}$ are strong representative functions of $\overline{S}$ and $\overline{T}$, hence $h_S^*(A^*y^*,\overline{u^{**}}) \geq \langle \overline{u^{**}},A^*y^*\rangle$ with equality if, and only if, $(\overline{u^{**}},A^*y^*) \in G(\overline{S})$ and $h_T^*(y^*,y - A^{**}\overline{u^{**}}) \geq \langle y - A^{**}\overline{u^{**}},y^*\rangle$ with equality if, and only if, $(y - A^{**}\overline{u^{**}},y^*) \in G(\overline{T})$. Hence, we obtain that $\langle y^*,y\rangle = h^*(y^*,y) = h_S^*(A^*y^*,\overline{u^{**}}) + h_T^*(y^*,y - A^{**}\overline{u^{**}}) \geq \langle \overline{u^{**}},A^*y^*\rangle + \langle y - A^{**}\overline{u^{**}},y^*\rangle = \langle y^*,y\rangle$ with equality if, and only if, $(\overline{u^{**}},A^*y^*) \in G(\overline{S})$ and $(y - A^{**}\overline{u^{**}},y^*) \in G(\overline{T})$. Hence, we have $A^*y^* \in \overline{S}\overline{u^{**}}$ which leads to $\overline{u^{**}} \in \overline{S}^{-1}A^*y^*$ and $y^* \in \overline{T}(y - A^{**}\overline{u^{**}})$ which leads to $y - A^{**}\overline{u^{**}} \in \overline{T}^{-1}y^*$. Hence, $y = A^{**}\overline{u^{**}} + (y - A^{**}\overline{u^{**}}) \in A^{**}\overline{S}^{-1}A^*y^* + \overline{T}^{-1}y^*$, or equivalently $(y,y^*) \in G(\overline{S}||^A\overline{T})$. ∎

Under the additional assumption that the domain of Gossez's closure of S is a subset of X, we obtain sufficient conditions for the maximal monotonicity of Gossez type (D) for the generalized parallel sum $S||^AT$. One can notice, that $D(\overline{S}) \subseteq X$ is particulary fulfilled when X is a reflexive Banach space.

Theorem 3.2. *Consider $A : X \longrightarrow Y$ a linear and continuous operator and let us denote by A^* its adjoint operator, and by A^{**} its biadjoint operator. Let $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ be two maximal monotone operators of Gossez type* (D) *with strong representative functions h_S and h_T respectively, such that $pr_{X^*}(dom(h_S)) \cap A^*(pr_{Y^*}(dom(h_T))) \neq \emptyset$. Consider the function $h : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$, $h(y,y^*) = cl_{\|\cdot\|\times\|\cdot\|_*}(h_S \Box_1^A h_T)(y,y^*)$. Assume that $D(\overline{S}) \subseteq X$, and that one of the following conditions is fulfilled.*

(a) $0 \in {}^{ic}(pr_{X^*}(dom(h_S)) - A^*pr_{Y^*}(dom(h_T)))$.

(b) *The set $\{(x^*,y^*,A^{**}x^{**} + y^{**},r) : h_S^*(x^*,x^{**}) + h_T^*(y^*,y^{**}) \leq r\}$ is closed regarding the set $\Delta_{A^*}^{Y^*} \times Y^{**} \times \mathbb{R}$ in the $(X^*,w^*) \times (Y^*,w^*) \times (Y^{**},w^*) \times \mathbb{R}$ topology.*

Then h is a strong representative function of $S||^AT$ and $S||^AT$ is a maximal monotone operator of Gossez type (D)*.*

Proof. We need only to show, that $\overline{S}||^A\overline{T} = S||^AT$, whenever $D(\overline{S}) \subseteq X$. Indeed $(y,y^*) \in G(\overline{S}||^A\overline{T})$, if, and only if, there exists $v_1^{**} \in A^{**}\overline{S}^{-1}A^*y^*$ and $v_2^{**} \in \overline{T}^{-1}y^*$ such that $y = v_1^{**} + v_2^{**}$. This is further equivalent to the existence of $u^{**} \in \overline{S}^{-1}A^*y^*$ and $v_2^{**} \in \overline{T}^{-1}y^*$ such that $v_1^{**} = A^{**}u^{**}$ and $y = v_1^{**} + v_2^{**}$. But then $(u^{**},A^*y^*) \in G(\overline{S})$, and from $D(\overline{S}) \subseteq X$ we have $(u^{**},A^*y^*)\in G(S)$, hence $A^{**}u^{**}=Au^{**}\in Y$, which leads to $v_1^{**}\in Y$ and $v_2^{**}\in Y$. Thus, $y=v_1^{**}+v_2^{**}\in AS^{-1}A^*y^*+T^{-1}y^*$, or. equivalently $(y,y^*)\in G(S||^AT)$. ∎

Remark 3.1. Concerning the two sufficient conditions for maximal monotonicity considered in Theorem 3.1 and Theorem 3.2, one can notice, according to Theorem 2.1, that condition (b) is fulfilled whenever condition (a) is fulfilled. In [13] an example is provided, where the latter fails, while condition (b) is valid (see [13], Example 5.1).

Remark 3.2. According to Remark 2.1,

$$^{\mathrm{ic}}(\mathrm{pr}_{X^*}(\mathrm{dom}(hs))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(h_T)))=\mathrm{ri}(\mathrm{pr}_{X^*}(\mathrm{dom}(hs))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(h_T))).$$

Remark 3.3. In [32] (Corollary 14), in a reflexive Banach space context, it is shown that if $A:X\longrightarrow Y$ is a linear operator and $S,T:X\rightrightarrows X^*$ are two maximal monotone operators with representative functions h_S and h_T, then $0\in\mathrm{icr}(A(D(S))-D(T))$ assures the maximal monotonicity of $S+A^*TA$. In this case we have ${}^{\mathrm{ic}}(A(D(S))-D(T))=\mathrm{icr}(A(D(S))-D(T))={}^{\mathrm{ic}}(A(\mathrm{pr}_X(\mathrm{dom}(h_S)))-\mathrm{pr}_Y(\mathrm{dom}(h_T))$.

Obviously, in a reflexive Banach space context, S, respectively T are maximal monotone if and only if S^{-1}, respectively T^{-1} are maximal monotone, as well that $h_S^{\top}$ respectively $h_T^{\top}$ are representative functions for S^{-1}, respectively T^{-1}, and $D(S^{-1})=R(S)$, respectively $D(T^{-1})=R(T)$. Hence, $0\in\mathrm{icr}(A^*(R(T))-R(S))$ assures the maximal monotonicity of $S||^AT$, and in this case we have ${}^{\mathrm{ic}}(A^*(R(T))-R(S))=\mathrm{icr}(A^*(R(T))-R(S))={}^{\mathrm{ic}}(A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(h_T)))-\mathrm{pr}_{X^*}(\mathrm{dom}(h_S))$. Hence, our regularity condition $(RC)^{\square_1^A}$ is actually equivalent to $0\in{}^{\mathrm{ic}}(A^*(R(T))-R(S))$, that is $\bigcup_{\lambda>0}\lambda(A^*(R(T))-R(S))$ is a closed linear subspace of X^*. It is worthwhile to mention that this result was also established by Simons in [39] in general Banach spaces. In what follows we obtain some similar results in nonreflexive Banach spaces for maximal monotone operators of Gossez type *(D)*.

Theorem 3.3. *Let X and Y be nonzero real Banach spaces, let X^* and Y^* be their dual spaces and let $A:X\longrightarrow Y$ be a linear and continuous operator and $A^*:Y^*\longrightarrow X^*$ its adjoint operator. Let $S:X\rightrightarrows X^*$ and $T:Y\rightrightarrows Y^*$ be two maximal monotone operators of Gossez type* (D)*, with strong representative functions h_S, respectively h_T, such that $pr_{X^*}(dom(h_S))\cap A^*(pr_{Y^*}(dom(h_T)))\neq\emptyset$. Then it holds:*

$$^{ic}\left(R(S)-A^*(R(T))\right)\subseteq{}^{ic}\left(coR(S)-A^*(coR(T))\right)\subseteq$$

$$^{ic}\left(pr_{X^*}(dom(h_S))-A^*(pr_{Y^*}(dom(h_T)))\right)=ri\left(pr_{X^*}(dom(h_S))-A^*(pr_{Y^*}(dom(h_T)))\right).$$

Proof. Let us denote by $C:=\mathrm{pr}_{X^*}(\mathrm{dom}(h_S))-A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(h_T)))$, and by $D:=R(S)-A^*(R(T))$. Obviously $D\subseteq\mathrm{co}D=\mathrm{co}R(S)-A^*(\mathrm{co}R(T))\subseteq C$ and according to Remark 3.2 we have ${}^{\mathrm{ic}}C=\mathrm{ri}C$.

Since, $\mathrm{co}D\subseteq C$, one has $\mathrm{aff}(\mathrm{co}D)=\mathrm{aff}(D)\subseteq\mathrm{aff}(C)$. To complete our proof, observe that it is enough to prove, that $\mathrm{aff}(C)\subseteq\mathrm{cl}(\mathrm{aff}(D))$. For this we will use an idea of L. Yao, (see [43]) and Proposition 1.2. Obviously $C\subseteq\mathrm{pr}_{X^*}(\mathrm{dom}(\varphi_S))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(\varphi_T))$, where φ_S, respectively φ_T denote the Fitzpatrick functions of the operators S, respectively T. It can be easily realized that is enough to prove, that

$$\mathrm{pr}_{X^*}(\mathrm{dom}(\varphi_S))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(\varphi_T))\subseteq\mathrm{cl}(\mathrm{aff}(D)).$$

We can assume, that $(0,0)\in G(S)$ and $(0,0)\in G(T)$. Suppose that there exists $(u^*-A^*v^*)\in\mathrm{pr}_{X^*}(\mathrm{dom}(\varphi_S))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(\varphi_T))$ such that $(u^*-A^*v^*)\notin\mathrm{cl}(\mathrm{aff}(D))$. Then, according to strong separation theorem, there exists $\delta \in \mathbb{R}$ and $p^{**} \in X^{**}$, such that

$$\langle p^{**}, u^* - A^*v^* \rangle > \delta > \sup\{\langle p^{**}, x^* \rangle : x^* \in \text{cl}(\text{aff}(D))\}.$$

We show next, that $\langle p^{**}, x^* \rangle = 0$ for all $x^* \in \text{aff}(D)$. First of all, observe, that $\sup\{\langle p^{**}x^* \rangle : x^* \in \text{cl}(\text{aff}(D))\} \geq 0$, since $0 \in \text{cl}(\text{aff}(D))$, hence $\delta > 0$. Suppose, that there exists $x^* \in \text{aff}(D)$, such that $\langle p^{**}x^* \rangle \neq 0$. Then, since $\text{aff}(D)$ is a linear space, we have

$$\delta > \left\langle p^{**}, \frac{\pm\delta}{\langle p^{**}, x^* \rangle} x^* \right\rangle = \pm\delta, \text{ impossible.}$$

Hence, $\langle p^{**}, x^* - A^*y^*) \rangle = 0$, for all $x^* \in R(S), y^* \in R(T)$. By taking $y^* = 0$ we obtain $\langle p^{**}, x^* \rangle = 0$ for all $x^* \in R(S)$, and from here results that $\langle p^{**}, A^*y^* \rangle = \langle A^{**}p^{**}, y^* \rangle = 0$, for all $y^* \in R(T)$. Let us denote by $q^{**} = A^{**}p^{**}$. Obviously $\langle p^{**}, 0 \rangle = 0$, respectively $\langle q^{**}, 0 \rangle = 0$. On the other hand, we have $\langle (0, p^{**}), (x, x^*) \rangle = 0$, for all $(x, x^*) \in G(S)$, respectively $\langle (0, q^{**}), (y, y^*) \rangle = 0$, for all $(y, y^*) \in G(T)$.

Since S and T are maximal monotone operators of Gossez type (D), according to Theorem 4.4 from [25], S and T have a unique maximal monotone extension to $X^{**} \times X^*$, respectively $Y^{**} \times Y^*$, which are their Gossez's monotone closure, $\overline{S}$, respectively $\overline{T}$. We show next, that $\langle (0, p^{**}), (x^{**}, x^*) \rangle = 0$, for all $(x^{**}, x^*) \in G(\overline{S})$, respectively $\langle (0, q^{**}), (y^{**}, y^*) \rangle = 0$, for all $(y^{**}, y^*) \in G(\overline{T})$. Let $(x^{**}, x^*) \in G(\overline{S})$. Then, there exists $(x_\alpha, x_\alpha^*) \in G(S)$, such that $x_\alpha \rightharpoonup^{w^*} x^{**}$ and $x_\alpha^* \longrightarrow^{\|\cdot\|} x^*$. Obviously, since $(x_\alpha, x_\alpha^*) \in G(S)$, we have $\langle (0, p^{**}), (x_\alpha, x_\alpha^*) \rangle = 0$, for every α. On the other hand $\langle (0, p^{**}), (x_\alpha, x_\alpha^*) \rangle = \langle 0, x_\alpha \rangle + \langle p^{**}, x_\alpha^* \rangle$, and $\langle p^{**}, x_\alpha^* \rangle \longrightarrow \langle p^{**}, x^* \rangle$, hence

$$\langle (0, p^{**}), (x^{**}, x^*) \rangle = 0.$$

So we have $\langle (0, p^{**}), (x^{**}, x^*) \rangle = 0$, for all $(x^{**}, x^*) \in G(\overline{S})$, and can be proved in similar way, that $\langle (0, q^{**}), (y^{**}, y^*) \rangle = 0$, for all $(y^{**}, y^*) \in G(\overline{T})$.

According to Proposition 1.2, $\langle (0, p^{**}), (x^{**}, x^*) \rangle = 0$, for all $(x^{**}, x^*) \in \text{dom}(\varphi_{\overline{S}})$, respectively $\langle (0, q^{**}), (y^{**}, y^*) \rangle = 0$, for all $(y^{**}, y^*) \in \text{dom}(\varphi_{\overline{T}})$. But $u^* \in \text{pr}_{X^*}(\text{dom}(\varphi_S))$, respectively $v^* \in \text{pr}_{Y^*}(\text{dom}(\varphi_T))$ and it is well known, that the restriction to $X \times X^*$ of $\varphi_{\overline{S}}$ is φ_S and the restriction to $Y \times Y^*$ of $\varphi_{\overline{T}}$ is φ_T, hence $u^* \in \text{pr}_{X^*}(\text{dom}(\varphi_{\overline{S}}))$ and $v^* \in \text{pr}_{Y^*}(\text{dom}(\varphi_{\overline{T}}))$.

Hence, $0 = \langle p^{**}, u^* \rangle - \langle q^{**}, v^* \rangle = \langle p^{**}, u^* - A^*v^* \rangle) > \delta > 0$, contradiction. Thus, since $D \subseteq \text{co}D \subseteq C$, we have $\text{aff}(D) = \text{aff}(\text{co}D) \subseteq \text{aff}(C) \subseteq \text{cl}(\text{aff}(D))$, hence if $u^* \in {}^{\text{ic}}(D)$ then $u^* \in {}^{\text{ic}}(\text{co}D)$ and $u^* \in {}^{\text{ic}}(C)$. ∎

Let us mention, that the proof of Theorem 3.3 is an adaptation of the proof of Theorem 3.3 from [13]. Theorem 3.3 gives rise to two supplementary interior point type conditions for the maximal monotonicity of $\overline{S}||^A\overline{T}$.

Corollary 3.1. Let $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ be two maximal monotone operators of Gossez type *(D)*, and let $A : X \longrightarrow Y$ be a linear and continuous operator, with its adjoint denoted by A^*. Assume, that $R(S) \cap A^*(R(T)) \neq \emptyset$. If

$$0 \in {}^{\text{ic}}\big(R(S) - A^*(R(T))\big)$$

or

$$0 \in {}^{\text{ic}}\big(\text{co}R(S) - A^*(\text{co}R(T))\big),$$

then the generalized parallel sum $\overline{S}||^A\overline{T}$ is maximal monotone operator of Gossez type *(D)*.

Proof. Since, $R(S)\cap A^*(R(T))\neq\emptyset$, one has $\mathrm{pr}_{X^*}(\mathrm{dom}(\varphi_S))\cap A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(\varphi_T)))\neq\emptyset$, where φ_S, respectively φ_T are the Fitzpatrick functions of S, respectively T. Obviously φ_S, respectively φ_T are strong representative functions of S, respectively T. According to Theorem 3.3,

$$^{\mathrm{ic}}\big(R(S)-A^*(R(T))\big)\subseteq{}^{\mathrm{ic}}\big(\mathrm{co}R(S)-A^*(\mathrm{co}R(T))\big)\subseteq{}^{\mathrm{ic}}\big(\mathrm{pr}_{X^*}(\mathrm{dom}(\varphi_S))-A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(\varphi_T)))\big),$$

hence $0\in{}^{\mathrm{ic}}\big(R(S)-A^*(R(T))\big)$ or $0\in{}^{\mathrm{ic}}\big(\mathrm{co}R(S)-A^*(\mathrm{co}R(T))\big)$ implies $0\in{}^{\mathrm{ic}}\big(\mathrm{pr}_{X^*}(\mathrm{dom}(\varphi_S))-A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(\varphi_T)))\big)$. The conclusion follows from Theorem 3.1. ∎

Under the assumption $D(\overline{S})\subseteq X$, the inclusions in Theorem 3.3 become equalities. Hence, concerning on the maximal monotonicity of the generalized parallel sum $S||^AT$, we have the following result.

Theorem 3.4. *Let X and Y be nonzero real Banach spaces, let X^* and Y^* be their dual spaces and let $A:X\longrightarrow Y$ be a linear and continuous operator and $A^*:Y^*\longrightarrow X^*$ its adjoint operator. Let $S:X\rightrightarrows X^*$ and $T:Y\rightrightarrows Y^*$ be two maximal monotone operators of Gossez type* (D)*, with strong representative functions h_S, respectively h_T, such that $pr_{X^*}(dom(h_S))\cap A^*(pr_{Y^*}(dom(h_T)))\neq\emptyset$. Assume that $D(\overline{S})\subseteq X$. Then, the following hold.*

1° $ri(R(S)-A^*(R(T)))={}^{ic}(R(S)-A^*(R(T)))=$
$ri(coR(S)-A^*(coR(T)))={}^{ic}(coR(S)-A^*(coR(T)))=$
$ri(pr_{X^*}(dom(h_S))-A^*(pr_{Y^*}(dom(h_T))))={}^{ic}(pr_{X^*}(dom(h_S))-A^*(pr_{Y^*}(dom(h_T))))$.

2° *The following statements are equivalent.*
  (a) $0\in ri(R(S)-A^*(R(T)))$,
  (b) $0\in{}^{ic}(R(S)-A^*(R(T)))$,
  (c) $0\in ri(coR(S)-A^*(coR(T)))$,
  (d) $0\in{}^{ic}(coR(S)-A^*(coR(T)))$,
  (e) $0\in ri(pr_{X^*}(dom(h_S))-A^*(pr_{Y^*}(dom(h_T))))$,
  (f) $0\in{}^{ic}(pr_{X^*}(dom(h_S))-A^*(pr_{Y^*}(dom(h_T))))$.

3° *Every condition from 2° assures that the generalized parallel sum $S||^AT$ is a maximal monotone operator of Gossez type* (D).

Proof. Obviously 2° follows from 1°, and 3° follows from 2° and Theorem 3.2. Let us prove 1°. Let us denote by $C:=\mathrm{pr}_{X^*}(\mathrm{dom}(h_S))-A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(h_T)))$, and by $D:=R(S)-A^*(R(T))$. Then $\mathrm{co}R(S)-A^*(\mathrm{co}R(T))=\mathrm{co}D$. Obviously $D\subseteq C$, and we prove that ${}^{\mathrm{ic}}(C)\subseteq D$. Let $(u^*-A^*v^*)\in{}^{\mathrm{ic}}(C)$. Then $0\in{}^{\mathrm{ic}}(C-(u^*-A^*v^*)$, and consider the functions $\tilde f:X\times X^*\longrightarrow\overline{\mathbb{R}}$, $\tilde f(x,x^*)=h_S(x,x^*+u^*)-\langle u^*,x\rangle$, and $\tilde g:Y\times Y^*\longrightarrow\overline{\mathbb{R}}$, $\tilde g(y,y^*)=h_T(y,y^*+v^*)-\langle v^*,y\rangle$.

Let $\tilde S:X\rightrightarrows X^*$ defined by $G(\tilde S)=\{(x,x^*)\in X\times X^*:\tilde f(x,x^*)=\langle x^*,x\rangle\}$ and $\tilde T:Y\rightrightarrows Y^*$ defined by $G(\tilde T)=\{(y,y^*)\in Y\times Y^*:\tilde g(y,y^*)=\langle y^*,y\rangle\}$. It can be easily observed, that $G(\tilde S)=G(S)-(0,u^*)$ and $G(\tilde T)=G(T)-(0,v^*)$. Obviously $\tilde S$ and $\tilde G$ are maximal monotone operators of Gossez type (D), and $\tilde f$, respectively $\tilde g$ are their strong representative functions, hence according to Theorem 3.2, the condition $0\in{}^{\mathrm{ic}}(\mathrm{pr}_{X^*}(\mathrm{dom}(\tilde f))-A^*\mathrm{pr}_{Y^*}(\mathrm{dom}(\tilde g)))={}^{\mathrm{ic}}(C-(u^*-A^*v^*))$ ensures the maximal monotonicity of $\tilde S||^A\tilde T$. Hence, $G(\tilde S||^A\tilde T)\neq\emptyset$, thus there exists $y^*\in(A\tilde S^{-1}A^*+\tilde T^{-1})^{-1}(y)$ for some $y\in Y$. Hence, there exists $y_1,y_2\in Y$ such that $(y^*,y_1)\in G(A\tilde S^{-1}A^*)$ and $(y_2,y^*)\in G(\tilde T)$. Since $G(\tilde T)=G(T)-(0,v^*)$ we have

$$(0,v^*)\in G(T)-(y_2,y^*)\Rightarrow A^*v^*\in A^*(R(T))-A^*y^*.\ (*)$$

Since $y_1 \in A\tilde{S}^{-1}A^*(y^*)$, there exists $x^* \in X^*$, such that $y_1 \in A\tilde{S}^{-1}(x^*)$ and $x^* = A^*y^*$. Thus, there exists $x \in \tilde{S}^{-1}(x^*)$ and $y_1 = Ax$. Hence, $(x,x^*) \in G(\tilde{S}) = G(S) - (0,u^*)$ and we obtain, that

$$u^* \in R(S) - x^* = R(S) - A^*y^*. \, (**)$$

From $(*)$ and $(**)$ we have $u^* - A^*v^* \in ((R(S) - A^*y^*) - (A^*(R(T)) - A^*y^*)) = D$. Hence, ${}^{\mathrm{ic}}(C) \subseteq D$.

If ${}^{\mathrm{ic}}(C) = \mathrm{ri}(C) = \emptyset$, then by Theorem 3.3 it holds ${}^{\mathrm{ic}}(D) = {}^{\mathrm{ic}}(\mathrm{co}D) = {}^{\mathrm{ic}}(C) = \mathrm{ri}(C) = \emptyset$, consequently $\mathrm{ri}(D) = \mathrm{ri}(\mathrm{co}D) = \emptyset$.

If ${}^{\mathrm{ic}}(C) \neq \emptyset$ we have ${}^{\mathrm{ic}}(C) \subseteq D \subseteq \mathrm{co}D \subseteq C$. Hence ${}^{\mathrm{ic}}(D) = {}^{\mathrm{ic}}(\mathrm{co}D) = {}^{\mathrm{ic}}(C) = \mathrm{ri}(C)$. Moreover, it holds $\mathrm{aff}({}^{\mathrm{ic}}(C)) = \mathrm{aff}(C)$ and as $\mathrm{ri}(C) = {}^{\mathrm{ic}}(C) \subseteq D \subseteq \mathrm{co}D \subseteq C$, we have $\mathrm{aff}(C) = \mathrm{aff}(D)$, these sets being closed. Thus $\mathrm{ri}(C) = \mathrm{ri}(D) = \mathrm{ri}(\mathrm{co}D)$. ▮

4. The maximal monotonicity of the parallel sum $S||T$

In the sequel, unless is otherwise specified, X is a nonzero Banach space, and X^* respectively X^{**} denote its dual, respectively its bidual.

Consider the monotone operators $S : X \rightrightarrows X^*$ and $T : X \rightrightarrows X^*$. Their parallel sum is defined as

$$S||T : X \rightrightarrows X^*, S||T := (S^{-1} + T^{-1})^{-1}.$$

Let us denote by $\overline{S}$, respectively $\overline{T}$ the Gossez monotone closure of S, respectively T. Then their parallel sum may be introduced as

$$\overline{S}||\overline{T} : X \rightrightarrows X^*, \overline{S}||\overline{T} := (\overline{S}^{-1} + \overline{T}^{-1})^{-1}.$$

In the literature there are only few regularity conditions, (and even those in reflexive Banach spaces), that assure the maximal monotonicity of the parallel sum of two maximal monotone operators (see [2, 26, 31, 33]). Relying on the results from the previous sections, we are able to give both closedness type and interior point type regularity conditions that ensure the maximal monotonicity of the parallel sum of two maximal monotone operators. Let us mention that some of these results were also established by Simons in [39], and Boţ and László in [13].

As a particular case of Theorem 3.1, when $X = Y, A \equiv id_X$, we obtain the following result.

Theorem 4.1. *Let $S : X \rightrightarrows X^*$ and $T : X \rightrightarrows X^*$ be two maximal monotone operators of Gossez type* (D)*, with strong representative functions h_S and h_T respectively, such that $pr_{X^*}(dom(h_S)) \cap pr_{X^*}(dom(h_T)) \neq \emptyset$. Assume that one of the following conditions is fulfilled.*

(a) $0 \in {}^{ic}(pr_{X^*}(dom(h_S)) - pr_{X^*}(dom(h_T)))$.
(b) *The set $\{(x^*, y^*, x^{**} + y^{**}, r) : h_S^*(x^*, x^{**}) + h_T^*(y^*, y^{**}) \leq r\}$ is closed regarding the set $\Delta_{X^*} \times X^{**} \times \mathbb{R}$ in the $(X^*, w^*) \times (X^*, w^*) \times (X^{**}, w^*) \times \mathbb{R}$ topology.*

Then the function $h : X \times X^* \longrightarrow \overline{\mathbb{R}}, h(x,x^*) = cl_{\|\cdot\| \times \|\cdot\|_*}(h_S \Box_1 h_T)(x,x^*)$ is a strong representative function of $\overline{S}||\overline{T}$ and the parallel sum $\overline{S}||\overline{T}$ is maximal monotone operator of Gossez type (D).

Under the additional assumption that the domain of Gossez's closure of S is a subset of X, as a particular case of Theorem 3.2, we obtain sufficient conditions for the maximal monotonicity of Gossez type (D) for the parallel sum $S||T$. One can notice, that $D(\overline{S}) \subseteq X$ is particulary fulfilled when X is a reflexive Banach space.

Theorem 4.2. *Let $S : X \rightrightarrows X^*$ and $T : X \rightrightarrows X^*$ be two maximal monotone operators of Gossez type* (D) *with strong representative functions h_S and h_T respectively, such that $pr_{X^*}(dom(h_S)) \cap pr_{X^*}(dom(h_T)) \neq \emptyset$. Consider the function $h : X \times X^* \longrightarrow \overline{\mathbb{R}}$, $h(x,x^*) = cl_{\|\cdot\|\times\|\cdot\|_*}(h_S \Box_1 h_T)(x,x^*)$. Assume that $D(\overline{S}) \subseteq X$, and that one of the following conditions is fulfilled.*

(a) $0 \in {}^{ic}(pr_{X^*}(dom(h_S)) - pr_{X^*}(dom(h_T)))$.

(b) *The set $\{(x^*, y^*, x^{**} + y^{**}, r) : h_S^*(x^*, x^{**}) + h_T^*(y^*, y^{**}) \le r\}$ is closed regarding the set $\Delta_{X^*} \times X^{**} \times \mathbb{R}$ in the $(X^*, w^*) \times (X^*, w^*) \times (X^{**}, w^*) \times \mathbb{R}$ topology.*

Then h is a strong representative function of $S||T$ and $S||T$ is a maximal monotone operator of Gossez type (D).

As particular instances of Theorem 3.3 and Corollary 3.1 we have the following result.

Theorem 4.3. *Let X be a nonzero real Banach spaces, let X^* be its dual space and let $S : X \rightrightarrows X^*$ and $T : X \rightrightarrows X^*$ be two maximal monotone operators of Gossez type* (D)*, with strong representative functions h_S, respectively h_T, such that $pr_{X^*}(dom(h_S)) \cap pr_{X^*}(dom(h_T)) \neq \emptyset$.*

(a) *Then it holds:*

$$ {}^{ic}\big(R(S) - R(T)\big) \subseteq {}^{ic}\big(coR(S) - coR(T)\big) \subseteq $$

$$ {}^{ic}\big(pr_{X^*}(dom(h_S)) - pr_{X^*}(dom(h_T))\big) = ri\big(pr_{X^*}(dom(h_S)) - pr_{X^*}(dom(h_T))\big). $$

(b) *If*

$$ 0 \in {}^{ic}\big(R(S) - R(T)\big) $$

or

$$ 0 \in {}^{ic}\big(coR(S) - coR(T)\big), $$

then the parallel sum $\overline{S}||\overline{T}$ is maximal monotone operator of Gossez type (D).

Let us mention that the condition $0 \in {}^{\mathrm{ic}}\big(R(S) - R(T)\big)$ which ensures that the parallel sum $\overline{S}||\overline{T}$ is maximal monotone operator of Gossez type *(D)* was also obtained by Simons in [39], as well that Theorem 4.3 was also obtained by Boţ and László in [13]. Under the assumption $D(\overline{S}) \subseteq X$, the inclusions in Theorem 4.3 become equalities. Hence, as a particular instance of Theorem 3.4, concerning on the maximal monotonicity of the parallel sum $S||T$, we have the following result.

Theorem 4.4. *Let X be a nonzero real Banach spaces, let X^* be its dual space and let $S : X \rightrightarrows X^*$ and $T : X \rightrightarrows X^*$ be two maximal monotone operators of Gossez type* (D)*, with strong representative functions h_S, respectively h_T, such that $pr_{X^*}(dom(h_S)) \cap pr_{X^*}(dom(h_T)) \neq \emptyset$. Assume that $D(\overline{S}) \subseteq X$. Then, the following hold.*

1° $ri(R(S) - R(T)) = {}^{ic}(R(S) - R(T)) =$
$ri(coR(S) - coR(T)) = {}^{ic}(coR(S) - coR(T)) =$
$ri(pr_{X^*}(dom(h_S)) - pr_{X^*}(dom(h_T))) = {}^{ic}(pr_{X^*}(dom(h_S)) - pr_{X^*}(dom(h_T)))$.

2° *The following statements are equivalent.*

(a) $0 \in ri(R(S) - R(T))$,

(b) $0 \in {}^{ic}(R(S) - R(T))$,

(c) $0 \in ri(coR(S) - coR(T))$,

(d) $0 \in {}^{ic}(coR(S) - coR(T))$,
(e) $0 \in ri(pr_{X^*}(dom(h_S)) - pr_{X^*}(dom(h_T)))$,
(f) $0 \in {}^{ic}(pr_{X^*}(dom(h_S)) - pr_{X^*}(dom(h_T)))$.

3° *Every condition from* 2° *assures that the parallel sum* $S||T$ *is a maximal monotone operator of Gossez type* (D).

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